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Higher-dimensional weak amenability

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Abstract. Bade, Curtis and Dales have introduced the idea of weak amenability. A commutative Banach algebra $\mathfrak{A}$ is weakly amenable if there are no non-zero continuous derivations from $\mathfrak{A}$ to $\mathfrak{A}^*$. We extend this by defining an alternating n-derivation to be an alternating n-linear map from $\mathfrak{A}$ to $\mathfrak{A}^*$ which is a derivation in each of its variables. Then we say that $\mathfrak{A}$ is n-dimensionally weakly amenable if there are no non-zero continuous alternating n-derivations on $\mathfrak{A}$. Alternating n-derivations are the same as alternating Hochschild cocycles. Since such a cocycle is a coboundary if and only if it is 0, the alternating n-derivations form a subspace of $H^n(\mathfrak{A},\mathfrak{A}^*)$. The hereditary properties of n-dimensional weak amenability are studied; for example, if J is a closed ideal in $\mathfrak{A}$ such that $\mathfrak{A}/J$ is m-dimensionally weakly amenable and J is n-dimensionally weakly amenable then $\mathfrak{A}$ is $(m+n-1)$-dimensionally weakly amenable. Results of Bade, Curtis and Dales are extended to n-dimensional weak amenability. If $\mathfrak{A}$ is generated by n elements then it is $(n+1)$-dimensionally weakly amenable. If $\mathfrak{A}$ contains enough regular elements a with $\|a^m\| = o(m^{n/(n+1)})$ as $m \to \pm\infty$ then $\mathfrak{A}$ is n-dimensionally weakly amenable. It follows that if $\mathfrak{A}$ is the algebra $\mathrm{lip}_\alpha(X)$ of Lipschitz functions on the metric space X and $\alpha < n/(n+1)$ then $\mathfrak{A}$ is n-dimensionally weakly amenable. When X is the product of n copies of the circle then $\mathfrak{A}$ is n-dimensionally weakly amenable if and only if $\alpha < n/(n+1)$.

1. Introduction. Throughout this paper $\mathfrak{A}$ denotes a commutative Banach algebra and $\mathfrak{X}$ a symmetric Banach $\mathfrak{A}$-bimodule, that is, we have $ax = xa$ for all $a \in \mathfrak{A}$, $x \in \mathfrak{X}$. Following [1], $\mathfrak{A}$ is *weakly amenable* if, for all $\mathfrak{X}$, all derivations from $\mathfrak{A}$ into $\mathfrak{X}$ are zero. In this paper we extend this by saying that $\mathfrak{A}$ is *n-dimensionally weakly amenable* [Definition 2.1] if, for all $\mathfrak{X}$, all alternating n-cocycles from $\mathfrak{A}$ into $\mathfrak{X}$ are zero. By an *n-cocycle* we mean a continuous n-linear map from $\mathfrak{A}$ into $\mathfrak{X}$ whose Hochschild coboundary is 0 (cf. [5]). For $n = 1$ this reduces to weak amenability in the sense of Bade, Curtis and Dales. In Section 2 we show that an alternating n-cocycle is the same as an alternating linear map which is a derivation in each of its variables. This enables us to show how the values of an alternating n-cocycle are related to its values on the generators of an algebra and show in particular that if $\mathfrak{A}$ has n-generators then it is $(n+1)$-dimensionally

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weakly amenable. We also show that we need consider only the case $\mathfrak{X} = \mathfrak{A}^*$ (Corollary 2.11). In Section 3 we consider the hereditary properties of n-dimensional weak amenability. In particular, we show that a closed ideal J in an n-dimensionally weakly amenable algebra is n-dimensionally weakly amenable if and only if J/J_2 has dimension less than n, where J_2 is the closed linear span of $\{j_1 j_2 : j_1, j_2 \in J\}$. In Section 4 we extend the results of [1]. We show that n-dimensional weak amenability is implied by growth conditions and give a full account of conditions under which Beurling and Lipschitz algebras are n-dimensionally weakly amenable.

2. Alternating cocycles and n-derivations. Most of the work in this section and the next is algebraic and would apply to abstract modules over abstract algebras, though we make no further reference to this. We shall denote the closed linear span of the k-fold products in an algebra $\mathfrak{A}$ by $\mathfrak{A}_k$, the space of continuous n-linear maps from $\mathfrak{A}$ to $\mathfrak{X}$ by $\mathcal{L}^n(\mathfrak{A}, \mathfrak{X})$ and the coboundary operator from $\mathcal{L}^n(\mathfrak{A}, \mathfrak{X})$ to $\mathcal{L}^{n+1}(\mathfrak{A}, \mathfrak{X})$ by δ^{n+1} (or just δ when it is clear which δ^n we mean). The symmetric group S_n acts on the n-fold tensor product $\mathfrak{A} \widehat{\otimes} \ldots \widehat{\otimes} \mathfrak{A}$ on the right by

$$t\sigma = a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(n)},$$

where $t = a_1 \otimes \ldots \otimes a_n$.

The adjoint action makes $\mathcal{L}^n(\mathfrak{A}, \mathfrak{X})$ a left S_n-module. An element T of $\mathcal{L}^n(\mathfrak{A}, \mathfrak{X})$ is *alternating* if

$$\sigma T = (\operatorname{Sgn} \sigma) T$$

for all $\sigma \in S_n$. The higher-dimensional analogue of weak amenability is then

DEFINITION 2.1. The commutative Banach algebra $\mathfrak{A}$ is *n-dimensionally weakly amenable* if for every symmetric Banach $\mathfrak{A}$-bimodule $\mathfrak{X}$, every bounded alternating n-cocycle from $\mathfrak{A}$ into $\mathfrak{X}$ is 0.

Since every element of $\mathcal{L}^1(\mathfrak{A}, \mathfrak{X})$ is alternating, 1-dimensional weak amenability is just weak amenability.

The alternating cocycles have a much simpler description.

DEFINITION 2.2. An element T of $\mathcal{L}^n(\mathfrak{A}, \mathfrak{X})$ is an *n-derivation* if for $k = 1, \ldots, n$, $a_1, \ldots, a_n, a'_k \in \mathfrak{A}$,

$$T(a_1, \ldots, a_k a'_k, \ldots, a_n) = a_k T(a_1, \ldots, a'_k, \ldots, a_n) + T(a_1, \ldots, a_k, \ldots, a_n) a'_k.$$

Thus T is an n-derivation if and only if it is a derivation in each variable. If T is alternating then it is an n-derivation if and only if it is a derivation in one of the variables.

THEOREM 2.3. *Let $\mathfrak{A}$ be a commutative Banach algebra and $\mathfrak{X}$ a symmetric Banach $\mathfrak{A}$-bimodule. Then $T \in \mathcal{L}^n(\mathfrak{A}, \mathfrak{X})$ is an alternating n-derivation if and only if it is an alternating n-cocycle.*

Proof. Suppose that T is an alternating n-derivation. We have

$$\begin{aligned}(\delta T)(a_0, \ldots, a_n) = a_0 T(a_1, \ldots, a_n) &\\ + \sum_{j=1}^{n} (-1)^j T(a_0, \ldots, a_{j-1} a_j, \ldots, a_n) &\\ + (-1)^{n+1} T(a_0, \ldots, a_{n-1}) a_n.&\end{aligned}$$

If we express each term in $\sum$ as a sum of two terms using the n-derivation rule, all the terms in δT cancel, so $\delta T = 0$.

Conversely, if T is an alternating n-cocycle and $a_0, \ldots, a_n \in \mathfrak{A}$, define a_j for the other values of j to make $\{a_j : j \in \mathbb{Z}\}$ an $(n+1)$-periodic sequence. Put

$$t_j = (-1)^{nj} T(a_j a_{j+1}, a_{j+2}, \ldots, a_{j+n}), \quad s_j = (-1)^{nj} T(a_{j+1}, \ldots, a_{j+n}).$$

Because $n(n+1)$ is even, $\{s_j\}$ and $\{t_j\}$ are $(n+1)$-periodic. The formula $\delta T = 0$ gives, using the alternating property of T,

$$t_{j+1} + \ldots + t_{j+n} = s_{j+1} - s_j.$$

Adding these equations for $j = 1, \ldots, n$ and subtracting the $j = 0$ equation $n - 1$ times gives

$$nt_0 = n(s_0 - s_1),$$

from which we see that T is a derivation in the first variable. As T is alternating, it is an n-derivation.

COROLLARY 2.4. (i) *$\mathfrak{A}$ is n-dimensionally weakly amenable if and only if $\mathfrak{A}^1$ is.*

(ii) *If $\mathfrak{A}$ is n-dimensionally weakly amenable then* $\dim(\mathfrak{A}/\mathfrak{A}_2) < n$.

Proof. (i) If $\mathfrak{A}^1$ is n-dimensionally weakly amenable and T is an alternating n-derivation on $\mathfrak{A}$ with values in $\mathfrak{X}$ then $\mathfrak{X}$ becomes a symmetric unital $\mathfrak{A}^1$-module and T extends to an alternating n-derivation on $\mathfrak{A}^1$ if we define $T(b_1, \ldots, b_n) = 0$ when any of the b_j lie in $\mathbb{C}1$, so $T = 0$.

If $\mathfrak{A}$ is n-dimensionally weakly amenable and T is an alternating n-derivation on $\mathfrak{A}^1$ with values in $\mathfrak{X}$ then $T|\mathfrak{A} = 0$ and $T(b_1, \ldots, b_n) = 0$ if any of the b_j is 1, by the derivation identity, so $T = 0$.

(ii) With trivial action, $\mathbb{C}$ is an $\mathfrak{A}$-module. If $\dim(\mathfrak{A}/\mathfrak{A}_2) \geq n$, let $a'_1, \ldots, a'_n$ be elements of $\mathfrak{A}$ which are linearly independent modulo $\mathfrak{A}_2$ and let $f_1, \ldots, f_n \in \mathfrak{A}^*$ with $f(\mathfrak{A}_2) = 0$ and $f_i(a'_j) = \delta_{ij}$. Then

$$T(a_1, \ldots, a_n) = \det f_i(a_j)$$

is a non-trivial continuous n-linear map into $\mathbb{C}$ which is an n-derivation because each side of the derivation identity is 0.

COROLLARY 2.5. *If $\mathfrak{A}$ is n-dimensionally weakly amenable then it is p-dimensionally weakly amenable for all $p > n$.*

Proof. An alternating p-derivation becomes an alternating n-derivation if we hold $p - n$ variables constant.

COROLLARY 2.6. *Let T be an alternating n-cocycle. Then*

$$T(b_1, \ldots, b_n) = \sum m_{1\sigma(1)} \ldots m_{n\sigma(n)} a_1^{m_{\cdot 1}} \ldots a_p^{m_{\cdot p}} a_{\sigma(1)}^{-1} \ldots a_{\sigma(n)}^{-1} T(a_{\sigma(1)}, \ldots, a_{\sigma(n)}),$$

where:

(i) *p is a positive integer and $a_1, \ldots, a_p \in \mathfrak{A}$.*

(ii) *$[m_{ij}]$ is an $n \times p$ matrix of integers with $m_{ij} \geq 0$ for all i if a_j is singular.*

(iii) *$m_{\cdot j} = \sum_{i=1}^n m_{ij}$ and $b_i = a_1^{m_{i1}} \ldots a_p^{m_{ip}}$.*

(iv) *The sum is over all functions σ from $\{1, \ldots, n\}$ to $\{1, \ldots, p\}$.*

(v) *If a_j is singular then $a_j^m a_j^{-1}$ is interpreted as a_j^{m-1} if $m > 0$ and as 0 if $m = 0$ (in which case the coefficient of the term is 0 anyway).*

Proof. It follows from the derivation identity that

$$T(c_1^k, c_2, \ldots, c_n) = k c_1^{k-1} T(c_1, c_2, \ldots, c_n)$$

for positive integers k and, if c_1 is non-singular, for all integers k. Repeated application of this gives

$$T(c_1^{k_1}, c_2^{k_2}, \ldots, c_n^{k_n}) = k_1 \ldots k_n c_1^{k_1 - 1} \ldots c_n^{k_n - 1} T(c_1, \ldots, c_n).$$

The n-derivation law also gives

$$T(b_1, \ldots, b_n) = \sum a_1^{m_{\cdot 1}} \ldots a_p^{m_{\cdot p}} a_{\sigma(1)}^{-m_{1\sigma(1)}} \ldots a_{\sigma(n)}^{-m_{n\sigma(n)}} T(a_{\sigma(1)}^{m_{1\sigma(1)}}, \ldots, a_{\sigma(n)}^{m_{n\sigma(n)}}),$$

which, together with the identity in the previous sentence and the fact that T is alternating, gives the required result.

COROLLARY 2.7. (i) *If $p < n$ then $T(b_1, \ldots, b_n) = 0$.*

(ii) *If $p = n$ then $T(b_1, \ldots, b_n) = \det[m_{ij}] a_1^{m_{\cdot 1} - 1} \ldots a_n^{m_{\cdot n} - 1} T(a_1, \ldots, a_n)$.*

Proof. (i) If $p < n$ then σ is not injective so that, because T is alternating, $T(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = 0$ for all σ.

(ii) If $p = n$ we can omit the terms for which σ is not injective. The result follows because for $\sigma \in S_n$, $\sigma T = (\operatorname{Sgn} \sigma) T$.

COROLLARY 2.8. *If $\mathfrak{A}$ is generated (polynomially or rationally) by n elements then $\mathfrak{A}$ is $(n+1)$-dimensionally weakly amenable.*

Proof. If $a_1, \ldots, a_n \in \mathfrak{A}$ and the polynomials in $a_1, \ldots, a_n$ are dense in $\mathfrak{A}$ the result follows from 2.7(i).

For the set of rational functions in $a_1, \ldots, a_n$, that is, the set of elements P/Q when P and Q are polynomials in $a_1, \ldots, a_n$ and Q is a regular element of $\mathfrak{A}$, the result follows in the same way.

In considering the connection between n-dimensional weak amenability and cohomology the following is useful.

PROPOSITION 2.9. *If T is an alternating n-coboundary, that is, an alternating element of $\delta\mathcal{L}^{n-1}(\mathfrak{A}, \mathfrak{X})$, then $T = 0$.*

Proof. Let $R \in \mathcal{L}^{n-1}(\mathfrak{A}, \mathfrak{X})$ with $\delta R = T$. We have seen that $\mathcal{L}^n(\mathfrak{A}, \mathfrak{X})$ is a unital S_n-module and so it is a unital $\ell^1(S_n)$-module. Define $z \in \ell^1(S_n)$ by

$$z = (n!)^{-1} \sum (\operatorname{Sgn} \sigma) \sigma.$$

Then z is idempotent and $S \in \mathcal{L}^n(\mathfrak{A}, \mathfrak{X})$ is alternating if and only if $zS = S$. If $\tau \in S_n$ then $z\tau = (\operatorname{Sgn} \tau) z$. Thus if S is symmetric in two variables so that $\tau S = S$ for some transposition then $zS = z\tau S = -zS$, that is, $zS = 0$. Similarly, if τ is the n-cycle $\tau(j) = j + 1 \pmod{n}$ then $z(S + (-1)^n \tau S) = zS - zS = 0$ because $z\tau = (-1)^{n+1} z$. Thus $zT = z\delta R = 0$ because the first and last terms in δR give 0 by the previous remark and all the other terms are symmetric in two of the variables and so give 0 as well. Hence $T = zT = 0$.

COROLLARY 2.10. *The quotient map $\ker \delta \to H^n(\mathfrak{A}, \mathfrak{X})$ is injective on the alternating cocycles.*

This shows that if $H^n(\mathfrak{A}, \mathfrak{X}) = 0$, or more generally if $H^n(\mathfrak{A}, \mathfrak{Y}) = 0$ for some symmetric $\mathfrak{A}$-module $\mathfrak{Y}$ for which there is a continuous $\mathfrak{A}$-module injection from $\mathfrak{X}$ into $\mathfrak{Y}$, then there are no non-zero alternating n-cocycles from $\mathfrak{A}$ into $\mathfrak{X}$.

COROLLARY 2.11 (cf. [1, Theorem 1.5]). *If every alternating n-cocycle from $\mathfrak{A}$ into $\mathfrak{A}^*$ is 0 then $\mathfrak{A}$ is n-dimensionally weakly amenable.*

Proof. If $\dim(\mathfrak{A}/\mathfrak{A}_2) \geq n$ and $f \in \mathfrak{A}^*$ is non-zero but $f|\mathfrak{A}_2 = 0$ then $\lambda \mapsto \lambda f$ is an $\mathfrak{A}$-module map from $\mathbb{C}$ with trivial action into $\mathfrak{A}^*$. If we compose this with the cocycle in the proof of 2.4(ii) we have a non-trivial alternating n-cocycle from $\mathfrak{A}$ to $\mathfrak{A}^*$. Thus $\dim(\mathfrak{A}/\mathfrak{A}_2) < n$.

Let T be an alternating n-cocycle from $\mathfrak{A}$ to $\mathfrak{X}$ and let $f \in \mathfrak{X}^*$. We define a map α from $\mathfrak{X}$ to $\mathfrak{A}^*$ by

$$\alpha(x) = f(ax).$$

It is easy to check that this is an $\mathfrak{A}$-module map, so $\alpha \circ T = 0$ because it is an alternating n-cocycle with values in $\mathfrak{A}^*$. This shows that

$$f(aT(a_1, \ldots, a_n)) = 0$$

for all $f \in \mathfrak{X}^*$ and $a, a_1, \ldots, a_n \in \mathfrak{A}$. Thus $aT(a_1, \ldots, a_n) = 0$ so that, by the derivation identity, $T(a_1, \ldots, a_n) = 0$ if one of the a_j lies in $\mathfrak{A}_2$. Thus T gives an alternating n-linear map on the space $\mathfrak{A}/\mathfrak{A}_2$ whose dimension is less than n. This shows $T = 0$.

3. Hereditary properties of n-dimensional weak amenability

THEOREM 3.1. *Let $\mathfrak{A}$ be a commutative Banach algebra and let J be a closed ideal in $\mathfrak{A}$.*

(i) *If $\mathfrak{A}$ is n-dimensionally weakly amenable then so is $\mathfrak{A}/J$. More generally, if $\mathfrak{B}$ is a Banach algebra and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a continuous algebra homomorphism with dense range then $\mathfrak{B}$ is n-dimensionally weakly amenable.*

(ii) *If $\mathfrak{A}$ is n-dimensionally weakly amenable then J is n-dimensionally weakly amenable if and only if $\dim(J/J_2) < n$.*

(iii) *If J is m-dimensionally weakly amenable and $\mathfrak{A}/J$ is n-dimensionally weakly amenable then $\mathfrak{A}$ is $(m+n-1)$-dimensionally weakly amenable.*

Note. The case $n = 1$ of (ii) is Corollary 1.3 of [4].

Proof of 3.1. (i) If T is an alternating n-cocycle on $\mathfrak{B}$ with values in a symmetric $\mathfrak{B}$-module $\mathfrak{Y}$ then, using ϕ, $\mathfrak{Y}$ becomes a symmetric $\mathfrak{A}$-module and composing T with ϕ gives an alternating n-derivation on $\mathfrak{A}$ which is 0 by the n-dimensional weak amenability of $\mathfrak{A}$. This shows that $T|\phi(\mathfrak{A})$ is zero, so $T = 0$ by continuity.

(ii) If $\dim(J/J_2) \geq n$ then by Corollary 2.4, J is not n-dimensionally weakly amenable. Assume that $\mathfrak{A}$ is n-dimensionally weakly amenable and $\dim(J/J_2) < n$. Let T be an alternating n-cocycle in J with values in J^*. For each $a_1, \ldots, a_n \in \mathfrak{A}$ we define $\Delta(a_1, \ldots, a_n)$, an n-linear map from J to J^*, by

$$\Delta(a_1, \ldots, a_n)(j_1, \ldots, j_n) = \sum \operatorname{sgn} \varrho F_\varrho,$$

where the sum is over all subsets ϱ of $\{1, \ldots, n\}$, $\operatorname{sgn} \varrho = (-1)^{|\varrho|}$ and

$$F_\varrho = (b_1 \ldots b_n) T(c_1 j_1, \ldots, c_n j_n)$$

where if $i \in \varrho$ then $b_i =$ identity and $c_i = a_i$ whereas if $i \notin \varrho$ then $b_i = a_i$ and $c_i =$ identity (b_i and c_i lie in $\mathfrak{A}^1$ if $\mathfrak{A}$ has no identity). If $a_1, \ldots, a_n \in J$ then

$$\Delta(a_1, \ldots, a_n)(j_1, \ldots, j_n) = T(a_1, \ldots, a_n) j_1 \ldots j_n,$$

and Δ is introduced as a form of extension of T to $\mathfrak{A}$.

First of all we prove that for any k in J and $i \in \{1, \ldots, n\}$,

$$(*) \quad \Delta(a_1, \ldots, a_n)(j_1, \ldots, j_i k, \ldots, j_n) = \Delta(a_1, \ldots, a_n)(j_1, \ldots, j_i, \ldots, j_n) k.$$

We consider only the case $i = 1$ because the other cases are similar. If $\varrho \subseteq \{2, \ldots, n\}$ and $\varrho' = \varrho \cup \{1\}$, the terms on the left of $(*)$ corresponding to ϱ and ϱ' are

$$\begin{aligned}
&\operatorname{sgn} \varrho[(b_2 \ldots b_n) T(a_1 j_1 k, c_2 j_2, \ldots, c_n j_n) \\
&\qquad - (a_1 b_2 \ldots b_n) T(j_1 k, c_2 j_2, \ldots, c_n j_n)] \\
&= \operatorname{sgn} \varrho[(a_1 j_1 b_2 \ldots b_n) T(k, c_2 j_2, \ldots, c_n j_n) \\
&\qquad + (b_2 \ldots b_n) T(a_1 j_1, c_2 j_2, \ldots, c_n j_n) k \\
&\qquad - (a_1 b_2 \ldots b_n j_1) T(k, c_2 j_2, \ldots, c_n j_n) \\
&\qquad - (a_1 b_2 \ldots b_n) T(j_1, c_2 j_2, \ldots, c_n j_n) k] \\
&= \operatorname{sgn} \varrho[(b_2 \ldots b_n) T(a_1 j_1, c_2 j_2, \ldots, c_n j_n) \\
&\qquad - (a_1 b_2 \ldots b_n) T(j_1, c_2 j_2, \ldots, c_n j_n)] k,
\end{aligned}$$

which are the terms on the right of $(*)$ corresponding to ϱ and ϱ', so $(*)$ is proved.

Now let $\widetilde{J}$ be the closed subspace of $J \widehat{\otimes} \ldots \widehat{\otimes} J$ (n factors) spanned by tensors $j_1 \otimes \ldots \otimes j_n$ where all the j_i are in J and at least one is in J_2, and denote the restriction of $\Delta(a_1, \ldots, a_n)$ to $\widetilde{J}$ by $\widetilde{\Delta}(a_1, \ldots, a_n)$. We shall show that $\widetilde{\Delta}$ is an alternating n-cocycle from $\mathfrak{A}$ into $\mathcal{L}(\widetilde{J}, J^*)$ with the symmetric $\mathfrak{A}$-module structure

$$(aS)(t) = aS(t), \quad a \in \mathfrak{A},\ S \in \mathcal{L}(\widetilde{J}, J^*),\ t \in \widetilde{J}.$$

We have seen that S_n acts on the right of $J \widehat{\otimes} \ldots \widehat{\otimes} J$ and clearly $\widetilde{J}$ is a submodule, so $\mathcal{L}(\widetilde{J}, J^*)$ is a left S_n-module. By $(*)$,

$$\begin{aligned}
\Delta(a_1, \ldots, a_n)(j_1 k, j_2, \ldots, j_n) &= \Delta(a_1, \ldots, a_n)(j_1, j_2, \ldots, j_n) k \\
&= \Delta(a_1, \ldots, a_n)(j_1, j_2 k, \ldots, j_n) \\
&= \Delta(a_1, \ldots, a_n)(j_1 j_2, k, j_3, \ldots, j_n) \\
&= \Delta(a_1, \ldots, a_n)(j_2, j_1 k, \ldots, j_n).
\end{aligned}$$

Thus $(\sigma\Delta(a_1, \ldots, a_n))(j_1 k, j_2, \ldots, j_n) = \Delta(a_1, \ldots, a_n)(j_1 k, j_2, \ldots, j_n)$ if $\sigma = (1, 2)$ and similarly if $\sigma = (1, m)$. Since the transpositions $(1, m)$ generate S_n, this holds for all σ in S_n. It follows that

$$\Delta(a_1, \ldots, a_n)(t\sigma) = \Delta(a_1, \ldots, a_n)(t)$$

for any simple tensor $t = j_1 \otimes \ldots \otimes j_n$ where at least one of the j_i lies in J_2, so that $\sigma\Delta(a_1, \ldots, a_n) = \Delta(a_1, \ldots, a_n)$ for all σ in S_n. It follows from the definition of Δ and the alternating property of T that

$$\Delta(a_{\sigma(1)}, \ldots, a_{\sigma(n)})(j_{\sigma(1)}, \ldots, j_{\sigma(n)}) = \operatorname{Sgn} \sigma \Delta(a_1, \ldots, a_n)(j_1, \ldots, j_n)$$

so, by what we have just shown,

$$\widetilde{\Delta}(a_{\sigma(1)},\dots,a_{\sigma(n)})(t)=\operatorname{Sgn}\sigma\,\widetilde{\Delta}(a_1,\dots,a_n)(t)$$

for any $t=j_1\otimes\ldots\otimes j_n$ with at least one j_i in J_2 and hence for all t in $\widetilde{J}$. This shows that $\widetilde{\Delta}$ is alternating.

We now show that $\widetilde{\Delta}$ is a derivation in its first variable. Let $a_1,\dots,a_n,a_1'\in\mathfrak{A}$, $j_1,\dots,j_n,j_1'\in J$. As in the definition of $\widetilde{\Delta}$ let $\varrho\subseteq\{1,\dots,n\}$ and $(b_1',c_1')=(1,a_1')$ or $(a_1',1)$ depending on whether $1\in\varrho$ or not. The derivation condition for T shows that

$$\begin{aligned}&(b_1\dots b_nb_1')T(c_1c_1'j_1j_2',c_2j_2,\dots,c_nj_n)\\&\quad=a_1b_2\dots b_nj_1T(c_1'j_1',c_2j_2,\dots,c_nj_n)+a_1'b_1\dots b_nj_1'T(c_1j_1,\dots,c_nj_n).\end{aligned}$$

Multiplying by $\operatorname{sgn}\varrho$ and summing, the first terms give

$$a_1j_1\Delta(a_1',a_2,\dots,a_n)(j_1',j_2,\dots,j_n)=a_1\Delta(a_1',a_2,\dots,a_n)(j_1j_1',j_2,\dots,j_n)$$

and the second terms give $a_1'\Delta(a_1,\dots,a_n)(j_1j_1',j_2,\dots,j_n)$.

Thus $\Delta(a_1a_1',a_2,\dots,a_n)=a_1\Delta(a_1',\dots,a_n)+a_1'\Delta(a_1,\dots,a_n)$ on tensors of the form $j_1j_1'\otimes j_2\otimes\ldots\otimes j_n$ and hence, using the symmetry of $\widetilde{\Delta}(a_1,\dots,a_n)$ in the J variables, for any t in the spanning set for $\widetilde{J}$ and hence for all t in $\widetilde{J}$.

Thus $\widetilde{\Delta}$ is an alternating n-cocycle and hence is 0. Using the fact that if $k_1,\dots,k_n\in J$ then

$$\Delta(k_1,\dots,k_n)(j_1,\dots,j_n)=T(k_1,\dots,k_n)j_1\dots j_n$$

this shows that $T(k_1,\dots,k_n)j=0$ whenever j is the product of $n+1$ elements from J, and hence

$$(**)\qquad (T(k_1,\dots,k_n),j)=0$$

(where $(\cdot,\cdot)$ denotes the pairing between J and J^*) whenever j is the product of $n+2$ factors from J and, more generally, if $j\in J_{n+2}$. Using the derivation law we see that $(**)$ holds for all j in J if one of the k_i is in J_{n+2}. Thus $(**)$ holds if any of the $j,k_1,\dots,k_n$ is in J_{n+2}, and so gives an alternating n-cocycle from J/J_{n+2} into $(J/J_{n+2})^*$. If $j_1,\dots,j_m$ form a basis of J modulo J_2 then $m<n$ by hypothesis and J/J_{n+2} is generated by the m elements j_i+J_{n+2}, so that $T=0$.

(iii) Put $p=m+n-1$ and let T be an alternating p-cocycle from $\mathfrak{A}$ into $\mathfrak{A}^*$. We shall prove by downwards induction on l that for $0\le l\le m$ if at least l of the variables $a_1,\dots,a_p$ lie in J then $T(a_1,\dots,a_p)=0$. The case $l=m$ follows from the m-dimensional weak amenability of J because if we restrict m of the variables to lie in J and fix the other $n-1$ then we have an alternating m-cocycle on J.

Suppose that the result holds for some $l>0$. Let $a_1,\dots,a_{l-1}\in J$ and for all $a_l,\dots,a_p$ put

$$T_0(a_l,\dots,a_p)=T(a_1,\dots,a_p).$$

Then T_0 is 0 if any of $a_l,\dots,a_p$ lie in J. By the derivation rule, if $j\in J$ then

$$T_0(a_l,\dots,a_p)j=T_0(a_l,\dots,a_pj)-T_0(a_l,\dots,a_{p-1}j)a_p=0.$$

Thus the values of T_0 lie in $J_{\mathfrak{A}}^{\perp}=(\mathfrak{A}/J_{\mathfrak{A}})^*$, where $J_{\mathfrak{A}}=$ closed linear span $\{aj:a\in\mathfrak{A},\ j\in J\}$. As $\mathfrak{A}/J_{\mathfrak{A}}$ and hence $(\mathfrak{A}/J_{\mathfrak{A}})^*$ are symmetric $\mathfrak{A}$-modules on which J acts trivially, they become $(\mathfrak{A}/J)$-modules and T_0 gives a $(p-l+1)$-cocycle from $\mathfrak{A}/J$ to $(\mathfrak{A}/J_{\mathfrak{A}})^*$, where $p-l+1\ge n$. It is easy to check that T_0 is alternating and is a derivation in each of its variables because T is. Thus, because $\mathfrak{A}/J$ is n-dimensionally weakly amenable, $T_0=0$. Thus $T(a_1,\dots,a_n)=0$ if $a_1,\dots,a_{l-1}\in J$ and hence, by the alternating property, if any $l-1$ of $a_1,\dots,a_p$ lie in J. This completes the induction and the case $l=0$ gives $T=0$ as required.

EXAMPLE 3.2. To show that the dimension $m+n-1$ in (iii) above cannot be improved in general, consider the $m+n-2$ zero algebra $\mathfrak{A}$. Because $\dim(\mathfrak{A}/\mathfrak{A}_2)=m+n-2$, $\mathfrak{A}$ is not $(m+n-2)$-dimensionally weakly amenable, but if J is an $(m-1)$-dimensional subspace then it is an $(m-1)$-generated ideal and so is m-dimensionally weakly amenable, and $\mathfrak{A}/J$ is n-dimensionally weakly amenable for the same reasons.

For amenability, a closed ideal in an amenable Banach algebra is amenable if it has a closed complementary subspace (or even if it is weakly complemented [3, Theorem 3.7]. Thus a closed ideal which has finite dimension or codimension in an amenable algebra is amenable. For n-amenability the only corresponding result is

PROPOSITION 3.3. *Let $\mathfrak{A}$ be a weakly amenable Banach algebra and J a closed ideal of finite codimension. Then J is weakly amenable.*

Proof. Consider first of all the case where J is maximal. We have $\mathfrak{A}_2=\mathfrak{A}$, so J is modular and $\mathfrak{A}/J\simeq\mathbb{C}$. By Theorem 3.1, we need to show that $J_2=J$, so we suppose $J_2\ne J$. Dividing out by J_2 we can assume $J_2=0$. Thus $\mathfrak{A}$ is a singular extension of the $\mathbb{C}$-module J by $\mathbb{C}$. The action of $\mathbb{C}$ on J is determined by the operator P from J to J given by $P(j)=1\cdot j$. The associative law shows that P is idempotent. Any P-invariant subspace of J is an ideal in $\mathfrak{A}$ so, dividing out by a P-invariant hyperplane, we can assume that J is one-dimensional and the action of $\mathfrak{A}/J$ on J is either $\lambda\cdot j=0$ or $\lambda\cdot j=\lambda j$ (with $j\in J$, $\lambda\in\mathfrak{A}/J=\mathbb{C}$). By [6, Theorem 2.3.9, p. 58] the element 1 of $\mathfrak{A}/J$ can be lifted to an idempotent in $\mathfrak{A}$ which generates a subalgebra complementary to J. Thus $\mathfrak{A}$ is isomorphic to $\mathbb{C}^2$ with one of the products $(\lambda,\mu)(\lambda',\mu')=(\lambda\lambda',0)$ or $(\lambda\lambda',\lambda\mu'+\lambda'\mu)$. However, the first case is impossible because $\mathfrak{A}_2 = \mathfrak{A}$, and the second by Corollary 2.4(i) because it is the algebra obtained by adjoining a unit to the one-dimensional zero algebra. Thus $J = J_2$ and J is weakly amenable.

For the general case we can form a chain

$$\mathfrak{A} = J_0 \supseteq J_1 \supseteq \ldots \supseteq J_n = J$$

of closed ideals where each J_i is maximal in J_{i-1}; this works without the assumption of modularity because J has finite codimension. It then follows by induction using what we have proved that J_i is weakly amenable for each value of i.

EXAMPLE 3.4. The non-unital algebra $\mathfrak{A}$ generated by a single element S with $S^4 = 0$ is 2-dimensionally weakly amenable because it is singly generated. The ideal $\mathfrak{A}_2$ is a 2-dimensional zero algebra and so is not 2-dimensionally weakly amenable though it has finite dimension and codimension.

EXAMPLE 3.5. We give an example of a weakly amenable Banach algebra with a one-dimensional annihilator ideal J such that J is not weakly amenable. To do this it is enough to find a weakly amenable Banach algebra with non-trivial annihilator ideal J because any subspace of J is an ideal and dividing out by a closed subspace of codimension 1 we get the example we want.

We begin by defining a norm $\|\ \|_0$ on the subspace c_{00} of c_0 algebraically spanned by the standard basis vectors $e_1, e_2, \ldots$ Let $K \subset c_{00}$ be the absolutely convex subset generated by $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ where $f_j = 2^j(e_j - e_{j+1})$. Then K generates a seminorm $\|\ \|_0$ on c_{00}. To see that $\|\ \|_0$ is a norm, consider the projection P_k of c_{00} onto the span E_k of $e_1, \ldots, e_k$ given by $P(e_j) = e_j$ if $1 \le j \le k$ and $P(e_j) = e_k$ if $j > k$. We have $P(f_j) = f_j$ if $j < k$ and $P(f_j) = 0$ if $j \ge k$. This shows that K is closed under P so P is a contraction and $K \cap E_k$ is the absolutely convex set K_k generated by $e_1, \ldots, e_k$ and $f_1, \ldots, f_{k-1}$. The seminorm on E_k generated by K_k is thus the restriction of $\|\ \|_0$ and clearly is a norm. Any x in c_{00} lies in one of the E_k so $\|\ \|_0$ is a norm on c_{00}. We have $\|e_k\|_0 = 1$ for all k because the linear functional F on c_{00} defined by $F(a) = \sum_{j=1}^{\infty} a_j$ has $F(e_j) = 1$ and $F(f_j) = 0$ for all j, showing $\|F\| \le 1$ and hence $\|e_j\|_0 \ge 1$. As $e_j \in K$ we have $\|e_j\|_0 = 1$. Let B be the operator on c_{00} given by $Be_j = 2^{3j}e_j$ and define a new norm on c_{00} by $\|a\| = \|Ba\|_0$. We will show that $\|\ \|$ is an algebra norm with respect to the pointwise product. To do this it suffices to show that the unit ball $B^{-1}K$ is closed under multiplication, and this follows if we show that the product of any two elements $B^{-1}e_j = 2^{-3j}e_j$ and $B^{-1}f_k = 2^{-2k}(e_k - \frac{1}{8}e_{k+1})$ is in $B^{-1}K$. Most of these products are 0; the exceptions are

$$\begin{aligned}
(B^{-1}e_j)^2 &= 2^{-6j}e_j = 2^{-3j}B^{-1}e_j,\\
B^{-1}e_jB^{-1}f_j &= 2^{-5j}e_j = 2^{-2j}B^{-1}e_j,\\
B^{-1}e_{j+1}B^{-1}f_j &= -2^{-5j-6}e_{j+1} = -2^{-2j-3}B^{-1}e_{j+1},\\
B^{-1}f_kB^{-1}f_{k+1} &= -2^{-4k-5}e_{k+1} = -2^{-k-2}B^{-1}e_{k+1},\\
(B^{-1}f_k)^2 &= 2^{-4k}(e_k - \tfrac{1}{64}e_{k+1})\\
&= 2^{-4k}(e_k - \tfrac{1}{8}e_{k+1}) + 2^{-4k-6}7e_{k+1}\\
&= 2^{-2k}B^{-1}f_k + 2^{-k-3}\cdot 7\cdot B^{-1}e_{k+1},
\end{aligned}$$

where the last product has norm less than $2^{-2k} + 7\cdot 2^{-k-3} \le 2^{-2} + 7\cdot 2^{-4} < 1$.

Thus c_{00} with the norm $\|\ \|$ is a normed algebra, and we denote its completion by $\mathfrak{A}$. As c_{00} and hence $\mathfrak{A}$ is generated by the idempotents e_k, $\mathfrak{A}$ is weakly amenable. The elements $B^{-1}e_j$ $(j = 1, 2, \ldots)$ form a Cauchy sequence in $\mathfrak{A}$, because $\|B^{-1}e_j - B^{-1}e_{j+1}\| = \|e_j - e_{j+1}\|_0 = 2^{-j}\|f_j\| \le 2^{-j}$, and hence converge to an element r of $\mathfrak{A}$. As $\|B^{-1}e_j\| = \|e_j\|_0 = 1$ we have $\|r\| = 1$, in particular $r \ne 0$. For $j > k$, we have $e_kB^{-1}e_j = 0$, so $e_kr = 0$. This implies that $ar = 0$ for all a in $\mathfrak{A}$, completing the proof.

THEOREM 3.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be commutative Banach algebras with $\mathfrak{A}_2 = \mathfrak{A}$, $\mathfrak{B}_2 = \mathfrak{B}$. If $\mathfrak{A}$ is m-dimensionally weakly amenable and $\mathfrak{B}$ is n-dimensionally weakly amenable then $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$ is $(m+n-1)$-dimensionally weakly amenable.*

Proof. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are unital and T is a $p = (m+n-1)$-alternating cocycle on $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$. Because $\mathfrak{A} \widehat{\otimes} 1$ is a subalgebra of $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$ isomorphic with $\mathfrak{A}$, if $c_1, \ldots, c_p \in \mathfrak{A} \widehat{\otimes} \mathfrak{B}$ and m or more of the c_j lie in $\mathfrak{A} \otimes 1$ then $T(c_1, \ldots, c_p) = 0$ because $\mathfrak{A}$ is m-dimensionally weakly amenable. Similarly, $T(c_1, \ldots, c_p) = 0$ if n or more of the c_j lie in $1 \otimes \mathfrak{B}$. If $c_i = a_i \otimes b_i = (a_i \otimes 1)(1 \otimes b_i)$ $(1 \le i \le p)$ with $a_i \in \mathfrak{A}$ and $b_i \in \mathfrak{B}$ then using the derivation law in each variable, $T(c_1, \ldots, c_p)$ can be expressed as the sum of 2^p terms $T(c_1', \ldots, c_p')$ where $c_j' \in (\mathfrak{A} \otimes 1) \cup (1 \otimes \mathfrak{B})$. For each term, if fewer than m of the c_j' lie in $\mathfrak{A} \otimes 1$ then the rest, of which there are at least $p - (m-1) = n$, lie in $1 \otimes \mathfrak{B}$ and so, either way, the term is 0. Thus $T = 0$ and the result is proved.

If $\mathfrak{A}$ or $\mathfrak{B}$ do not have identities let $\mathfrak{A}^+ = \mathfrak{A}$ if $\mathfrak{A}$ has an identity and $\mathfrak{A}^+ = \mathfrak{A}^1$ otherwise, similarly for $\mathfrak{B}^+$. Because there are norm one projections from $\mathfrak{A}^+$ and $\mathfrak{B}^+$ onto $\mathfrak{A}$ and $\mathfrak{B}$ respectively, $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$ is embedded isometrically in $\mathfrak{A}^+ \widehat{\otimes} \mathfrak{B}^+$, and is an ideal. Since $\mathfrak{A}_2 = \mathfrak{A}$, $\mathfrak{B}_2 = \mathfrak{B}$ we see that $(\mathfrak{A} \widehat{\otimes} \mathfrak{B})_2 = \mathfrak{A} \widehat{\otimes} \mathfrak{B}$. Because $\mathfrak{A}$ is m-dimensionally weakly amenable, so is $\mathfrak{A}^+$; similarly, $\mathfrak{B}^+$ is n-dimensionally weakly amenable. So $\mathfrak{A}^+ \widehat{\otimes} \mathfrak{B}^+$ is $(m+n-1)$-dimensionally weakly amenable by what we have proved and so $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$ is $(m+n-1)$-dimensionally weakly amenable by Theorem 3.1.

EXAMPLE 3.7. The conditions $\mathfrak{A} = \mathfrak{A}_2$, $\mathfrak{B} = \mathfrak{B}_2$ are essential for this result. Of course, if $m = n = 1$ then automatically $\mathfrak{A} = \mathfrak{A}_2$ and $\mathfrak{B} = \mathfrak{B}_2$, so this condition can be omitted. However, if we take $\mathfrak{A} = \ell^1(\mathbb{Z})$, $\mathfrak{B} = \ell^1(\mathbb{N})$ with convolution multiplication in both cases then $\mathfrak{A} \widehat{\otimes} \mathfrak{B} = \ell^1(\mathbb{Z} \times \mathbb{N})$ and $(\mathfrak{A} \widehat{\otimes} \mathfrak{B})_2$ consists of functions supported on $\{(r,s) : r \in \mathbb{Z},\ s \geq 2\}$. Thus the codimension of $(\mathfrak{A} \widehat{\otimes} \mathfrak{B})_2$ in $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$ is infinite and hence $\mathfrak{A} \widehat{\otimes} \mathfrak{B}$ is not p-dimensionally weakly amenable for any value of p, whereas $\mathfrak{A}$ is 1-dimensionally weakly amenable (in fact amenable) and $\mathfrak{B}$ is 2-dimensionally weakly amenable because it is singly generated.

4. Growth criteria for n-dimensional weak amenability—Beurling and Lipschitz algebras. We now look at the extension of [1, Theorem 1.4] to n-dimensional weak amenability.

THEOREM 4.1. *Let $n \in \mathbb{N}$ and suppose that $\mathfrak{A}$ is unital.*

(a) *If $\mathfrak{A}$ contains a subset E such that*

(i) *E is a subgroup under addition,*
(ii) *$\mathfrak{A}$ is the smallest closed subalgebra generated rationally by E,*
(iii) *for each a in E,*
$$\|e^{ma}\| = o(m^{n/(n+1)}) \quad \text{as } m \to \infty,$$
then $\mathfrak{A}$ is n-dimensionally weakly amenable.

(b) *If $\mathfrak{A}$ contains a subset G such that*

(i) *G is a group under multiplication,*
(ii) *$\mathfrak{A}$ is the smallest closed subalgebra rationally generated by G,*
(iii) *for each a in G,*
$$\|a^m\| = o(m^{n/(n+1)}) \quad \text{as } m \to \infty,$$
then $\mathfrak{A}$ is n-dimensionally weakly amenable.

Proof. (a) Let T be an alternating n-derivation on $\mathfrak{A}$ with values in $\mathfrak{A}^*$. Then, by the derivation identity, for $a_1, \ldots, a_m \in E$,
$$T(e^{ma_1}, \ldots, e^{ma_n}) = m^n e^{m(a_1+\ldots+a_n)} T(a_1, \ldots, a_n).$$
This shows that
$$\begin{aligned}\|T(a_1, \ldots, a_n)\| &= m^{-n} \|e^{-m(a_1+\ldots+a_n)} T(e^{ma_1}, \ldots, e^{ma_n})\| \\ &\leq m^{-n} \|T\|\, \|e^{ma_0}\|\, \|e^{ma_1}\| \ldots \|e^{ma_n}\|,\end{aligned}$$
where $a_0 = -(a_1 + \ldots + a_n) \in E$. Using the growth condition on $\|e^{ma}\|$ we see that the right converges to zero as $m \to \infty$, so $T(a_1, \ldots, a_n) = 0$. By Corollary 2.6, this implies that $T = 0$.

(b) This is proved in the same way using the identity
$$a_1^{-1} \ldots a_n^{-1} T(a_1, \ldots, a_n) = m^{-n} (a_1^{-1} \ldots a_n^{-1})^{-m} T(a_1^m, \ldots, a_n^m)$$
for elements $a_1, \ldots, a_n$ of G.

The same argument gives a number of different results—for example, (i)–(iii) of (a) could be replaced by
$$\inf_{\lambda \neq 0} \lambda^{-n} \|e^{-\lambda(a_1+\ldots+a_n)}\|\, \|e^{\lambda a_1}\| \ldots \|e^{\lambda a_n}\| = 0$$
for all $a_1, \ldots, a_n$ in $\mathfrak{A}$.

We now consider the n-dimensional weak amenability of Beurling algebras. These are defined in [1]. Even if we restrict ourselves to abelian groups which are discrete, there is a range of such groups and a considerable variety of weights on each group. They are of theoretical importance because any commutative unital Banach algebra $\mathfrak{A}$ is a quotient of a Beurling algebra (just take G to be the group of regular elements, $\omega(g) = \|g\|$, and the identity map on G extends to a continuous algebra homomorphism of $\ell^1(G, \omega)$ into $\mathfrak{A}$ which is surjective because each a in $\mathfrak{A}$ can be expressed as $a = (a - \lambda 1) + \lambda 1$, which is the sum of two elements from G if $|\lambda| > \|a\|$). However, this is more a comment on the complexity of quotients of Beurling algebras than a simplification of Banach algebra theory. Theorem 4.1(b) immediately shows that if the weights satisfy $\omega(g^m) = o(m^{n/(n+1)})$ as $m \to \infty$, for all g in G, then $\ell^1(G, \omega)$ is n-dimensionally weakly amenable, and Corollary 2.8 shows that if G is generated by n elements then $\ell^1(G, \omega)$ is $(n+1)$-dimensionally weakly amenable. We shall consider only the case $G = \mathbb{Z}^p$. For $p = 1$, $\ell^1(\mathbb{Z}, \omega)$ is 2-dimensionally weakly amenable by the above remarks and is weakly amenable if and only if $\sup k/(\omega(k)\omega(-k)) = \infty$. This last result includes [1, Theorems 2.2 and 2.3] and is the case $p = 1$ of Theorem 4.2 below. Though we will not prove it, $\ell^1(\mathbb{Z}^p, \omega)$ is weakly amenable if and only if $\sup \|k\|/(\omega(k)\omega(-k)) = \infty$, where the supremum is over $k \in \mathbb{Z}^p$ and $\|\ \|$ is the Euclidean norm on $\mathbb{R}^p$ (or in fact any other norm, as they are all equivalent). Theorem 4.2 can be extended in the same spirit to give a necessary and sufficient condition for $\ell^1(\mathbb{Z}^p, \omega)$ to be n-dimensionally weakly amenable when $n < p$.

THEOREM 4.2. *The Beurling algebra $\mathfrak{A} = \ell^1(\mathbb{Z}^p, \omega)$ is p-dimensionally weakly amenable if and only if the weights satisfy*
$$\sup_K \det K / (\omega(k_1) \ldots \omega(k_p) \omega(-k_\bullet)) = \infty,$$
where the sup is over all $p \times p$ matrices of integers K; $k_1, \ldots, k_p$ are the rows of K and $k_\bullet = k_1 + \ldots + k_p$.

Proof. Let $z_1, \ldots, z_p$ be the standard generators of $\mathbb{Z}^p$. By Corollaries 2.6 and 2.7, an alternating p cycle from $\mathfrak{A}$ to $\mathfrak{A}^*$ is determined by

$T(z_1,\ldots,z_p)\in\mathfrak{A}^*$ and thus by the numbers

$$t(l)=(z_1^{-1}\ldots z_p^{-1}T(z_1,\ldots,z_p),l),\qquad l\in\mathbb{Z}^p.$$

Hence

$$(T(k_1,\ldots,k_p),l)=t(k_\bullet+l)\det K$$

where $k_\bullet=k_1+\ldots+k_p$ and K is any $p\times p$ matrix of integers. Here we are considering $G=\mathbb{Z}^p$ as a subset of $\mathfrak{A}$ but need to bear in mind that the algebra multiplication restricted to G is the group operation—that is, addition.

We have

$$|(T(k_1,\ldots,k_p),l)|\le\|T\|\omega(k_1)\ldots\omega(k_p)\omega(l).$$

This shows that

$$(*)\qquad \sup_{K,l}\left|\frac{t(k_\bullet+l)\det K}{\omega(k_1)\ldots\omega(k_p)\omega(l)}\right|<\infty.$$

Thus if $\mathfrak{A}$ is not p-dimensionally weakly amenable there are values of the $t(l)$, not all zero, for which this supremum is finite. Conversely, if there are such values of the $t(l)$ then the above equations define $(T(k_1,\ldots,k_p),l)$ in such a way that it can be extended to $(T(a_1,\ldots,a_p),b)$ for all $a_1,\ldots,a_p$, b in $\mathfrak{A}$ and so gives a p-linear map T from $\mathfrak{A}$ to $\mathfrak{A}^*$ with $\|T\|$ not greater than the supremum. It is straightforward to verify that T is an alternating p-derivation. Thus $\mathfrak{A}$ is p-dimensionally weakly amenable if and only if the only function t making the supremum in $(*)$ finite is $t=0$.

Replacing l by $l-k_\bullet$ we see that $\mathfrak{A}$ is p-dimensionally weakly amenable if and only if there is no non-zero t with

$$\sup_l\left(\sup_K\frac{|\det K|}{\omega(k_1)\ldots\omega(k_p)\omega(l-k_\bullet)}\right)|t(l)|<\infty.$$

If

$$\sup_K|\det K|/(\omega(k_1)\ldots\omega(k_p)\omega(-k_\bullet))<\infty$$

then this inequality is satisfied with $t(l)=0$ except for $l=0$, so $\mathfrak{A}$ is not p-dimensionally weakly amenable. If this inequality is satisfied for some non-zero t then choosing l with $t(l)\ne0$ we have $\sup_K|\det K|/(\omega(k_1)\ldots\omega(k_p)\times\omega(l-k_\bullet))<\infty$. However,

$$\begin{aligned}\omega(l)^{-1}\sup_K|\det K|/(\omega(k_1)\ldots\omega(k_n)\omega(-k_\bullet))&\\ &=\sup_K\det K/(\omega(k_1)\ldots\omega(k_n)\omega(-k_\bullet)\omega(l))\\ &\le\sup_K\det K/(\omega(k_1)\ldots\omega(k_n)\omega(l-k_\bullet))<\infty,\end{aligned}$$

which concludes the proof.

We now consider the Lipschitz algebras $\mathfrak{A}=\operatorname{lip}_\alpha K$ introduced in [1, §3]. In the proof of [1, Theorem 3.10] the authors show (without using the hypothesis that $\alpha<1/2$) that if h is a real-valued function in $\operatorname{Lip}_1 K$ then $\|\exp(inh)\|=O(n^\alpha)$ as $n\to\infty$. Thus using Theorem 4.1(a) with E as the set of such elements h we see that if $\alpha<n/(n+1)$ then $\operatorname{lip}_\alpha K$ is n-dimensionally weakly amenable. If $\mathfrak{A}$ is generated by n elements, so that K is homeomorphic with a compact subset of $\mathbb{C}^n$, then $\operatorname{lip}_\alpha K$ is $(n+1)$-dimensionally weakly amenable. As for weak amenability, to go beyond this depends rather more delicately on the metric, so we only get further results in special cases.

THEOREM 4.3. *Let $K=\mathbb{T}^n$ and let $\alpha\in(0,1)$. Then $\mathfrak{A}=\operatorname{lip}_\alpha\mathbb{T}^n$ is n-dimensionally weakly amenable if and only if $\alpha\le n/(n+1)$.*

Proof. Put $\sigma=n/(n+1)$. First of all we show that $\mathfrak{A}$ is not n-dimensionally weakly amenable if $\alpha>\sigma$. Define an $(n+1)$-linear functional τ on $C^1(\mathbb{T}^n)$ and an n-linear map T from C^1 to $(C^1)^*$ by

$$\tau(f_0,\ldots,f_n)=(f_0,T(f_1,\ldots,f_n))=\int f_0\det\frac{\partial f_i}{\partial x_j}\,dx,$$

where $f_0,\ldots,f_n\in C^1(\mathbb{T}^n)$ and dx is Lebesgue measure on $\mathbb{T}^n$ (normalised to give a total mass of 1). These are continuous multilinear maps. If $f_j=\exp(i(x\cdot m_j))$, where $m_j\in\mathbb{Z}^n$ $(j=0,\ldots,n)$, then

$$\tau(f_0,\ldots,f_n)=\begin{cases}0 & \text{if } m_0+\ldots+m_n\ne0,\\ (i)^n\det m_{jk} & \text{if } m_0+\ldots+m_n=0.\end{cases}$$

This is an alternating function of $m_1,\ldots,m_n$ and an elementary calculation shows that in fact it is an alternating function of $m_0,\ldots,m_n$. Because the trigonometric polynomials are dense in C^1, τ is an alternating $(n+1)$-linear functional. Another elementary calculation shows that T is an n-derivation which is clearly not zero. Thus C^1 is not n-dimensionally weakly amenable.

We shall use interpolation procedures to show that τ is continuous in the $\operatorname{lip}_\alpha$ norm and so gives an alternating n-derivation from $\mathfrak{A}$ to $\mathfrak{A}^*$. Denote by S the set of elements s of $(\mathbb{R}^+)^{n+1}$ for which there is a constant C (depending on s) with

$$\tau(f_0,\ldots,f_n)\le C\|f_0\|_{n+1}^{s_0}\cdots\|f_n\|_{n+1}^{s_n},$$

where $\|\ \|_p^r$ is the norm in the Sobolev space H_p^r on $\mathbb{T}^n$ [2, Definition 6.2.2]. Clearly, $H_p^0=L_p$ and by [2, Theorem 6.2.3], $C^1\subseteq H_p^1$ is a continuous embedding onto a dense subspace. A simple calculation, together with the definition of τ, shows that for $f_0,\ldots,f_n\in C^1$,

$$|\tau(f_0,\ldots,f_n)|\le C\|f_0\|_{n+1}^0\|f_1\|_{n+1}^1\cdots\|f_n\|_{n+1}^1.$$

Thus $(1,0,0,\ldots,0)\in S$. By the alternating property of τ we see that all the standard basis vectors of $\mathbb{R}^{n+1}$ lie in S. The interpolation theorems [2, Theorems 4.4.1 and 6.4.5(7)] show that S is convex (in 6.4.5 the restriction $s_0 \neq s_1$ is clearly not necessary if $p_0 = p_1$ since the interpolation spaces are all the same in this case). Thus S contains $(\sigma, \ldots, \sigma)$, that is, τ is continuous with respect to the H^{σ}_{n+1} norm on each variable.

To complete the proof of this part of the theorem we show that if $0 < s < \alpha \leq 1$ then $\mathrm{lip}_\alpha \subseteq B^s_{n+1,1} \subseteq H^s_{n+1}$ with continuous embeddings. The first of these follows from [2, Theorem 6.2.5] with $N = 0$, $m = 1$, $p = n+1$ and $q = 1$. We have $\omega^1_{n+1}(t, f) \leq \|f\|_\alpha t^\alpha$, so that $0 \leq t^{-s}\omega^1_{n+1}(t, f)t^{-1} \leq \|f\|_\alpha t^{\alpha-s-1}$, which is integrable over $[0, \pi]$, showing that $\| \ \|_\alpha$ dominates $\| \ \|^s_{n+1,1}$. The second embedding is [2, Theorem 6.2.4(9)].

To show that $\mathrm{lip}_\alpha \mathbb{T}^n$ is not n-dimensionally weakly amenable if $\alpha = n/(n+1)$ first of all we show that τ above is not bounded with respect to this Lipschitz norm. Let $l \in \mathbb{N}$ and $r = 2^{-\alpha}$, where $\alpha = n/(n+1)$, and put $h(\theta) = \sum_{j=1}^{l} r^j \exp(2^j i\theta)$ $(\theta \in \mathbb{R})$. As $0 < \alpha < 1$ we have $1/2 < r < 1$, so

$$\|h\|_\infty \leq \frac{1}{1-r}$$

independently of l. Also, for $k \in \{1, \ldots, l\}$,

$$|h(\theta) - h(\phi)| = \Big(\sum_{j=1}^{k} r^j 2^j\Big)|\theta - \phi| + \sum_{j=k+1}^{l} 2r^j \leq \frac{(2r)^k}{2r-1}|\theta - \phi| + \frac{2r^{k+1}}{1-r}.$$

If $|\theta - \phi| \leq \pi$, choose k with

$$2^{-k}\pi \leq |\theta - \phi| \leq 2^{-k+1}\pi \quad \text{if } |\theta - \phi| \geq 2^{-l}\pi$$

and $k = l$ if $|\theta - \phi| < 2^{-l}\pi$. We have $(2r)^k = 2^{-k(\alpha-1)} \leq |\theta - \phi|^{\alpha-1}(2\pi)^{1-\alpha}$ and if $|\theta - \phi| \geq 2^{-l}\pi$ then $2r^{k+1} = 2r \cdot 2^{-k\alpha} \leq 2r\pi^{-\alpha}|\theta - \phi|^\alpha$, so that

$$|h(\theta) - h(\phi)| \leq \left[\frac{(2\pi)^{1-\alpha}}{2r-1} + \frac{2r\pi^{-\alpha}}{1-r}\right]|\theta - \phi|^\alpha.$$

By periodicity this holds for all θ, ϕ in $\mathbb{R}$ and shows that $h \in \mathrm{lip}_\alpha \mathbb{T}$ and the lip_α norm of h is bounded independently of l. We now define functions in $\mathrm{lip}_\alpha \mathbb{T}^n$ by

$$f_0(\theta_1, \ldots, \theta_n) = h(-(\theta_1 + \ldots + \theta_n)),$$
$$f_m(\theta_1, \ldots, \theta_n) = h(\theta_m), \quad m = 1, \ldots, n.$$

Clearly, the Fourier series of these functions are

$$\sum_{j=1}^{l} r^j \exp[-2^j i(\theta_1 + \ldots + \theta_n)] \quad \text{and} \quad \sum_{j=1}^{l} r^j \exp(2^j i\theta_m).$$

A straightforward calculation using the orthogonality of the functions $\exp(j\theta)$

shows that

$$\tau(f_0, \ldots, f_n) = \sum_{j=1}^{l} r^{(n+1)j} \cdot 2^{nj}.$$

Now let $l \to \infty$. As the lip_α norms of $f_0, \ldots, f_n$ are bounded and the right hand side tends to infinity because $2r > 1$ this shows that τ is not bounded with respect to the lip_α norm.

Now suppose T is a continuous alternating n-derivation from $\mathfrak{A}$ into $\mathfrak{A}^*$. As in our discussion of the Beurling algebras, T gives rise to, and is determined by, a complex-valued function t on $\mathbb{Z}^n$ defined by

$$t(l) = (z_1^{-1} \ldots z_n^{-1} T(z_1, \ldots, z_n), f_0),$$

where

$$f_0(\theta_1, \ldots, \theta_n) = \exp(i(l_1\theta_1 + \ldots + l_n\theta_n)), \quad z_j(\theta_1, \ldots, \theta_n) = \exp(i\theta_j).$$

Let $\phi \in \mathbb{T}^n$ and denote the corresponding translation operator, which is an isometric algebra isomorphism, on $\mathfrak{A}$ by S_ϕ. Putting

$$T_\phi(f_1, \ldots, f_n) = S^*_\phi T(S_\phi f_1, \ldots, S_\phi f_n)$$

we get another alternating n-derivation and, because $S_\phi z_j = (\exp(i\phi_j))z_j$, the corresponding t-function is

$$t_\phi(l) = (\exp(i(l \cdot \phi)))t(l).$$

Now consider $W = \int T_\phi \, d\phi$, which exists because $S_\phi a$ is a continuous function of ϕ for each a in $\mathfrak{A}$. It is clearly also an alternating n-derivation and the corresponding t-function ω is given by $\omega(0) = t(0)$ and $\omega(l) = 0$ if $l \neq 0$. For trigonometric polynomials $f_0, \ldots, f_n$,

$$(W(f_1, \ldots, f_n), f_0) = \tau(f_0, \ldots, f_n)t(0)$$

since this obviously holds if $f_j = z_j$ $(j = 1, \ldots, n)$ and $f_0 = \exp(i(l \cdot \theta))$, and so for any trigonometric polynomial f_0. As each side represents an alternating n-derivation from the trigonometric polynomials to their algebraic dual and, by Corollary 2.7, these are completely determined by their values on $z_1, \ldots, z_n$, the equation is established. However, the left is continuous in the lip_α norm and τ is not, so $t(0) = 0$. If we had put $W = (\exp(i(k \cdot \phi))) \int (\exp(-i(k \cdot \phi)))T_\phi \, d\phi$ for some $k \in \mathbb{Z}^n$, the same argument would apply except that we would have $\omega(0) = t(k)$ and conclude that $t(k) = 0$. Thus all the $t(k)$ are 0 and so T is 0.

COROLLARY 4.4. *Let $m, n \in \mathbb{N}$ and $\alpha \in (0, 1)$. Then $\mathfrak{A} = \mathrm{lip}_\alpha \mathbb{T}^m$ is n-dimensionally weakly amenable if and only if $m < n$, or $m \geq n$ and $\alpha \leq n/(n+1)$.*

Proof. By Corollaries 2.5 and 2.8, because $\mathfrak{A}$ has m generators, it is n-dimensionally weakly amenable if $n = m + 1$ and hence for all $n > m$.

Suppose that $m \geq n$ and $\alpha \leq n/(n+1)$. $\mathfrak{A}$ is generated by the m generators $z_j(\theta) = \exp(i\theta_j)$, $j = 1, \ldots, m$, and, by considering functions which are constant on the other θ_k, each n-element subset of these generators lies in a subalgebra isomorphic with $\mathrm{lip}_\alpha \mathbb{T}^n$, which is n-dimensionally weakly amenable. Thus if T is an alternating n-derivation with values in a symmetric Banach $\mathfrak{A}$-module, by restricting to the subalgebra we see that T is zero for any n-generators, and so by Corollary 2.6, $T = 0$.

Suppose that $m \geq n$ and $\alpha > n/(n+1)$. By restricting to $\{\theta : \theta \in \mathbb{T}^m,\ \theta_{n+1} = \ldots = \theta_m = 0\}$ we see that $\mathrm{lip}_\alpha \mathbb{T}^n$ is a quotient of $\mathrm{lip}_\alpha \mathbb{T}^m$ and so by Theorem 3.1(i) the latter is not n-dimensionally weakly amenable because the former is not.

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Hereditarily finitely decomposable Banach spaces

by

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Abstract. A Banach space is said to be HD_n if the maximal number of subspaces of X forming a direct sum is finite and equal to n. We study some properties of HD_n spaces, and their links with hereditarily indecomposable spaces; in particular, we show that if X is complex HD_n, then $\dim(\mathcal{L}(X)/\mathcal{S}(X)) \leq n^2$, where $\mathcal{S}(X)$ denotes the space of strictly singular operators on X. It follows that if X is a real hereditarily indecomposable space, then $\mathcal{L}(X)/\mathcal{S}(X)$ is a division ring isomorphic either to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, the quaternionic division ring.

1. Introduction

General introduction. The problems discussed in this article came as natural questions after the article of W. T. Gowers and B. Maurey ([GM]) solving the unconditional basic sequence problem. In a Banach space X, a sequence $(e_n)_{n\in\mathbb{N}}$ is said to be a *basis* if every vector x in X can be written uniquely in the form $x = \sum_{i=0}^{\infty} \lambda_i e_i$, where the λ_i's are scalars. It is an *unconditional basis* if there is a constant C such that the inequalities

$$\Big\|\sum_{i\in E} \lambda_i e_i\Big\| \leq C \Big\|\sum_{i=0}^{\infty} \lambda_i e_i\Big\|$$

hold for all subsets E of $\mathbb{N}$ and all coefficients λ_i. A sequence is a *basic sequence* (resp. an *unconditional basic sequence*) if it is a basis (resp. an unconditional basis) of its closed linear span. Details about these notions can be found in [LT].

A lot of classical spaces have an unconditional basis (spaces l_p, for $p \geq 1$, have one) but for example L_1 does not have one; an example of a Banach space without a basis is even harder to find, but was given by P. Enflo in 1973 ([E]). On the other hand, it is a classical result that every Banach space contains a basic sequence; but the question "Does every Banach space contain an unconditional basic sequence?" has remained unsolved for many

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