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An uncertainty principle related to the Poisson summation formula

by

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Abstract. We prove a class of uncertainty principles of the form

$$\|S_g f\|_1 \le C(\|x^a f\|_p + \|\omega^b \widehat{f}\|_q),$$

where $S_g f$ is the short time Fourier transform of f. We obtain a characterization of the range of parameters a, b, p, q for which such an uncertainty principle holds. Counter-examples are constructed using Gabor expansions and unimodular polynomials. These uncertainty principles relate the decay of f and $\widehat{f}$ to their behaviour in phase space. Two applications are given: (a) If such an inequality holds, then the Poisson summation formula is valid with absolute convergence of both sums. (b) The validity of an uncertainty principle implies sufficient conditions on a symbol σ such that the corresponding pseudodifferential operator is of trace class.

1. Introduction. In its essence the mathematical uncertainty principle states that a function cannot be well concentrated simultaneously with its Fourier transform. This principle has found so many different expressions that the context of diverse books and articles on the uncertainty principle is almost disjoint, see for example [1, 2, 4, 5, 11, 20]. Our starting point is the uncertainty principle in the form [11]

$$\frac{1}{2\pi}\|f\|_2^2 \le \|xf\|_2^2 + \|\omega \widehat{f}\|_2^2, \tag{1}$$

where the Fourier transform is normalized as

$$\mathcal{F}f(\omega) = \widehat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x\cdot\omega}\, dx.$$

Here the quantities on the right hand side are related to the uncertainty of position and momentum. This uncertainty principle is often expressed by

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saying that the smallest physically meaningful cell in phase space $\mathbb{R}^d \times \mathbb{R}^d$ has volume 1.

In this paper we investigate the question of how more stringent conditions on the decay of f and $\widehat{f}$ affect the lower bound in (1). For this we measure the decay or the concentration of a function by the weighted L^p-norms

$$(2)\qquad \|f\|_{p,a} = \Big(\int_{\mathbb{R}^d} |f(x)|^p(1+|x|)^{ap}\,dx\Big)^{1/p}.$$

In agreement with the physical intuition we want to bound $\|f\|_{p,a} + \|\widehat{f}\|_{q,b}$ from below by a quantity that measures the phase-space content of f. For this we use the short time Fourier transform as an appropriate concept for the phase-space density of f.

Fix a non-zero function $g \in \mathcal{S}$, a so-called *window*; then the short time Fourier transform of f with respect to g is the continuous function on $\mathbb{R}^d \times \mathbb{R}^d$

$$(3)\qquad S_g f(x,y) = \int_{\mathbb{R}^d} \overline{g}(t-x)\, e^{-2\pi i y\cdot t} f(t)\,dt.$$

It is one of the fundamental problems of mathematical phase-space analysis to understand the interplay between the behaviour of f and $\widehat{f}$ and that of $S_g f$. If f is given, we have no direct information on $\widehat{f}$, and conversely, $\widehat{f}$ conveys no direct information on f. The short time Fourier transform is a simultaneous phase-space representation of f that combines information on the position x and the momentum ω. In engineering one speaks of time-frequency analysis and Gabor theory, f providing information exclusively about the temporal behaviour of a signal, and $\widehat{f}$ only giving the frequency spectrum of f.

Our main result is the following type of uncertainty principle.

THEOREM 1. (a) *If*

$$\Big(\frac{a}{b} - \frac{1}{p'}\Big)\Big(\frac{b}{d} - \frac{1}{q'}\Big) > \max\Big(\frac{1}{pq}, \frac{1}{2p}, \frac{1}{2q}, \frac{1}{4}\Big),$$

then there exists a constant C, depending on a, b, p, q and g, such that the uncertainty principle

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |S_g f(x,y)|\,dx\,dy \le C(\|f\|_{p,a} + \|\widehat{f}\|_{q,b})$$

holds for all $f \in L^p_a \cap \mathcal{F}L^q_b$.

(b) *Conversely, if*

$$\Big(\frac{a}{b} - \frac{1}{p'}\Big)\Big(\frac{b}{d} - \frac{1}{q'}\Big) < \max\Big(\frac{1}{pq}, \frac{1}{2p}, \frac{1}{2q}, \frac{1}{4}\Big),$$

such an inequality cannot hold for all $f \in L^p_a \cap \mathcal{F}L^q_b$.

To see that Theorem 1 yields an improved lower bound, we note that by Plancherel's theorem $\|S_g f\|_2 = \|f\|_2\|g\|_2$ and obviously $\|S_g f\|_\infty \le \|f\|_2\|g\|_2$. This gives

$$(4)\qquad \|f\|_2^2\|g\|_2^2 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |S_g f(x,y)|^2\,dx\,dy \le \|S_g f\|_\infty \iint |S_g f(x,y)|\,dx\,dy$$
$$\le \|f\|_2\|g\|_2 \iint_{\mathbb{R}^d\times\mathbb{R}^d} |S_g f(x,y)|\,dx\,dy$$

and thus $\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |S_g f(x,y)|\,dx\,dy \ge \|f\|_2\|g\|_2$. Hence, for fixed $g \in L^2(\mathbb{R}^d)$, Theorem 1 implies the uncertainty principle

$$(5)\qquad \|f\|_2 \le C(\|f\|_{p,a} + \|\widehat{f}\|_{q,b}).$$

This inequality is similar in spirit to the generalized uncertainty principles of Cowling and Price [2]. In a different direction Lieb has obtained rather deep inequalities involving the short time Fourier transform [18].

In the sequel we will use another, purely mathematical interpretation of Theorem 1. For this we consider the space S_0 of all functions $f \in L^1(\mathbb{R}^d)$ such that $S_g f$ is integrable on $\mathbb{R}^d \times \mathbb{R}^d$ with norm

$$(6)\qquad \|f\|_{S_0} = \iint_{\mathbb{R}^d\times\mathbb{R}^d} |S_g f(x,y)|\,dx\,dy$$

Then Theorem 1 describes optimal conditions such that $L^p_a \cap \mathcal{F}L^q_b$ is embedded in S_0. We refer to [6, 8, 10] for detailed information on this function space. Since S_0 is defined rather implicitly, the theorem gives easy sufficient conditions for membership in S_0.

The conditions on the parameters a, b, p, q are greatly inspired by a beautiful paper of J.-P. Kahane and P.-G. Lemarié-Rieusset [16], who find a similar, but not identical, range of parameters in their investigation of the Poisson summation formula. At first glance the uncertainty principle and Poisson summation formula are totally unrelated, but as a first application of Theorem 1 we show that the uncertainty principle implies that the Poisson summation formula holds in a strong sense. In this way we recover a part of the results of Kahane and Lemarié with a completely different method.

A second application concerns pseudodifferential operators. In Section 5 we show that the uncertainty principle of Theorem 1 translates into easy sufficient conditions on symbols to generate pseudodifferential operators of trace class.

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2. The short time Fourier transfrom, $S_0(\mathbb{R}^d)$, and Poisson summation formula. In this section we collect the necessary facts about the short time Fourier transform and S_0.

Unless stated otherwise, we work in $\mathbb{R}^d$. As usual, $x \cdot \omega = \sum_{i=1}^d x_i \omega_i$ for $x, \omega \in \mathbb{R}^d$ and $x^2 = x \cdot x$.

We write $L^p_a \cap \mathcal{F}L^q_b(\mathbb{R}^d)$ for the Banach space of all functions f satisfying $f \in L^p_a(\mathbb{R}^d)$ and $\widehat{f} \in L^q_b(\mathbb{R}^d)$. Since we deal with function spaces, it follows from the closed graph theorem that an inclusion $B_1 \subseteq B_2$ implies a continuous embedding $B_1 \hookrightarrow B_2$ with a norm inequality $\|f\|_{B_2} \le C\|f\|_{B_1}$. For future reference we note the following embeddings, which are easily verified using Hölder's inequality.

LEMMA 1. (a) *$L^p_a(\mathbb{R}^d) \hookrightarrow L^1(\mathbb{R}^d)$ if and only if $a > d/p'$, where p' is the conjugate index, i.e. $1/p + 1/p' = 1$.*

(b) *For $p > 2$, $L^p_a(\mathbb{R}^d) \hookrightarrow L^2_c(\mathbb{R}^d)$ if and only if $c < a + d/p - d/2$.*

Writing $T_x f(t) = f(t-x)$ for translations, $M_y f(t) = e^{2\pi i y \cdot t} f(t)$ for the modulation operator, and $\widetilde{f}(t) = \overline{f(-t)}$ for the inversion, the short time Fourier transform (3) can also be expressed in the following forms:

$$S_g f(x,y) = \langle f, M_y T_x g\rangle = \langle \widehat{f}, T_y M_{-x}\widehat{g}\rangle = e^{-2\pi i x\cdot y} f * (M_y g)^{\sim}(x) = \widehat{f} * (M_{-x}\widehat{g})^{\sim}(y) = e^{2\pi i x\cdot y} S_{\widehat{g}}\widehat{f}(y,-x). \tag{7}$$

In this paper we are interested in what conditions on f and $\widehat{f}$ imply the integrability of $S_g f$, in other words, embeddings into $S_0 = \{f \in L^1(\mathbb{R}^d) : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |S_g f(x,y)|\,dx\,dy = \|f\|_{S_0} < \infty\}$.

Although S_0 is not so widely known, it is an important Banach space in harmonic analysis. We refer to [6, 8, 10] for some background and details. Now we summarize the properties of S_0 which will be used in the sequel.

LEMMA 2. (a) *$S_0(\mathbb{R}^d)$ is a Banach algebra under pointwise multiplication and convolution. Its definition is independent of the choice of $g \in \mathcal{S}$. Different windows g yield equivalent norms.*

(b) $S_0 \hookrightarrow L^1 \cap L^2(\mathbb{R}^d)$.

(c) *S_0 is the minimal non-trivial Banach space contained in L^1 which is isometrically invariant under the operators $T_x M_y$, $x, y \in \mathbb{R}^d$.*

(d) $\mathcal{F}S_0 = S_0$. (*This follows from* (7).)

Clearly, a function or distribution is uniquely determined by its short time Fourier transform. As a consequence of the special properties of the group generated by T_x and M_y, an explicit inversion formula holds [14, 10].

LEMMA 3. *Suppose that $\|g\|_2 = 1$. Then*

$$f = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S_g f(x,y)\, M_y T_x g\,dx\,dy. \tag{8}$$

If $f, g \in L^2$, the truncated integral $\int_{|x|\le R}\int_{|y|\le R}$ converges to f in L^2. If $f, g \in S_0$, the integral is absolutely convergent in S_0.

S_0 is a natural domain for the Poisson summation formula. We present the argument briefly. For this we introduce Wiener's algebra

$$W(\mathbb{R}^d) = \Big\{ f \text{ continuous on } \mathbb{R}^d : \sum_{k\in\mathbb{Z}^d} \max_{x\in[0,1]^d} |f(x+k)| < \infty \Big\}. \tag{9}$$

LEMMA 4. *If both $f \in W(\mathbb{R}^d)$ and $\widehat{f} \in W(\mathbb{R}^d)$, then the Poisson summation formula*

$$\sum_{k\in\mathbb{Z}^d} f(x+k) = \sum_{k\in\mathbb{Z}^d} \widehat{f}(k) e^{2\pi i k\cdot x} \tag{10}$$

holds for all $x \in \mathbb{R}^d$ with absolute convergence of both sums.

This is clear, since the Fourier coefficients of the periodic function on the left hand side are exactly $\widehat{f}(k)$. Since $f, \widehat{f} \in W$, the Fourier series converges absolutely and thus both sums must be equal.

According to Katznelson's counter-example [17] it is not sufficient to assume $f, \widehat{f}$ being continuous and in L^1. A minimal assumption under which the Poisson summation formula makes sense pointwise is $f \in W \cap \mathcal{F}W$.

PROPOSITION 1. *$S_0(\mathbb{R}^d)$ is embedded in $W \cap \mathcal{F}W$.*

Proof. Writing $\chi = \chi_{[0,1]^d}$, the Wiener norm can be written as

$$\|f\|_W = \sum_{k\in\mathbb{Z}^d} \|f \cdot T_k\chi\|_\infty.$$

Now choose $g \in \mathcal{S}$ non-negative and with compact support, such that $\chi \le T_x g$ for $x \in [0,1]^d$. Then we estimate

$$\|f\|_W \le \sum_{k\in\mathbb{Z}^d} \int_{[0,1]^d} \|f \cdot T_{x+k} g\|_\infty\,dx = \int_{\mathbb{R}^d} \|f \cdot T_x g\|_\infty\,dx \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\widehat{f} * M_{-x}\widehat{g}(y)|\,dx\,dy = \|f\|_{S_0}.$$

In the last step we have used the Riemann–Lebesgue lemma and (7). Since S_0 is invariant under the Fourier transform, we also have $\|\widehat{f}\|_W \le C\|\widehat{f}\|_{S_0} \le C'\|f\|_{S_0}$. ∎

COROLLARY 1. *If $f \in S_0$, the Poisson summation formula holds pointwise with absolute convergence of both sums.*

3. S_0-uncertainty principles. In [16] J.-P. Kahane and P.-G. Lemarié-Rieusset have characterized the range of parameters a, b, p, q such that $f \in L^p_a \cap \mathcal{F}L^q_b$ implies the validity of the Poisson summation formula. Since S_0 is in a sense the largest "nice" Banach space for which the Poisson summation formula holds, it is natural to conjecture a relation between the Poisson summation formula, uncertainty principles, and embeddings of $L^p_a \cap \mathcal{F}L^q_b$ into S_0. In this section we prove the sufficient conditions of Theorem 1 for the embedding of $L^p_a \cap \mathcal{F}L^q_b$ into S_0.

THEOREM 2. *If*

$$\Big(\frac{a}{d}-\frac{1}{p'}\Big)\Big(\frac{b}{d}-\frac{1}{q'}\Big) > \max\Big(\frac{1}{pq},\frac{1}{2p},\frac{1}{2q},\frac{1}{4}\Big),$$

then $L^p_a \cap \mathcal{F}L^q_b$ *is embedded in* $S_0(\mathbb{R}^d)$.

Proof. We first prove the theorem for $1 \le p,q \le 2$, in which case a more precise inequality can be derived.

PROPOSITION 2. *Suppose that* $1 \le p,q \le 2$,

$$\Big(\frac{a}{d}-\frac{1}{p'}\Big)\Big(\frac{b}{d}-\frac{1}{q'}\Big) > \frac{1}{pq},$$

and $f,g \in L^p_a \cap \mathcal{F}L^q_b$. *Then*

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |S_g f(x,y)|\,dx\,dy \le C(p,q)(\|f\|_{p,a}\|g\|_{p,a} + \|\widehat{f}\|_{q,b}\|\widehat{g}\|_{q,b}). \tag{11}$$

Proof. For the proof we only use Hölder's and the Hausdorff–Young inequalities, and Minkowski's inequality in the form

$$\Big(\int\Big|\int F(x,y)\,dx\Big|^r dy\Big)^{1/r} \le \int\Big(\int |F(x,y)|^r\,dy\Big)^{1/r} dx, \tag{12}$$

and the obvious submultiplicativity of the weights

$$(1+(|x|+|y|))^s \le (1+|x|)^s(1+|y|)^s \tag{13}$$

for $s \ge 0, x,y \in \mathbb{R}^d$. We write $F = S_g f$ and $F_y(x) = F(x,y)$, if we consider F a function of x only.

Step 1. For given $r > 0$ we partition $\mathbb{R}^d \times \mathbb{R}^d$ into $A_r = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x| \ge |y|^{1/r}\}$ and $B_r = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |y| \ge |x|^r\}$. Then

$$\int_{\mathbb{R}^{2d}} |S_g f| = \int_{A_r} |F| + \int_{B_r} |F|$$

The integral over A_r is estimated by

$$\int_{A_r} |F| = \int_{\mathbb{R}^d} dy \int_{|x|\ge|y|^{1/r}} |F_y(x)|(1+|x|)^a(1+|x|)^{-a}\,dx$$

$$\le \int_{\mathbb{R}^d} \|F_y\|_{p,a}\cdot\Big(\int_{|x|\ge|y|^{1/r}} (1+|x|)^{-ap'}\,dx\Big)^{1/p'} dy$$

$$\le \Big(\int_{\mathbb{R}^d} \|F_y\|^{p'}_{p,a}\,dy\Big)^{1/p'}\Big(\int_{\mathbb{R}^d}\Big[\int_{|x|\ge|y|^{1/r}} (1+|x|)^{-ap'}\,dx\Big]^{p/p'} dy\Big)^{1/p}.$$

In the latter integral we have $\int_{|x|\ge|y|^{1/r}}(1+|x|)^{-ap'}\,dx = \mathcal{O}(|y|^{(-ap'+d)/r})$ and thus the double integral is of order

$$\mathcal{O}\Big(\int_{|y|\ge1} |y|^{((-ap'+d)/r)(p/p')}\,dy\Big).$$

This is finite if and only if

$$\frac{-ap'+d}{r}\cdot\frac{p}{p'} < -d,$$

that is, if and only if

$$\frac{a}{d}-\frac{1}{p'} > \frac{r}{p}.$$

Therefore we obtain

$$\int_{A_r} |F(x,y)|\,dx\,dy \le C\Big(\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d} |S_g f(x,y)|^p(1+|x|)^{ap}\,dx\Big]^{p'/p} dy\Big)^{1/p'}. \tag{14}$$

If we replace (p,a,r) by $(q,b,1/r)$ and interchange x and y, the analogous estimate

$$\int_{B_r} |F(x,y)|\,dx\,dy \le C'\Big(\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d} |S_g f(x,y)|^q(1+|y|)^{bq}\,dy\Big]^{q'/q} dx\Big)^{1/q'}. \tag{15}$$

holds, provided that

$$\frac{b}{d}-\frac{1}{q'} > \frac{1}{rq}.$$

So far we have shown that if

$$\Big(\frac{a}{d}-\frac{1}{p'}\Big)\Big(\frac{b}{d}-\frac{1}{q'}\Big) > \frac{1}{pq},$$

then $\int |S_g f|$ is bounded by the two mixed norms in (14) and (15).

Step 2. We estimate the mixed norms by $\|f\|_{p,a}$ and $\|\widehat{f}\|_{q,b}$ respectively. For this we use the following lemma, which will be proved later.

LEMMA 5. *Suppose that* $f,g \in L^p_a(\mathbb{R}^d)$ *for* $1 \le p \le 2$ *and* $a \ge 0$. *Then*

$$\Big(\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d} |S_g f(x,y)|^p(1+|x|)^{ap}\,dx\Big]^{p'/p} dy\Big)^{1/p'} \le C_p\|f\|_{p,a}\|g\|_{p,a}$$

with a constant C_p *depending only on* p.

With Lemma 5 we can finish the proof of Proposition 2 easily. The right hand side of (14) is bounded by $C\|g\|_{p,a}\|f\|_{p,a}$; and since $|S_g f(x,y)| =$

$|S_{\hat g}\widehat{f}(y,-x)|$, Lemma 5 also applies to the right hand side of (15), to yield the bound $\|\widehat{g}\|_{q,b}\|\widehat{f}\|_{q,b}$. Combining these estimates proves (11). ∎

Proof of Lemma 5. Set

$$I(p)=\Big(\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d}|F(x,y)|^p(1+|x|)^{ap}\,dx\Big]^{p'/p}dy\Big)^{p/p'}.$$

Using Minkowski's inequality (12) with $r=p'/p\ge 1$ yields

$$I(p)\le\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|F(x,y)|^{pr}(1+|x|)^{apr}\,dy\Big)^{1/r}dx$$
$$=\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|F(x,y)|^{p'}\,dy\Big)^{p/p'}(1+|x|)^{ap}\,dx.$$

Using (7), the inner integral is just $\|\widehat{f}*(M_{-x}\widehat{g})^{\sim}\|_{p'}$. Since $p\le 2$, this p'-norm can be majorized with the Hausdorff–Young inequality by $C_p\|f\cdot T_{-x}\overline{g}\|_p$. We continue as follows:

$$I(p)\le C_p^p\int\Big(\int|f(t)g(t+x)|^p\,dt\Big)((1+|x|))^{ap}\,dx$$
$$=C_p^p\int|f(t)|^p\Big(\int|g(u)|^p(1+|u-t|)^{ap}\,du\Big)\,dt\le C_p^p\|f\|_{p,a}^p\|g\|_{p,a}^p,$$

where we have used the submultiplicativity (13) in the last step. Thus the lemma is proved. ∎

Finally we prove the remaining cases of Theorem 2, where Lemma 5 is not applicable.

First assume that $p>2$ and $1\le q\le 2$. In this case we use the embedding $L_a^p\hookrightarrow L_c^2$ for $c<a+d/p-d/2$ (Lemma 1(b)). By what we have already proved in Proposition 2, $f\in S_0(\mathbb{R}^d)$, provided that

$$\frac{1}{2q}<\Big(\frac{c}{d}-\frac12\Big)\Big(\frac{b}{d}-\frac{1}{q'}\Big)<\Big(\frac{a}{d}-\frac{1}{p'}\Big)\Big(\frac{b}{d}-\frac{1}{q'}\Big).$$

The case $1\le p\le 2, q>2$ is treated similarly.

If both $p>2$ and $q>2$, then $L_a^p\hookrightarrow L_c^2$ for $c<a+d/p-d/2$ and $L_b^q\hookrightarrow L_\gamma^2$ for $\gamma<a+d/q-d/2$. Again by Proposition 2, $f\in S_0$, whenever

$$\frac14<\Big(\frac{c}{d}-\frac12\Big)\Big(\frac{\gamma}{d}-\frac12\Big)<\Big(\frac{a}{d}-\frac{1}{p'}\Big)\Big(\frac{b}{d}-\frac{1}{q'}\Big)$$

Theorem 2 is completely proved. ∎

Remarks. 1. If one is willing to accept greater generality and more notation, the steps of the proof can be recast as embeddings of certain function spaces. For this we define the *modulation spaces* which were introduced by H. Feichtinger [8]. They are a tool to measure the time-frequency content of a distribution f in terms of S_gf. Fix $g\in\mathcal{S}(\mathbb{R}^d)$ and let w be a positive weight function that satisfies $w(x+u,y+v)\le C(1+|u|+|v|)^a w(x,y)$ for some constants $C>0$, $a\ge 0$ and all $u,v,x,y\in\mathbb{R}^d$. The modulation spaces $M_w^{p,q}(\mathbb{R}^d)$ are defined by the norms

$$\|f\|_{p,q,w}=\Big(\int_{\mathbb{R}^d}\Big[\int_{\mathbb{R}^d}|S_gf(x,y)|^p\,w(x,y)^p\,dx\Big]^{q/p}dy\Big)^{1/q}\tag{16}$$

Different windows $g\in\mathcal{S}$ define the same space and yield equivalent norms. In Step 1 of the proof we have essentially shown an embedding of the form

$$M_{w_a}^{p,p'}\cap\mathcal{F}M_{w_b}^{q,q'}\hookrightarrow S_0(\mathbb{R}^d),\tag{17}$$

where $w_a(x,y)=(1+|x|)^a$. Similarly, with this notation Lemma 5 expresses the embedding

$$L_a^p(\mathbb{R}^d)\hookrightarrow M_{w_a}^{p,p'}\quad\text{if } 1\le p\le 2.\tag{18}$$

An alternative proof of Lemma 5 could be given by means of interpolation of modulation spaces. One verifies that $L_a^1\hookrightarrow M_{w_a}^{1,\infty}$ and $L_a^2=M_{w_a}^{2,2}$. Interpolating one obtains the lemma.

2. The S_0-uncertainty principle allows for many variations. One might try other weight functions and obtain embeddings and uncertainty principles respectively in this way. The following result seems of some interest because it sheds some light on the case of the critical parameters in Theorem 1.

Suppose that

$$\Big(\frac{a}{d}-\frac{1}{p'}\Big)\Big(\frac{b}{d}-\frac{1}{q'}\Big)=\frac{1}{pq},$$

where $1\le p,q\le 2$, *and that* $\alpha>1/p$ *and* $\beta>1/q$. *If* $f(x)(\log(1+|x|))^\alpha\in L_a^p$ *and* $\widehat{f}(\omega)(\log(1+|\omega|))^\beta\in L_b^q$, *then* $f\in S_0(\mathbb{R}^d)$.

We omit the proof, since it does not involve any new ideas and follows exactly the same steps as the proof of Theorem 2.

3. Another interesting problem is how the "excess" decay measured by $(a/d-1/p')(b/d-1/q')-1/(pq)$ affects the phase-space concentration of f. In this respect we can show the following result, whose proof is again similar to that of Proposition 2.

If $1\le p,q\le 2$ *and*

$$\Big(\frac{a}{d}-\frac{1}{p'}\Big)\Big(\frac{b}{d}-\frac{1}{q'}\Big)>\Big(\frac{\alpha}{d}+\frac{1}{q}\Big)\Big(\frac{\beta}{d}+\frac{1}{p}\Big),$$

then

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|S_gf(x,y)|(1+|x|)^\alpha(1+|y|)^\beta\,dx\,dy\le C(\|f\|_{p,a+\alpha}+\|\widehat{f}\|_{q,b+\beta}).\tag{19}$$

4. A reasonable next project would be the study of more general embeddings of the type $L_a^p\cap\mathcal{F}L_b^q\hookrightarrow M_w^{r,s}$. It is not difficult to derive sufficient

conditions for such embeddings, but in the light of the counter-examples in the next section it could be quite challenging to find the optimal exponents.

4. Counter-examples. In this section we show that the conditions in Theorem 1 are essentially sharp.

We can rely in part on the counter-examples to the Poisson summation formula which have been given by Kahane and Lemarié [16], and which in turn extend Katznelson's construction [17].

Case 1: $1 \le p, q \le 2$ and $(a/d - 1/p')\,(b/d - 1/q') < 1/(pq)$. In dimension $d = 1$, by [16], there exists a continuous function $f \in L^p_a \cap \mathcal{F}L^q_b(\mathbb{R})$ so that $\sum_{k\in\mathbb{Z}} f(k) \neq \sum_{k\in\mathbb{Z}} \widehat{f}(k)$. According to Corollary 1 such an f cannot be in $S_0(\mathbb{R})$.

If $d > 1$, we choose an $f \in L^p_{a/d} \cap \mathcal{F}L^q_{b/d}(\mathbb{R}) \setminus S_0(\mathbb{R})$ of one variable, and we set $F(x) = \prod_{i=1}^d f(x_i)$; then $F \in L^p_a \cap \mathcal{F}L^q_b(\mathbb{R}^d)$, but $F \notin S_0(\mathbb{R}^d)$.

Case 2: $1 \le p < 2$, $q > 2$ and $(a/d - 1/p')\,(b/d - 1/q') < 1/(2p)$. For $d = 1$, there is a continuous function $f \in L^p_a \cap \mathcal{F}L^q_b(\mathbb{R})$ such that $f \notin W(\mathbb{R})$ (cf. [16]). By Proposition 1, f cannot be in $S_0(\mathbb{R})$. The extension to higher dimensions is again done by tensor products.

The case $1 \le q \le 2$, $p > 2$ and $(a/d - 1/p')\,(b/d - 1/q') < 1/(2q)$ is symmetrical. Thus we only have to consider afresh

Case 3. $p, q > 2$ and $(a/d - 1/p')\,(b/d - 1/q') < 1/4$. To treat this case, we need some facts about norm estimates of Gabor sums and on unimodular polynomials, which may be of independent interest.

Lemma 6. *Suppose $f \in S_0(\mathbb{R}^d)$. Then there exists an $R > 0$ and $C = C(R, f)$ so that for $r > R$,*

$$\Big\| \sum_{k,l\in\mathbb{Z}^d} a_{kl} M_{lr} T_{kr} f \Big\|_{S_0} \ge C \sum_{k,l\in\mathbb{Z}^d} |a_{kl}|.$$

Proof. Set $f_r = \sum_{k,l\in\mathbb{Z}^d} a_{kl} M_{lr} T_{kr} f$. Given $\varepsilon > 0$, there exists an $R > 0$ such that

$$\int\limits_{|x|\ge R/2} \int\limits_{|y|\ge R/2} |S_g f(x,y)|\,dx\,dy < \varepsilon.$$

Note that

$$S_g f_r(x,y) = \sum_{k,l\in\mathbb{Z}^d} a_{kl} e^{2\pi i r k\cdot(y - rl)} S_g f(x - rk, y - rl).$$

By periodizing the integral $\iint_{\mathbb{R}^{2d}}$ over a cube $C_r = [-r/2, r/2]^{2d}$, we can write

$$\begin{aligned}\|f_r\|_{S_0} &= \iint |S_g f_r(x,y)|\,dx\,dy \\ &= \sum_{m,n\in\mathbb{Z}^d} \iint\limits_{C_r} \Big| \sum_{k,l\in\mathbb{Z}^d} a_{kl} e^{2\pi i r k\cdot(y + rn - rl)} S_g f(x + r(m-k), y + r(n-l)) \Big|\,dx\,dy.\end{aligned}$$

Therefore

$$\begin{aligned}\|f_r\|_{S_0} &\ge \sum_{m,n} \iint\limits_{C_r} |a_{mn}|\,|S_g f(x,y)|\,dx\,dy \\ &\quad - \sum_{m,n} \iint\limits_{C_r} \sum_{(k,l)\neq(m,n)} |a_{kl}| \cdot |S_g f(x + r(m-k), y + r(n-l))|\,dx\,dy \\ &\ge \sum_{m,n\in\mathbb{Z}^d} |a_{mn}| \Big(\iint\limits_{C_r} |S_g f(x,y)|\,dx\,dy - \varepsilon \Big),\end{aligned}$$

which proves the lemma for ε small enough and $r \ge R$. ∎

A related statement can be found in [19].

Lemma 7. *Let $g \in \mathcal{S}$, $(\alpha_k) \subseteq \mathbb{C}$ be a sequence of complex numbers and $P_n(x)$ be a sequence of functions of period 1. Then we have, for every integer $R > 0$,*

$$\Big\| \sum_{k=1}^{\infty} \alpha_k P_k(Rx) T_{Rk} g(x) \Big\|_{p,a} \le C R^{d/p} \Big(\sum_{k=1}^{\infty} |\alpha_k|^p \|P_k\|_p^p (1 + |k|)^{ap} \Big)^{1/p}.$$

where naturally $\|P_k\|_p^p = \int_{[0,R]^d} |P_k(x)|^p\,dx$.

Proof. Let $f(x) = \sum_{k=1}^{\infty} \alpha_k P_k(Rx) T_{Rk} g(x)$ and $\int_{\mathbb{R}^d} = \sum_{n\in\mathbb{Z}^d} \int_{Rn + [0,R]^d}$. We obtain

$$\|f\|_{p,a}^p = \int\limits_{[0,R]^d} \sum_{n\in\mathbb{Z}^d} \Big| \sum_k \alpha_k P_k(Rx + Rn) g(x + R(n-k)) |^p (1 + |x + Rn|)^{ap} \Big|\,dx.$$

Majorizing $\sup_{x\in[0,R]^d}(1 + |x + Rn|)$ by $C(1 + |Rn|)$, we see that for fixed x the integrand is just the discrete ℓ^p_a-norm $\|c_x * d_x\|_{p,a}^p$, where c_x and d_x are the sequences $c_x(k) = \alpha_k P_k(R(x + n)) = \alpha_k P_k(Rx)$ and $d_x(k) = g(x + Rk)$, $k \in \mathbb{Z}^d$. Since $\|c_x * d_x\|_{p,a} \le \|c_x\|_{p,a} \|d_x\|_{1,a}$, we obtain the desired estimate from the fact that $\|f\|_{p,a}^p$ is bounded by

$$\begin{aligned}&C \int\limits_{[0,R]^d} \Big(\sum_k |\alpha_k|^p |P_k(Rx)|^p (1 + |Rk|)^{ap} \Big) \Big(\sum_k |g(x + Rk)| (1 + |Rk|)^a \Big)^p dx \\ &\le C R^d \Big(\sup_{x\in[0,R]^d} \sum_k |g(x + Rk)| (1 + |Rk|)^a \Big)^p \sum_k |\alpha_k|^p \|P_k\|_p^p (1 + R|k|)^{ap}. \ ∎\end{aligned}$$

Furthermore, we assume the existence of trigonometric polynomials p_n on $\mathbb{R}$, so-called *unimodular polynomials*, with the following properties:

(20) $$p_n(t)=\sum_{k=1}^{n} a_k e^{2\pi ikt}\qquad \text{with } |a_k|=1$$

and

(21) $$\|p_n\|_\infty \le C\sqrt{n}$$

with a constant C independent of the degree n. The best known examples are the Rudin–Shapiro polynomials. We refer to Kahane's fundamental paper [15] for further information and references on unimodular polynomials.

We can now provide a counter-example for the remaining Case 3.

PROPOSITION 3. *If $p,q\ge 2$ and $(a/d-1/p')(b/d-1/q')<1/4$, there exists a sequence $f_n\in S_0(\mathbb{R}^d)$ such that*

$$\|f_n\|_{S_0}/(\|f_n\|_{p,a}+\|\widehat{f}_n\|_{q,b})>Cn^{\varepsilon}$$

for some $\varepsilon>0$. Consequently, $L^p_a\cap\mathcal{F}L^q_b$ is not embedded in S_0.

Proof. Let $g\in S_0(\mathbb{R}^d)$ be arbitrary and choose an integer $R>0$ so that the assertion of Lemma 6 holds. We choose a sequence of unimodular polynomials p_n satisfying (20) and (21) and set $P_n(x)=\prod_{j=1}^{d}p_n(Rx_j)$. Then $P_n(x)=\sum_{k\in C_n}a_ke^{2\pi ik\cdot x}$, where the summation is over all integer vectors in the cube $C_n=\{x\in\mathbb{R}^d:0<x_i\le n\}$. Furthermore,

$$\|P_n\|_\infty=\|p_n\|_\infty^d\le Cn^{d/2}$$

and $|a_k|=1$ for $k\in C_n\cap\mathbb{Z}^d$, and $\widehat{P}_n=\sum_{C_n}a_k\delta_{Rk}$ is a discrete measure.

Set $k=[n^s]+1$, with $s>0$ to be chosen later, and define

(22) $$f_n=P_k\cdot(\widehat{P}_n*g)=\sum_{\substack{l\in C_k\\ m\in C_n}}a_la_mM_{Rl}T_{Rm}g=P_k\sum_{m}a_mT_{mR}g.$$

By Lemma 6,

(23) $$\|f_n\|_{S_0}\ge C\sum_{\substack{l\in C_k\\ m\in C_n}}|a_l|\cdot|a_m|=Ck^dn^d\ge Cn^{(s+1)d}.$$

On the other hand, using Lemma 7 we obtain

$$\|f_n\|_{p,a}\le\|P_k\|_p\Big(\sum_{m\in C_n}|a_m|^p(1+R|m|)^{ap}\Big)^{1/p},$$

which is easily seen to be of the order

(24) $$\|f_n\|_{p,a}=\mathcal{O}(k^{d/2}n^{a+d/p})=\mathcal{O}(n^{d(s/2+a/d+1/p)}).$$

A similar argument applied to $\widehat{f}_n=\widehat{P}_k*(P_n\cdot\widehat{g})=\sum_l a_lP_n(\cdot-R^2l)T_{Rl}\widehat{g}$ yields

(25) $$\|f_n\|_{q,b}=\mathcal{O}(k^{b+d/q}n^{d/2})=\mathcal{O}(n^{d/2+ds(b/d+1/q)}).$$

Combining (23)–(25), we obtain

$$\frac{\|f_n\|_{S_0}}{(\|f_n\|_{p,a}+\|\widehat{f}_n\|_{q,b})}\ge C\frac{n^{(s+1)d}}{(n^{d(s/2+1+a/d-1/p')}+n^{d(1/2+s(b/d-1/q')+s)})}.$$

This expression dominates Cn^{ε} if and only if

$$s+1>\frac{s}{2}+1+\frac{a}{d}-\frac{1}{p'}\quad\text{and}\quad s+1>\frac{1}{2}+s\Big(\frac{b}{d}-\frac{1}{q'}\Big)+s$$

if and only if

(26) $$\frac{a}{d}-\frac{1}{p'}<\frac{s}{2}\quad\text{and}\quad\frac{b}{d}-\frac{1}{q'}<\frac{1}{2s}.$$

Since $(a/d-1/p')(b/d-1/q')<1/4$ we can choose such an s and the result follows. ■

Remark. Since clearly $L^\infty_{d+\varepsilon}\cap\mathcal{F}L^\infty_{d+\varepsilon}(\mathbb{R}^d)\hookrightarrow W(\mathbb{R}^d)$, Proposition 3 shows that $S_0(\mathbb{R}^d)$ is a proper subspace of $W(\mathbb{R}^d)\cap\mathcal{F}W(\mathbb{R}^d)$. This was already proved by Losert [19].

Our previous considerations still leave open the critical case

$$\Big(\frac{a}{d}-\frac{1}{p'}\Big)\Big(\frac{b}{d}-\frac{1}{q'}\Big)=\max\Big(\frac{1}{pq},\frac{1}{2p},\frac{1}{2q},\frac{1}{4}\Big).$$

We conjecture that in this case $L^p_a\cap\mathcal{F}L^q_b$ is not embedded into S_0. Here is a partial result in this direction. It also improves the considerations of Kahane and Lemarié.

PROPOSITION 4. *If $q=2$, $3/2<p\le 2$ and $(a/d-1/p')(b/d-1/2)\le 1/(2p)$, then there exists a continuous $f\in L^p_a\cap\mathcal{F}L^q_b$ which is not in S_0 and for which the Poisson summation formula fails.*

Proof. Again it is sufficient to prove the statement in dimension $d=1$. For if $f\in L^p_{a/d}(\mathbb{R})\cap L^q_{b/d}(\mathbb{R})\setminus S_0(\mathbb{R})$, then $f(x_1)\ldots f(x_d)\in L^p_a\cap\mathcal{F}L^q_b(\mathbb{R}^d)\setminus S_0(\mathbb{R}^d)$. Thus assume that $a-1/p'=r/p$ and $b-1/2=1/(2r)$ for some $r>0$.

Let $D_n(x)=\sum_{|k|\le n}e^{2\pi ikx}$ be the Dirichlet kernel. Choose $\alpha_k=(k^{r+1}\log k)^{-1}$ and set

(27) $$f(x)=\sum_{k=1}^{\infty}\alpha_kD_{[k^r]}(x)g(x-k)=\sum_{k=1}^{\infty}\alpha_k\sum_{j=1}^{[k^r]}M_jT_kg(x).$$

Then

$$\widehat{f}(\omega)=\sum_{j=1}^{\infty}\sum_{k\ge j^{1/r}}^{\infty}\alpha_kM_{-k}T_j\widehat{g}(\omega)=\sum_{j=1}^{\infty}\mu_j(\omega)T_j\widehat{g}(\omega).$$

By Lemma 7,

$$\|\widehat{f}\|_{2,b}^2 \le C\sum_j \|\mu_j\|_2^2(1+|j|)^{2b}.$$

In our case $2b = 1 + 1/r$ and

$$\|\mu_j\|_2^2 = \sum_{k\ge j^{1/r}}^{\infty} |k^{r+1}\log k|^{-2} \le C(\log j)^{-2} j^{-2-1/r}.$$

Therefore

$$\|\widehat{f}\|_{2,b}^2 \le C\sum_j (\log j)^{-2} j^{-2-1/r} j^{1+1/r} < \infty.$$

Again by Lemma 7 we estimate the p-norm of f as

$$\|f\|_{p,a}^p \le \sum_{k=1}^{\infty} \alpha_k^p \, \|D_{[k^r]}\|_p^p \, (1+|k|)^{ap}.$$

If $p \le 2$, then $\|D_n\|_p^p \le C(\log n)^{2-p} n^{p-1}$, and since $ap = r + p + 1$ we estimate

$$\|f\|_{p,a}^p \le \sum_{k=1}^{\infty} k^{-p(r+1)} (\log k)^{-p} (\log k^r)^{2-p} k^{r(p-1)} k^{r+p-1}$$
$$\le C\sum_{k=1}^{\infty} (\log k)^{2(1-p)} k^{-1}.$$

Therefore $f \in L_a^p$ if and only if $2(1-p) < -1$, i.e. $p > 3/2$. (For $p > 2$, f is always in L_a^p by a similar estimate using $\|D_k\|_p^p \le Cn^{p-1}$.)

Now we specify g. Let g be of the form $g = \psi * \psi^\sim$ for some $\psi \ge 0$ in $\mathcal{S}$ with compact support. Then f as in (27) is continuous, since the sum is locally finite, and

$$f(n) = \sum_{k=1}^{\infty} (k^{r+1}\log k)^{-1} D_{[k^r]}(n) g(n-k) \ge \frac{1}{n\log n} g(0)$$

and thus $\sum_n f(n) = \infty$. We have therefore constructed a continuous function $f \in L_a^p \cap L_b^2(\mathbb{R})$, but $f \notin S_0$ and $f \notin W(\mathbb{R})$. ∎

5. Pseudodifferential operators of trace class. In this section we give an application of the previous results to pseudodifferential operators. We refer to [11, 21] for an exposition of the general theory and we will adhere to the conventions used in Folland [11].

For our purpose a pseudodifferential operator $\sigma(D,X)$ with a symbol $\sigma(p,q)$, $(p,q) \in \mathbb{R}^d \times \mathbb{R}^d$, is just a superposition of translation and modulation operators $M_q T_{-p}$. In the Weyl correspondence the operator is defined as

(28)
$$\sigma(D,X) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \widehat{\sigma}(p,q) e^{i\pi p\cdot q} M_q T_{-p}\, dp\, dq$$

or pointwise as

$$\sigma(D,X)f(t) = \iint \widehat{\sigma}(p,q) e^{i\pi p\cdot q} e^{2\pi i q\cdot t} f(t+p)\, dp\, dq$$
$$= \iint \sigma\Big(\xi, \frac{x+y}{2}\Big) e^{2\pi i (t-y)\cdot\xi} f(y)\, dy d\xi.$$

We are only interested in symbols $\sigma \in S_0$, therefore the integral converges absolutely and defines a bounded operator on any L^p, $p \ge 1$.

The importance of S_0 in this context stems from the following theorem, which seems to be due to H. Feichtinger [9]. Since a published proof is not available, we sketch a simple one.

THEOREM 3. *If $\sigma \in S_0(\mathbb{R}^d \times \mathbb{R}^d)$, then $\sigma(D,X)$ is a trace-class operator on $L^2(\mathbb{R}^d)$.*

The proof follows easily from a calculation with "elementary" pseudodifferential operators and the inversion formula for the short time Fourier transform.

LEMMA 8. *Set $\Phi(p,q) = e^{-\pi(p^2+q^2)/2}$, $p,q \in \mathbb{R}^d$, $\phi(t) = e^{-\pi t^2}$, $t \in \mathbb{R}^d$, and write $x = (x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and $y = (y_1,y_2) \in \mathbb{R}^d \times \mathbb{R}^d$. Let $\Pi_{x,y}$ be the pseudodifferential operator corresponding to the symbol $\mathcal{F}^{-1}(M_y T_x \Phi)$. Then*

(29) $\Pi_{x,y} f = e^{2\pi i x_2\cdot y_2} \langle f, M_{-y_1-x_2/2} T_{-y_2+x_1/2}\phi\rangle M_{-y_1+x_2/2} T_{-y_2-x_1/2}\phi,$

in other words, $\Pi_{x,y}$ is a rank one operator.

Proof. Formula (29) seems to be implicit in many arguments on pseudodifferential operators. It follows from [11], pp. 31–33, and is also explicit in [13]. For completeness we give an elementary proof without fancy arguments, using only the formula

(30)
$$(e^{-\pi t^2/2})^\wedge(\omega) = 2^{d/2} e^{-2\pi\omega^2} \quad \text{for } t, \omega \in \mathbb{R}^d.$$

With the assumption $f \in L^1 \cap L^2$, all integrals are absolutely integrable and we can integrate in any order. In (28) only $\widehat{\sigma} = M_y T_x \Phi$ occurs, and thus after separating the parts depending on p and q, we obtain the explicit expression

$$\Pi_{x,y} f(t) = \iint \exp(2\pi i(p\cdot y_1 + q\cdot y_2))$$
$$\times \exp\Big(-\frac{\pi}{2}[(p-x_1)^2 + (q-x_2)^2]\Big) e^{i\pi p\cdot q} e^{2\pi i q\cdot t} f(t+p)\, dp\, dq$$

$$= \int \left(\int \exp\left(-\frac{\pi}{2}(q-x_2)^2\right) \exp\left(2\pi i q \cdot \left(t + y_2 + \frac{p}{2}\right)\right) dq \right)$$
$$\times \exp\left(-\frac{\pi}{2}(p-x_1)^2\right) \exp(2\pi i p \cdot y_1) f(t+p)\, dp.$$

The inner integral is the Fourier transform of a Gaussian at $t+y_2+p/2$ and with (30) it is

$$2^{d/2} \exp\left(-2\pi\left(t+y_2+\frac{p}{2}\right)^2\right) \exp\left(2\pi i x_2 \cdot \left(t+y_2+\frac{p}{2}\right)\right).$$

Now we set $u = t+p$ in the remaining integral and obtain

$$\Pi_{x,y} f(t) = 2^{d/2} \int_{\mathbb{R}^d} \exp\left(-2\pi\left(y_2 + \frac{u+t}{2}\right)^2 - \frac{\pi}{2}(u-t-x_1)^2\right)$$
$$\times \exp\left(2\pi i \left[\left(y_2 + \frac{u+t}{2}\right) \cdot x_2 + (u-t)\cdot y_1\right]\right) f(u)\, du.$$

After sorting out terms depending on t and on u and completing some squares, we arrive after some book-keeping at formula (29). ■

Proof of Theorem 3. Since the translation and modulation operators are isometries on L^2, we have

$$(31) \qquad |\mathrm{tr}\, \Pi_{x,y}| \le \|\phi\|_2^2 \quad \text{for all } x, y \in \mathbb{R}^{2d} \times \mathbb{R}^{2d}.$$

Given a symbol $\sigma \in S_0(\mathbb{R}^{2d})$, and hence also $\widehat{\sigma} \in S_0(\mathbb{R}^{2d})$, we write

$$\widehat{\sigma} = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} S_\Phi \widehat{\sigma}(x,y)\, M_y T_x \Phi \, dx\, dy$$

by means of the inversion formula in Lemma 3. Therefore

$$\sigma(D,X) = \iint S_\Phi \widehat{\sigma}(x,y)\, \Pi_{x,y}\, dx\, dy.$$

This integral is absolutely convergent and thus

$$(32) \qquad |\mathrm{tr}\, \sigma(D,X)| \le \iint_{\mathbb{R}^{4d}} |S_\Phi \widehat{\sigma}(x,y)| \cdot |\mathrm{tr}\, \Pi_{x,y}|\, dx\, dy \le \|\phi\|_2^2 \|\widehat{\sigma}\|_{S_0}. \ ■$$

Combining the theorem with the embeddings proved earlier, we obtain sufficient conditions for the trace class of pseudodifferential operators that are perhaps more useful than the main theorem itself.

COROLLARY 2. *Suppose that* $\sigma \in L^p_a \cap \mathcal{F}L^q_b(\mathbb{R}^{2d})$ *and that*

$$\left(\frac{a}{2d} - \frac{1}{p'}\right)\left(\frac{b}{2d} - \frac{1}{q'}\right) > \max\left(\frac{1}{pq}, \frac{1}{2p}, \frac{1}{2q}, \frac{1}{4}\right).$$

Then $\sigma(D,X)$ *is of trace class.*

Remarks. 1. Theorem 3 and Corollary 2 extend the results by Daubechies [3] and Heil, Ramanathan, and Topiwala [13] to general p and q, but the range of parameters does not seem to be sharp even for $p = q = 2$.

2. In view of Lemma 8 it is trivial that every pseudodifferential operator $\sigma(D,X)$ with a symbol of the form

$$(33) \qquad \widehat{\sigma} = \sum_j c_j M_{x_j} T_{y_j} \Phi \quad \text{with} \quad \sum_j |c_j| < \infty$$

and $(x_j, y_j) \in \mathbb{R}^{4d}$ is of trace class. Following standard procedures one might consider the Banach space B of all symbols σ with a representation (33) and an appropriate norm. This space would furnish a natural but rather cumbersome symbol class for trace-class operators. However, an important theorem of Feichtinger [8] asserts that every $\sigma \in S_0$ has such a representation (33) with ℓ^1-coefficients. Thus the abstract extension B coincides with S_0 and Theorem 3 and Corollary 2 provide a more explicit and user-friendly description of trace-class symbols. This discrete representation also implies that every pseudodifferential operator with a symbol $\sigma \in S_0$ is a nuclear operator on L^p.

3. Using modulation spaces and their interpolation properties, one can easily obtain sufficient conditions for a pseudodifferential operator to be in the p-Schatten ideal I_p. The map $\sigma \to \sigma(D,X)$ maps $S_0 = M^{1,1}$ (with trivial weight) into the trace-class ideal I_1 and $L^2 = M^{2,2}$ onto the Hilbert–Schmidt operators I_2. By interpolation every symbol $\sigma \in M^{p,p}$ gives an operator in I_p for $1 \le p \le 2$.

4. Theorem 3 carries over to the Kohn–Nirenberg correspondence, where the operator

$$(34) \qquad \sigma_{KN}(D,X) = \iint \widehat{\sigma}(p,q) M_q T_{-p}\, dp\, dq$$

is associated to a symbol σ. Comparing (28) with (34), we see that we can switch between the two definitions by a simple multiplication with $e^{i\pi p\cdot q}$. Thus if $\widehat{\tau}(p,q) = e^{i\pi p\cdot q}\widehat{\sigma}$, then $\tau_{KN}(D,X) = \sigma(D,X)$. Since it can be shown that S_0 is invariant under the multiplication with the "chirp" $e^{i\pi p\cdot q}$ (cf. [6, 10]), Theorem 3 also holds for the Kohn–Nirenberg correspondence.

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