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Rotundity and smoothness of convex bodies in reflexive and nonreflexive spaces

by

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Abstract. For combining two convex bodies C and D to produce a third body, two of the most important ways are the operation $\overline{+}$ of forming the closure of the vector sum $C+D$ and the operation $\overline{\gamma}$ of forming the closure of the convex hull of $C\cup D$. When the containing normed linear space X is reflexive, it follows from weak compactness that the vector sum and the convex hull are already closed, and from this it follows that the class of all rotund bodies in X is stable with respect to the operation $\overline{+}$ and the class of all smooth bodies in X is stable with respect to both $\overline{+}$ and $\overline{\gamma}$. In our paper it is shown that when X is separable, these stability properties of rotundity (resp. smoothness) are actually equivalent to the reflexivity of X. The characterizations remain valid for each nonseparable X that contains a rotund (resp. smooth) body.

Introduction. Except where otherwise stated, X will always denote a normed linear space (i.e., a normed real vector space). For two subsets C and D of X, $C+D$ denotes the vector sum and $C\,\gamma\, D$ denotes the convex hull of $C\cup D$. The closures of $C+D$ and $C\,\gamma\,D$ are denoted by $C\mathbin{\overline{+}}D$ and $C\,\overline{\gamma}\,D$ respectively. As the term is used here, a *body* in X is a subset that is bounded, closed, convex, and has nonempty interior. When C and D are bodies, so are $C\mathbin{\overline{+}}D$ and $C\,\overline{\gamma}\,D$, and in fact $\overline{+}$ and $\overline{\gamma}$ are two of the most important and useful ways of combining two bodies to produce a third body. The present paper is concerned with the extent to which the basic properties of rotundity (= strict convexity) and smoothness of bodies are preserved by the operations $\overline{+}$ and $\overline{\gamma}$.

When X is reflexive and C and D are bodies in X, it follows from weak

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compactness that both $C + D$ and $C \gamma D$ are closed. In this case, rotundity and smoothness exhibit the following stability properties with respect to the operations $\bar{+}$ and $\bar{\gamma}$:

(i) if C and D are both rotund, so is $C \bar{+} D$;
(ii) if at least one of C and D is smooth, so is $C \bar{+} D$;
(iii) if C and D are both smooth, so is $C \bar{\gamma} D$.

The main result of the present paper (stated in its final Corollary 4.4) asserts that if X contains a rotund body, then condition (i) is equivalent to the reflexivity of X, and if X contains a smooth body, then each of the conditions (ii) and (iii) is equivalent to the reflexivity of X. The assumptions about the existence of rotund or smooth bodies are unnecessary when X is separable, for every separable normed linear space admits an equivalent norm in which the unit ball is both rotund and smooth [Da].

Our arguments are somewhat technical in nature, because we have phrased them so that the characterizations of reflexivity are valid not only for rotundity and the usual smoothness, but also for several relatives of smoothness. However, the reader can grasp the main ideas of the arguments by reading only the statements of Theorem 1.5 and the Extension Lemma 3.4, and then reading the statements and proofs of Theorems 4.1 and 4.3 and of Corollary 4.4.

1. Definitions and preliminary results

1.1. Definition. A *bornology* on X is a covering β of X by nonempty bounded subsets such that β is closed under scalar multiplication and under the formation of finite unions. A closed convex set $C \subset X$ with $0 \in \operatorname{int} C$ is *β-smooth at a point* $x \in \partial C$ if C's Minkowski functional (gauge functional) p_C is β-differentiable at x, i.e., if for all $S \in \beta$ the limit

$$\lim_{t\to 0} \frac{p_C(x+tv) - p_C(x)}{t}$$

exists uniformly for $v \in S$. A general closed convex set $C \subset X$ with nonempty interior is *β-smooth* if each of its translates (equivalently, some translate) C_0 with $0 \in \operatorname{int} C_0$ is β-smooth at each $x \in \partial C_0$.

1.2. Remarks. (a) Natural choices for β are all finite sets (*Gateaux smoothness*, or simply *smoothness*), all weakly compact sets (*Hadamard smoothness*), or all bounded sets (*Fréchet smoothness*).

(b) The assumption that $\bigcup \beta = X$ assures that each β-smooth set is smooth in the sense of admitting a unique supporting hyperplane at each boundary point. (This is in fact equivalent to Gateaux smoothness.)

(c) If C and D are closed convex sets in X such that $\emptyset \neq \operatorname{int} C \subset D$, and if C is β-smooth at a point $x \in \partial C \cap \partial D$, then D is also β-smooth at x. Indeed, suppose $0 \in \operatorname{int} C$. Choose $f \in \partial p_D(x)$ (the subdifferential of p_D at x; cf. [Ph]). Then necessarily $f = p'_C(x)$, since $p_D(x) = p_C(x)$ and $p_D(y) \le p_C(y)$ for all y. Let $q(y) = p_C(x) + f(y - x)$, and note that the functions p_C and q are both β-differentiable at x (q is affine!), that $q \le p_D \le p_C$, and that $q(x) = p_D(x) = p_C(x)$. Hence also p_D is β-differentiable at x.

(d) Suppose that Y and Z are normed linear spaces, and $K \subset Y$ and $L \subset Z$ are closed convex sets with $0 \in \operatorname{int} K$ and $0 \in \operatorname{int} L$. Suppose that β_Y and β_Z are bornologies on Y and Z respectively. With $1 < q < \infty$, consider the closed convex set

$$C = \{(y,z) \in Y \times Z \mid p_K(y)^q + p_L(z)^q \le 1\}$$

and the bornology β on $Y \times Z$ that consists of all finite unions of sets of the form $A \times B$ with $A \in \beta_Y$ and $B \in \beta_Z$. If K is β_Y-smooth and L is β_Z-smooth, then C is β-smooth. This follows in a standard way from the fact that $p_C(y,z)^q = p_K(y)^q + p_L(z)^q$.

1.3. Definition. Let τ be a *linear topology* on X, that is, a topology with respect to which the operations of vector addition and multiplication by scalars are jointly continuous. We shall say that a closed convex set $C \subset X$ with $\operatorname{int} C \neq \emptyset$ is *τ-LUR at a point* $x \in \partial C$ if it is true that $x_n \xrightarrow{\tau} x$ whenever $x_n \in C$ with $\operatorname{dist}\big(\frac{1}{2}(x_n + x), \partial C\big) \to 0$. This is a natural generalization of the *local uniform convexity* first studied by Lovaglia [Lo].

1.4. Remark. If C and D are closed convex sets in X such that $\emptyset \neq \operatorname{int} C \subset D$, then each point of the set $\partial C \cap \partial D$ that is an extreme point of D is also an extreme point of C.

At least for its main particular cases, the following Theorem 1.5 is part of the mathematical folklore. We include a proof for the sake of completeness and to show that it continues to hold in the more general setting employed here, although our main results do not concern τ-LUR.

1.5. Theorem. *Suppose that C and D are bodies in a reflexive space X, that τ is a linear topology on X and β is a bornology on X. Then $C + D$ and $C \gamma D$ are closed and the following implications hold*:

(i) *if C and D are both τ-LUR (resp. rotund), so is $C + D$;*
(ii) *if at least one of C and D is β-smooth, so is $C + D$;*
(iii) *if C and D are both β-smooth, so is $C \gamma D$.*

Proof. A simple weak-compactness argument implies that both $C + D$ and $C \gamma D$ are closed.

(i) To prove (i), we first establish the following

Claim. *For each $c \in C$ and $d \in D$,*

$$\operatorname{dist}(c + d, \partial(C + D)) \ge \operatorname{dist}(c, \partial C) + \operatorname{dist}(d, \partial D).$$

Let $r_c = \operatorname{dist}(c, \partial C)$ and $r_d = \operatorname{dist}(d, \partial D)$. Then, if $\overline{B}(x, r)$ denotes the closed ball of radius r centered at x, $\overline{B}(c, r_c) \subset C$ and $\overline{B}(d, r_d) \subset D$, whence $\overline{B}(c+d, r_c + r_d) \subset C + D$, and that proves the Claim.

Now if $x \in \partial(C+D)$, $\{x_n\} \subset C+D$, and $\operatorname{dist}(\frac{1}{2}(x+x_n), \partial(C+D)) \to 0$, take $c, c_n \in C$ and $d, d_n \in D$ such that $c + d = x$ and $c_n + d_n = x_n$ for all n. Then by the Claim, $c \in \partial C$, $d \in \partial D$, $\operatorname{dist}(\frac{1}{2}(c + c_n), \partial C) \to 0$ and $\operatorname{dist}(\frac{1}{2}(d + d_n), \partial D) \to 0$. If C and D are τ-LUR then $c_n \xrightarrow{\tau} c$ and $d_n \xrightarrow{\tau} d$. Hence also $x_n \xrightarrow{\tau} x$. The rotund case can be handled similarly.

(ii) Now suppose that D is β-smooth. Take $x \in \partial(C+D)$ and a nonzero $f \in X^*$ such that $f(x) = \sup f(C+D)$. Write $x = c+d$, where $c \in C$, $d \in D$. Then

$$f(c) + f(d) = f(x) = \sup f(C + D) = \sup f(C) + \sup f(D),$$

whence $f(c) = \sup f(C)$ and $f(d) = \sup f(D)$. Consequently,

$$f(x) = f(c) + \sup f(D) = \sup f(c + D)$$

and hence $x \in \partial(c+D) \cap \partial(C+D)$. Since clearly $c+D \subset C+D$, the body $C+D$ is β-smooth at x by Remark 1.2(c).

(iii) Suppose, finally, that both C and D are β-smooth, and that $x \in \partial(C \gamma D)$. Take a nonzero $f \in X^*$ with $f(x) = \sup f(C \gamma D))$ and write $x = (1-\lambda)c + \lambda d$ with $c \in C, d \in D, 0 \le \lambda \le 1$. Then the body $(1-\lambda)C + \lambda D$ is contained in $C \gamma D$ and

$$f(x) \le \sup f((1-\lambda)C + \lambda D) \le \sup f(C \gamma D) = f(x).$$

Consequently, $x \in \partial((1-\lambda)C + \lambda D) \cap \partial(C \gamma D)$. By (ii) above, the body $(1-\lambda)C + \lambda D$ is β-smooth at x. By Remark 1.2(c) again, the larger body $C \gamma D$ is also β-smooth at x. ∎

We want next to show that even in a nonreflexive normed linear space, Theorem 1.5 continues to hold when the bodies C and D are homothetic. A particular case of this result appears in [GKM]. We need the following simple lemma.

1.6. Lemma. *Suppose that X is a topological vector space, A is a compact subset of X, and B is a bounded closed subset of X. If T is a compact set of scalars and $f, g : T \to \mathbb{R}$ are continuous functions, then the set*

$$Q = \bigcup_{\lambda \in T} [f(\lambda)A + g(\lambda)B]$$

is closed in X.

Proof. Let $\{x_\nu\}$ be a net of elements of Q that converges to a point $x \in X$. We can write $x_\nu = f(\lambda_\nu)a_\nu + g(\lambda_\nu)b_\nu$, where $\lambda_\nu \in T$, $a_\nu \in A$ and $b_\nu \in B$. By compactness (and passing if necessary to a subnet), it is permissible to assume that $\lambda_\nu \to \lambda_0 \in T$ and $a_\nu \to a_0 \in A$. Thus $g(\lambda_\nu)b_\nu \to x - f(\lambda_0)a_0$. If $g(\lambda_0) = 0$, then $g(\lambda_\nu)b_\nu \to 0$, and hence

$$x \in f(\lambda_0)a_0 + g(\lambda_0)B \subset Q.$$

On the other hand, if $g(\lambda_0) \neq 0$, then

$$b_\nu \to \frac{x - f(\lambda_0)a_0}{g(\lambda_0)} =: b_0 \in B$$

and thus $x = f(\lambda_0)a_0 + g(\lambda_0)b_0 \in Q$. ∎

1.7. Proposition. *Suppose that C and D are bodies in a normed linear space X, with $D = u + \alpha C$ for some $u \in X$ and $\alpha > 0$. Then the sets $C + D$ and $C \gamma D$ are closed and the following implications hold:*

(i) *if C is rotund or τ-LUR or β-smooth then so is $C + D$;*
(ii) *if C is β-smooth then so is $C \gamma D$.*

Proof. It is clear that D has all the rotundity and smoothness properties of C. It follows from C's convexity that $C + D = u + (1 + \alpha)C$, and consequently $C + D$ is closed. By Lemma 1.6, $C \gamma D$ is closed, for

$$C \gamma D = \bigcup_{\lambda \in [0,1]} ((1-\lambda)C + \lambda D) = \bigcup_{\lambda \in [0,1]} (\lambda u + (1 - \lambda + \alpha\lambda)C).$$

The rest of the proof is identical to that of Theorem 1.5. ∎

While Theorem 1.5 establishes the stability of certain geometric properties of bodies in reflexive spaces, our main results are Theorem 4.3 and Corollary 4.4, which are concerned with the nonstability of these properties in nonreflexive spaces. The tools for the constructions that establish this nonstability are developed in the next two sections.

2. Some properties of projective transformations.

A *projective transformation* is a mapping of the form

$$x \mapsto \frac{Ax + b}{c(x) + d},$$

where A is a bounded linear transformation, b is a vector, c is a continuous linear functional, and d is a scalar. We refer the reader to [Gr], [MS], and [Kl2] for the basic properties of such transformations. Here we use only the special projective transformations for which A is the identity transformation and $b = 0$. The following proposition lists the properties that are needed here. Properties (a), (b) and (c) are standard.

2.1. Proposition. *Suppose that β is a bornology on a normed linear space X, and $\varphi \in X^* \setminus \{0\}$. Let*

$$H_1 = \varphi^{-1}((-\infty, 1)) \quad \text{and} \quad H_{-1} = \varphi^{-1}((-1, +\infty)).$$

Then the transformation

$$\Phi : x \mapsto \frac{x}{1-\varphi(x)}$$

has the following properties:

(a) *Φ is a homeomorphism of H_1 onto H_{-1}, with the inverse*

$$\Phi^{-1} : \widetilde{x} \mapsto \frac{\widetilde{x}}{1+\varphi(\widetilde{x})}.$$

(b) *If $x, y, z \in H_1$ and $z \in (x, y)$, then $\Phi(z) \in (\Phi(x), \Phi(y))$. Consequently, Φ maps convex (resp. affine) sets into convex (resp. affine) sets.*

(c) *When restricted to an arbitrary bounded set A such that $\sup \varphi(A) < 1$ (resp. $\inf \varphi(A) > -1$), Φ (resp. Φ^{-1}) is a Lipschitz transformation.*

(d) *If F and G are closed hyperplanes in X with $0 \notin F \cap G$, then φ can be chosen in such a way that $\Phi(F \cap H_1)$ and $\Phi(G \cap H_1)$ are parallel and $F \cap G \subset \varphi^{-1}(1)$.*

(e) *If $C \subset H_1$ is a closed convex set with $0 \in \operatorname{int} C$, then $\Phi(C)$ is β-smooth and/or rotund if and only if C is.*

Proof. (a) is obvious.

(b) If $z = \lambda x + \mu y$, $\lambda > 0$, $\mu > 0$, and $\lambda + \mu = 1$, then a direct computation shows that $\Phi(z) = \overline{\lambda}\Phi(x) + \overline{\mu}\Phi(y)$, where

$$\overline{\lambda} = \frac{\lambda(1-\varphi(x))}{1-\varphi(z)} > 0, \quad \overline{\mu} = \frac{\mu(1-\varphi(y))}{1-\varphi(z)} > 0,$$

and $\overline{\lambda} + \overline{\mu} = 1$.

(c) Suppose that $A \subset X$, $s = \sup\{\|a\| \mid a \in A\} < \infty$, and $d = 1 - \sup \varphi(A) > 0$. Then for each $x, y \in A$ we have

$$\begin{aligned}\|\Phi(x) - \Phi(y)\| &= \left\| \left(\frac{x}{1-\varphi(x)} - \frac{x}{1-\varphi(y)} \right) + \left(\frac{x}{1-\varphi(y)} - \frac{y}{1-\varphi(y)} \right) \right\| \\ &\le \frac{\|x\| \cdot |\varphi(x) - \varphi(y)|}{(1-\varphi(x))(1-\varphi(y))} + \frac{\|x-y\|}{1-\varphi(x)} \le \left(\frac{s\|\varphi\|}{d^2} + \frac{1}{d} \right) \|x-y\|.\end{aligned}$$

(d) By (a) and (b), the sets $\widetilde{F} = \Phi(F \cap H_1)$ and $\widetilde{G} = \Phi(G \cap H_1)$ are closed hyperplanes (relative to H_{-1}). There exist $f, g \in X^*$ such that $F = f^{-1}(1)$ and $G = g^{-1}(1)$. Put $\varphi = \frac{1}{2}(f+g)$, $\psi = \frac{1}{2}(f-g)$. It is easy to see that $\widetilde{F} \subset \psi^{-1}(1)$ and $\widetilde{G} \subset \psi^{-1}(-1)$. Hence $\widetilde{F}$ and $\widetilde{G}$ are parallel. Moreover, $\varphi(F \cap G) = \{1\}$.

(e) The Minkowski functional p_C is everywhere finite, since $0 \in \operatorname{int} C$. It is easy to see that $C \subset H_1$ implies $\varphi(x) < p_C(x)$. Keeping this in mind, we can write (for any $x \in X$)

$$\begin{aligned} p_C(x) &= \inf\{t > 0 \mid x/t \in C\} = \inf\{t > 0 \mid \Phi(x/t) \in \Phi(C)\} \\ &= \inf \left\{ t > 0 \,\middle|\, \frac{x}{t-\varphi(x)} \in \Phi(C) \right\}. \end{aligned}$$

Substituting $s := t - \varphi(x)$, we can write the last expression as

$$\inf\{s + \varphi(x) \mid s > 0,\ x \in s\Phi(C)\} = \varphi(x) + p_{\Phi(C)}(x).$$

Thus $p_C = \varphi + p_{\Phi(C)}$. This implies that C is β-smooth if and only if $\Phi(C)$ is β-smooth. The equivalence of the rotundity of C and that of $\Phi(C)$ follows directly from (a) and (b). ■

3. Rotundity, smoothness, and the Extension Lemma

3.1. Convention. From now on, by a *smoothness property* we shall mean β-smoothness for a bornology β on X that is closed under linear automorphisms of X and under bounded linear projections of X onto closed hyperplanes in X. (Note that this condition is satisfied by all three of the bornologies mentioned in Remark 1.2(a).) Moreover, we shall say that a norm on X has a smoothness property, or is β-smooth, or is rotund, whenever its unit ball has this attribute.

3.2. Observation. Suppose that C is a body in X with $0 \in \operatorname{int} C$. If Y is a closed subspace of codimension one in X, then $C \cap Y$ is a body in Y that has all the smoothness properties of C, and is rotund whenever C is.

3.3. Lemma. *Suppose that Y is a nonreflexive normed linear space and $\psi \in Y^* \setminus \{0\}$. Then there exists an equivalent norm $\|\cdot\|$ on Y that preserves all the smoothness properties of the original norm, is rotund whenever the original norm is, and is such that ψ does not attain its supremum on the unit ball of $(Y, \|\cdot\|)$.*

Proof. By a theorem of James [Ja] (cf. also [Di]), there exists a norm-one functional $\varphi \in Y^*$ that does not attain its supremum on the unit ball B_Y of Y. Then $\operatorname{dist}(B_Y, \varphi^{-1}(1)) = 0$ and $B_Y \cap \varphi^{-1}(1) = \emptyset$. If φ is a scalar multiple of ψ, we can take $\|\cdot\|$ equal to the original norm of Y. If φ, ψ are linearly independent, there exist $u, v \in Y$ such that $\varphi(u) = \psi(v) = 1$ and $\varphi(v) = \psi(u) = 0$. Define $T : Y \to Y$ by $Ty = y + (\varphi(y) - \psi(y))(v - u)$. Then $T^2 = I$, so T is an automorphism of Y. Hence $T(B_Y)$ is the unit ball of an equivalent norm $\|\cdot\|$ on Y, and $\|\cdot\|$ is rotund and/or β-smooth whenever B_Y is. It is easy to see that T maps $\varphi^{-1}(0)$ onto $\psi^{-1}(0)$ and u onto v, and hence also $\varphi^{-1}(1)$ onto $\psi^{-1}(1)$. Consequently, $\operatorname{dist}(T(B_Y), \psi^{-1}(1)) \le \|T\| \operatorname{dist}(B_Y, \varphi^{-1}(1)) = 0$ and $T(B_Y) \cap \psi^{-1}(1) = T(B_Y \cap \varphi^{-1}(1)) = \emptyset$. In other words, ψ does not attain its supremum on $T(B_Y)$. ■

The construction used to prove the following key lemma is technically a bit involved. However, the underlying idea is geometrically very simple: strategic use of projective transformations in conjunction with an adaptation of the ℓ^q norming process. The reader may find it helpful first to review the statement of 2.1, and then, with the aid of an easily made diagram, to go quickly through the proof of 3.4 for the case in which the space X is 2-dimensional. Then reread the proof of 3.4 in detail.

3.4. Extension Lemma. *Suppose that F, G, and Y are three hyperplanes in a normed linear space X such that no two of them are parallel. With $M_1 = F \cap G$, suppose that $M_1 \subset Y$ and the origin belongs to $Y \setminus M_1$. Let $K \subset Y \setminus M_1$ be a body in Y that has the origin as an interior point, and let z be a point in F such that $K \curlyvee \{z\}$ does not intersect G. Then there exists a body $C \subset X$ such that $C \cap Y = K$, $C \cap F = \{z\}$, $C \cap G = \emptyset$, and C has the same smoothness properties as K and is rotund if K is. If, in addition, $K \curlyvee \{-z\}$ does not intersect G, then C can be chosen so that $-z \in \partial C$.*

Proof. Choose $\varphi \in X^*$ as in Proposition 2.1(d) relative to F and G from our hypotheses, and consider the related projective transformation Φ. Using the same notation as in the proof of 2.1(d), note that the set $\widetilde{Y} = \Phi(Y \cap H_1)$ lies between $\widetilde{F}$ and $\widetilde{G}$, and hence $\widetilde{Y}$ is parallel to $\widetilde{F}$ and $\widetilde{G}$. By Proposition 2.1(c), the set $\widetilde{K} = \Phi(K)$ is at least linearly bounded (bounded whenever K has positive distance from M_1). The Minkowski functional $p_{\widetilde{K}}$ of K is everywhere finite on Y.

For any point $\widetilde{v} \in H_{-1} \setminus \widetilde{Y}$ and any real $q > 1$ consider the "ℓ^q half-extension" $\widetilde{C}^q_{\widetilde{v}}$ of K, defined as follows:

$$\widetilde{C}^q_{\widetilde{v}} = \{w \in X \mid w = \widetilde{y} + t\widetilde{v},\ \widetilde{y} \in \widetilde{K},\ t > 0,\ p_{\widetilde{K}}(\widetilde{y})^q + t^q \le 1\}.$$

Now set $\widetilde{z} = \Phi(z)$, and fix $\alpha > 0$ such that $-\alpha\widetilde{z}$ lies in H_{-1} on the same side of $\widetilde{G}$ as $\widetilde{Y}$. Put

$$\widetilde{C}^q = \widetilde{C}^q_{\widetilde{z}} \cup \widetilde{C}^q_{-\alpha\widetilde{z}}.$$

The set $\widetilde{C}^q$ is of course closed, its convexity is easy to verify, and we shall show that it has the same smoothness properties as K and is rotund whenever K is.

Suppose that K is rotund (resp. β-smooth for some bornology β). Then by Proposition 2.1(e), the set $\widetilde{K}$ is also rotund (resp. β-smooth). The closed convex sets

$$\widetilde{C}^q_{\widetilde{z}} \cup \widetilde{C}^q_{-\widetilde{z}} \quad \text{and} \quad \widetilde{C}^q_{\alpha\widetilde{z}} \cup \widetilde{C}^q_{-\alpha\widetilde{z}}$$

are rotund by a standard argument (resp. β-smooth by Remark 1.2(d)), and it then follows from Remark 1.4 (resp. Remark 1.2(c)) that the set $\widetilde{C}^q$ is rotund (resp. β-smooth).

We claim that, for some $q > 1$, $\Phi^{-1}(\widetilde{C}^q)$ gives the desired extension of K. To prove this, note that the boundedness of K implies that $\widetilde{K}$ is bounded away from $\varphi^{-1}(-1)$, say

$$\widetilde{K} \cup \{\widetilde{z}\} \cup \{-\alpha\widetilde{z}\} \subset \varphi^{-1}([2\varepsilon - 1, +\infty))$$

for some $\varepsilon > 0$. In this situation, there exists $\overline{q} > 1$ such that

$$(*) \qquad \widetilde{C}^{\overline{q}} \subset \varphi^{-1}([\varepsilon - 1, +\infty)).$$

In fact, let $\delta > 0$ be such that $\varepsilon - 1 \le (2\varepsilon - 1)(1 + \delta)$ (observe that $2\varepsilon - 1 < 0$ because $\widetilde{K}$ contains the origin). Since the function $(\xi, t, q) \mapsto \xi^q + t^q$ is uniformly continuous on $[0,1]^2 \times [1,2]$, there exists $\overline{q} > 1$ such that

$$p_{\widetilde{K}}(\widetilde{y}) + t - p_{\widetilde{K}}(\widetilde{y})^{\overline{q}} - t^{\overline{q}} \le \delta \quad \text{for all } \widetilde{y} \in \widetilde{K} \text{ and } t \in [0,1].$$

This implies that

$$p_{\widetilde{K}}(\widetilde{y}) \le \delta + 1 - t$$

whenever $t \ge 0$, $\widetilde{y} \in \widetilde{K}$, and either $\widetilde{y} + t\widetilde{z}$ or $\widetilde{y} + t(-\alpha\widetilde{z})$ belongs to $\widetilde{C}^{\overline{q}}$.

From the fact that $\varphi(\widetilde{y}/(p_{\widetilde{K}}(\widetilde{y})) \ge 2\varepsilon - 1$ it follows that

$$\varphi(\widetilde{y}) \ge p_{\widetilde{K}}(\widetilde{y})(2\varepsilon - 1) \ge (1 + \delta - t)(2\varepsilon - 1),$$

and consequently, for $\widetilde{y} \in \widetilde{K}$ and $t > 0$, we have

$$\varphi(\widetilde{y} + t\widetilde{z}) \ge (1 + \delta - t)(2\varepsilon - 1) + t(2\varepsilon - 1) = (1 + \delta)(2\varepsilon - 1) \ge \varepsilon - 1$$

whenever $\widetilde{y} + t\widetilde{z} \in \widetilde{C}^{\overline{q}}$, and similarly $\varphi(\widetilde{y} + t(-\alpha\widetilde{z})) \ge \varepsilon - 1$ whenever $\widetilde{y} + t(-\alpha\widetilde{z}) \in \widetilde{C}^{\overline{q}}$. This means that $(*)$ holds.

Now, set $C := \Phi^{-1}(\widetilde{C}^{\overline{q}})$. By Proposition 2.1(a) and 2.1(b), the set C is closed, convex and has nonempty interior. By Proposition 2.1(e), C has the same smoothness properties as K and is rotund whenever K is. Moreover, it is clear that $C \cap Y = K$, $C \cap F = \{z\}$, and $C \cap G = \emptyset$, for the mapping Φ is bijective. It remains to show that the set C is bounded. For $\widetilde{y} \in \widetilde{K}$, $t \in \mathbb{R}$ and $\widetilde{y} + t\widetilde{z} \in \widetilde{C}^{\overline{q}}$, we have $-\alpha \le t \le 1$ and

$$\|\Phi^{-1}(\widetilde{y} + t\widetilde{z})\| \le \frac{1 + \varphi(\widetilde{y})}{1 + \varphi(\widetilde{y}) + t\varphi(\widetilde{z})} \cdot \frac{\|\widetilde{y}\|}{1 + \varphi(\widetilde{y})} + \frac{|t| \cdot \|\widetilde{z}\|}{1 + \varphi(\widetilde{y} + t\widetilde{z})}.$$

The right-hand side is bounded because $(*)$ holds,

$$\Phi^{-1}(\widetilde{y}) = \frac{\widetilde{y}}{1 + \varphi(\widetilde{y})} \in K, \quad \text{and} \quad \frac{1 + \eta}{1 + \eta + t\varphi(\widetilde{z})} \to 1 \quad \text{as } \eta \to +\infty.$$

Finally, the last claim in the statement is easily proved by taking $\alpha = (1 - \varphi(z))/(1 + \varphi(z))$ in the above construction. Indeed, in this case $\Phi(-z) = -\alpha\widetilde{z}$ (clearly, $\alpha > 0$, since $-\alpha\widetilde{z}$ and $\widetilde{z}$ lie on opposite sides of $\widetilde{Y}$). ■

4. Main results. In what follows, C^π denotes the polar of C, i.e.,

$$C^\pi = \{f \in X^* \mid \sup f(C) \le 1\}.$$

Our notions of smoothness properties are the ones that were defined in Convention 3.1.

4.1. THEOREM. *Let X be a nonreflexive normed linear space that admits an equivalent norm $\|\cdot\|$ with some smoothness properties. Let M be a closed linear subspace of codimension 2 in X with the quotient mapping $Q_M : X \to X/M$. Then there are two bodies C and D in X that have the following properties*:

- $0 \in \operatorname{int} C$;
- *C and D have the same smoothness properties as $\|\cdot\|$ and are rotund if $\|\cdot\|$ is*;
- *each of the bodies $C \mp D$, $C \overline{\gamma} D$, C^π, $\overline{Q_M(C)}$ is nonsmooth and nonrotund.*

Proof. Let Y be a closed subspace of X such that $\operatorname{codim} Y = 1$ and $M \subset Y$. Let $\psi \in Y^*$ be such that $M = \psi^{-1}(0)$. Since Y, in the induced norm, is a nonreflexive normed linear space, Lemma 3.3 guarantees the existence of an equivalent norm on Y such that the unit ball B_Y has all the smoothness properties of the original norm, is rotund if the original norm is, and ψ does not attain its supremum on B_Y. Obviously, we can suppose that $\sup \psi(B_Y) = 1$.

Set $M_1 = \psi^{-1}(1)$ and choose an arbitrary point $u \in M_1$. Consider the body $L \subset Y$ that is symmetric with B_Y relative to the line through the origin and u (the kernel of the symmetry being M), i.e.,

$$L = \{y \in Y = M \oplus \mathbb{R}u \mid y = -m + tu,\ m \in M,\ t \in \mathbb{R},\ m + tu \in B_Y\}.$$

Let f and g be two different linear extensions of ψ, and set $F = f^{-1}(1)$, $G = g^{-1}(1)$. Take any point $z \in F$ such that $g(z) < 1$. Observe that the sets B_Y and L are both contained in the set $g^{-1}((-\infty, 1))$, for they contain the origin and do not intersect M_1. Consequently,

$$(B_Y \mathbin{\gamma} L \mathbin{\gamma} \{z\}) \cap G = \emptyset.$$

We are now in a position to apply the Extension Lemma 3.4, using its notation, to construct two bodies $C \subset X$ and $D \subset X$, assuming respectively $K = B_Y$ and $K = L$.

Trivially, the bodies $\frac{1}{2}(C \mp D)$ and $C \overline{\gamma} D$ contain z and are contained in the "angle between F and G containing B_Y". Moreover, they contain the point u. Indeed, there is a sequence of points $y_n = m_n + t_n u \in B_Y$ (with $m_n \in M$ and $t_n \in \mathbb{R}$) such that $\operatorname{dist}(y_n, M_1) \to 0$; then $-m_n + t_n u \in L$, $t_n u \in \frac{1}{2}(C + D) \cap (C \mathbin{\gamma} D)$ and $t_n \to 1$.

Neither the body $\frac{1}{2}(C \mp D)$ nor the body $C \overline{\gamma} D$ can be rotund or smooth, since both bodies contain the segment $[u, z]$ and both are supported by F and G at u. The same argument easily shows that the two-dimensional body $\overline{Q_M(C)}$ in X/M cannot be rotund or smooth. Because of the finite-dimensional situation, by duality, the same is true for the polar $\overline{Q_M(C)}^\pi$ in the space $(X/M)^* = M^\pi$. But this polar is a two-dimensional section of C^π, and thus C^π is neither rotund nor smooth. ■

4.2. Remark. The argument used in the last proof shows that, in our nonreflexive situation, the set B_Y cannot be symmetric with respect to any line through the origin that is not in M, with M as the kernel of the symmetry.

It would be interesting also to prove, under the hypotheses of Theorem 4.1, that there exists an equivalent norm $|\cdot|$ on X, rotund if $\|\cdot\|$ is and with the same smoothness properties as those of $\|\cdot\|$, such that both the polar $|\cdot|_{X^*}$ and the quotient $|\cdot|_{X/M}$ are nonsmooth and nonrotund. In other words, we would like to "symmetrize" the set C. However, we are able only to prove the following result on equivalent norms (Theorem 4.3).

The existence of renormings as claimed in Theorem 4.3 was already known in some particular cases or examples. In [Kl1], Theorem 4.3(i) was proved for separable X and the usual smoothness; a particular case of this is [Tr]. M. Talagrand [Ta] proved that $C[0, \Omega]$ (where Ω is the first uncountable ordinal) admits an equivalent Fréchet smooth norm whose dual is not rotund. Moreover, note that any rotund renorming of ℓ^1 has nonsmooth dual since ℓ^∞ is known to admit no smooth equivalent norm ([Da], cf. also [Di]).

4.3. THEOREM. *Let $(X, \|\cdot\|)$ be a nonreflexive normed linear space and M a closed subspace of codimension two in X. Then there exist two equivalent norms $|\cdot|_{\mathrm{s}}$ and $|\cdot|_{\mathrm{r}}$ on X such that*:

(i) *$|\cdot|_{\mathrm{s}}$ has all the smoothness properties of $\|\cdot\|$, but $(X/M, |\cdot|_{\mathrm{s}})$ is not smooth and $(X^*, |\cdot|_{\mathrm{s}})$ is not rotund.*

(ii) *$|\cdot|_{\mathrm{r}}$ is rotund whenever $\|\cdot\|$ is, but $(X/M, |\cdot|_{\mathrm{r}})$ is not rotund and $(X^*, |\cdot|_{\mathrm{r}})$ is not smooth.*

Proof. Let Y be a closed hyperplane that contains M. Take $\psi \in Y^*$ so that $M = \psi^{-1}(0)$ and $\|\psi\| = 1$. By Lemma 3.3 we can suppose that ψ does not attain its norm in $(Y, \|\cdot\|)$; in other words, the set $M_1 = \psi^{-1}(1)$ does not intersect the unit ball of $(Y, \|\cdot\|)$. Let $f, g \in X^*$ be two different extensions of ψ. Set $F = f^{-1}(1)$, $G = g^{-1}(1)$, and $K = B_Y$ (the unit ball of $(Y, \|\cdot\|)$). It is easy to see that there exists a point $z \in F$ such that $-1 < g(z) < 1$. Now we have the same situation as in the Extension Lemma, with $G \cap (K \mathbin{\gamma} \{-z\}) = \emptyset$. Thus there exists a body $C \subset X$ that "extends"

K, has z and $-z$ as boundary points, is contained in the "angle between F and G which contains K", and is β-smooth and/or rotund whenever $\|\cdot\|$ is.

(i) Define $|\cdot|_s = \frac{1}{2}(p_C + p_{-C})$. Then $|\cdot|_s$ is an equivalent norm on X which coincides with $\|\cdot\|$ on Y and has all the smoothness properties of $\|\cdot\|$ (it is also rotund if $\|\cdot\|$ is). By the Hahn–Banach theorem there exists $h \in X^* \setminus \{0\}$ such that $M_1 \subset h^{-1}(1)$ and $-C \subset h^{-1}((-\infty,1])$ (note that $M_1 \cap (-C) = \emptyset$ since $C \cap Y = K$ is symmetric). Observe that $f \le p_C$, $g \le p_C$ and $h \le p_{-C}$, because

$$C \subset f^{-1}((-\infty,1]) \cap g^{-1}((-\infty,1]) \quad \text{and} \quad -C \subset h^{-1}((-\infty,1]).$$

Then the functionals $\widetilde{f} = \frac{1}{2}(f+h)$ and $\widetilde{g} = \frac{1}{2}(g+h)$ are distinct and $M_1 \subset \widetilde{f}^{-1}(1) \cup \widetilde{g}^{-1}(1)$. Moreover,

$$\widetilde{f} = \tfrac{1}{2}(f+h) \le \tfrac{1}{2}(p_C + p_{-C}) = |\cdot|_s$$

and, similarly, $\widetilde{g} \le |\cdot|_s$. Thus, if we set $S := \{x \in X \mid |x|_s \le 1\}$, then

$$S \subset \widetilde{f}^{-1}((-\infty,1]) \cap \widetilde{g}^{-1}((-\infty,1]).$$

As in the proof of Theorem 4.1, we conclude that the body $\overline{Q_M(S)}$ is not smooth. However, this body is the unit ball of the space $(X/M, |\cdot|_s)$. Consequently, $(X^*, |\cdot|_s)$ is not rotund since it contains $(X/M, |\cdot|_s)^* = M^\pi$ as a subspace.

(ii) It is easy to see that the set $R = C \cap (-C)$ is a symmetric body with the following properties:

- R is symmetric;
- R is rotund if C is;
- $R \cap Y = C \cap Y = K$;
- $R \subset f^{-1}((-\infty,1])$;
- $z \in R$.

Then $|\cdot|_r = p_R$ is an equivalent norm on X which is rotund if $\|\cdot\|$ is. As in the proof of Theorem 4.1, the unit ball of $(X/M, |\cdot|_r)$—which is equal to $\overline{Q_M(R)}$—is not rotund. Thus $(X^*, |\cdot|_r)$ is not smooth, since it contains $(X/M, |\cdot|_r)^*$ as a subspace. ∎

For the following, recall once more that *body* means *convex body*.

4.4. Corollary. *Suppose that X is a normed linear space, and β is a bornology on X that is closed under linear automorphisms and under bounded linear projections onto closed hyperplanes in X. Then the following assertions, concerning arbitrary bodies C, D and equivalent norms $|\cdot|$ in X, are equivalent:*

(i) *X is reflexive;*

(ii) *$C + D$ is always closed;*

(iii) *$C \gamma D$ is always closed;*

(iv) *X contains a rotund body, and $C \mp D$ is rotund whenever C and D are;*

(v) *X contains a β-smooth body, and $C \mp D$ is β-smooth whenever C or D is;*

(vi) *X contains a β-smooth body, and $C \,\overline{\gamma}\, D$ is β-smooth whenever C and D are;*

(vii) *X has an equivalent rotund norm, and $(X^*, |\cdot|)$ is smooth whenever $(X, |\cdot|)$ is rotund;*

(viii) *X has an equivalent β-smooth norm, and $(X^*, |\cdot|)$ is rotund whenever $(X, |\cdot|)$ is β-smooth.*

Proof. By well-known renorming results (cf. [Di]), each reflexive space admits an equivalent norm that is LUR and Fréchet smooth, hence also rotund and β-smooth. By this fact, Theorem 1.5, and the well-known duality between the smoothness of a reflexive space and the rotundity of its dual (cf. [Di]), (i) implies each of the other conditions (ii)–(viii).

The opposite implications follow from Theorems 4.1 and 4.3. For (ii) and (iii), note that the bodies in the proof of Theorem 4.1 are constructed in such a way that the point u does not belong to either of the sets $\frac{1}{2}(C+D)$ and $C \cup D$ but it does belong to the closure of each set. ∎

4.5. Remark. In the spirit of Convention 3.1, one could define a *rotundity property* as the usual rotundity or the property τ-LUR for a linear topology τ on X that is finer than the weak topology of X. It was our initial desire to extend our results from simple rotundity to the rotundity properties; in other words, we would have liked to obtain for rotundity properties what we got for smoothness properties.

It is possible to prove that "τ-LUR" is a τ-local property—that is, it holds at a point x of ∂C if and only if it holds relative to some τ-neighborhood of x, and, using this, it is possible to add the following to the statement of Proposition 2.1:

(f) *If C is as in (e), then the set $C \cap \varphi^{-1}((a,b))$ is bounded for all $-\infty < a < b < 1$ if and only if the set $\Phi(C) \cap \varphi^{-1}((c,d))$ is bounded for all $-1 < c < d < +\infty$. Moreover, if this boundedness condition is satisfied, then $\Phi(C)$ is τ-LUR if and only if C is τ-LUR.*

Unfortunately, if τ is finer than the weak topology, the ℓ^q-extension C produced in our Extension Lemma 3.4 is never τ-LUR if $\operatorname{dist}(K, M_1) = 0$, which is the only situation interesting for us. Indeed, it is sufficient to take points $y_n \in K$ such that $\operatorname{dist}(y_n, M_1) \to 0$. Then $\operatorname{dist}(z + y_n/2, \partial C) \to 0$ since $z \in F$ and

$$\operatorname{dist}\left(\frac{z+y_n}{2}, \partial C\right) \le \operatorname{dist}\left(\frac{z+y_n}{2}, F\right) \le \frac{1}{2}\operatorname{dist}(y_n, F) \le \frac{1}{2}\operatorname{dist}(y_n, M_1).$$

But the points y_n do not τ-converge to z since $z \notin Y$ and

$$\overline{\{y_n\}}^{\tau} \subset \overline{\{y_n\}}^{\text{weak}} \subset Y.$$

Thus we close with the following.

4.6. OPEN PROBLEM. *Let X be a nonreflexive space that contains a τ-LUR body for a linear topology τ finer than the weak topology. Does then X contain two τ-LUR bodies C, D such that $C \mp D$ is not rotund?*

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Operators preserving orthogonality of polynomials

by

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Abstract. Let S be a degree preserving linear operator of $\mathbb{R}[X]$ into itself. The question is if, preserving orthogonality of some orthogonal polynomial sequences, S must necessarily be an operator of composition with some affine function of $\mathbb{R}$. In [2] this problem was considered for S mapping sequences of Laguerre polynomials onto sequences of orthogonal polynomials. Here we improve substantially the theorems of [2] as well as disprove the conjecture proposed there. We also consider the same questions for polynomials orthogonal on the unit circle.

Introduction. Call $\{p_n\}_{n=0}^{\infty} \subset \mathcal{P}$ where $\mathcal{P}$ is either $\mathbb{R}[X]$ or $\mathbb{C}[Z]$ a *polynomial system* (for short: PS) if $\deg p_n = n$, $n = 0, 1, \ldots$ A PS which is orthogonal with respect to a positive measure is here referred to as OGPS; if it is orthonormal the abbrevation is ONPS.

1. Let $\alpha \in \mathbb{R}$. Then, setting

$$\binom{\alpha}{0} = 1, \qquad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!},$$

the (*generalized*) *Laguerre polynomials* $L_n^{(\alpha)}$, $n = 0, 1, \ldots$, are defined as usual by

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}, \qquad x \in \mathbb{R}.$$

They satisfy the three-term recurrence relation

$$XL_n^{(\alpha)} = -(n+1)L_{n+1}^{(\alpha)} + (2n+1+\alpha)L_n^{(\alpha)} - (n+\alpha)L_{n-1}^{(\alpha)},$$
$$L_{-1}^{(\alpha)} = 0, \qquad n = 0, 1, \ldots$$

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