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Divergence of the Bochner–Riesz means in the weighted Hardy spaces

by

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Dedicated to Professor S. Igari on the occasion of his sixtieth birthday

Abstract. We costruct functions in H^1_w ($w \in A_1$) whose Fourier integral expansions are almost everywhere non-summable with respect to the Bochner–Riesz means of the critical order.

1. Introduction. Let w be a non-negative locally integrable function on $\mathbb{R}^n$. We say that $w \in A_1$ if there exists a constant $c \geq 0$ such that $M(w)(x) \leq cw(x)$ a.e., where M denotes the Hardy–Littlewood maximal operator. Let f be a measurable function on $\mathbb{R}^n$. We say that $f \in L^1_w$ if $\|f\|_{L^1_w} = \|fw\|_1 < \infty$, where $\|\cdot\|_1$ denotes the ordinary L^1-norm. Let $\Phi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space) satisfy $\int \Phi = 1$. The *weighted Hardy space* H^1_w ($w \in A_1$) is the class of functions $f \in L^1_w$ such that

$$\|f\|_{H^1_w} = \int \sup_{\varepsilon>0} |\Phi_\varepsilon * f(x)| w(x)\, dx < \infty,$$

where $\Phi_\varepsilon(x) = \varepsilon^n \Phi(\varepsilon x)$. (For the weighted Hardy spaces H^p_w, $p > 0$, see [12].) When $w = 1$ (a constant function), the space H^1_w will be denoted simply by H^1.

Let

$$S^\delta_R(f)(x) = \int (1 - R^{-2}|\xi|^2)^\delta_+ \widehat{f}(\xi) e^{2\pi i x\xi}\, d\xi$$

be the Bochner–Riesz means of order δ on $\mathbb{R}^n$. In this note we assume $n \geq 2$. Put $S_R(f) = S_R^{(n-1)/2}(f)$.

The following result is due to Stein [9] (see also [1], [8]).

THEOREM A. *There exists an* $f \in H^1$ *such that*

$$\limsup_{R\to\infty} |S_R(f)(x)| = \infty \quad \textit{almost everywhere.}$$

We shall prove the following results.

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THEOREM 1. *We consider $S_m(f)$ for $m \in \mathbb{N}$ (the set of positive integers). Then there exists an $f \in H^1_w \cap L^1$ such that*

$$\limsup_{m\to\infty} |S_m(f)(x)| = \infty \quad \text{almost everywhere.}$$

THEOREM 2. *We can find a $g \in H^1_w \cap L^1$ such that $S_m(g)$ ($m \in \mathbb{N}$) diverges almost everywhere but*

$$\sup_{R>0} |S_R(g)(x)| < \infty \quad \text{almost everywhere.} \tag{1.1}$$

Recalling Kolmogorov's theorem and Marcinkiewicz's theorem on pointwise divergence of 1-dimensional Fourier series (see [14, Chap. VIII] and [3]), we note that Theorem A and Theorem 1 are analogues of Kolmogorov's theorem (unbounded divergence) and that Theorem 2 is a Bochner–Riesz means version of Marcinkiewicz's theorem (bounded divergence).

Remark 1. Let $\{R_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that $\inf_{j\ge1} R_{j+1}/R_j \ge q > 1$. Then it is known that the lacunary maximal function $\sup_j |S_{R_j}(f)|$ satisfies

$$\sup_{\lambda>0} \lambda w(\{x \in \mathbb{R}^n : \sup_j |S_{R_j}(f)(x)| > \lambda\}) \le c_w \|f\|_{H^1_w},$$

where $w(E) = \int_E w(x)\,dx$, $w \in A_1$ (see [7] and also [4], [5]), but by Theorem 1 or Theorem 2 we see that the maximal function $\sup_{m\in\mathbb{N}} |S_m(f)|$, and hence $\sup_{R>0} |S_R(f)|$, does not satisfy the same estimate.

Remark 2. Let $0 < p < 1$ and $\delta(p) = n/p - (n+1)/2$. Then we have

$$\sup_{\lambda>0} \lambda^p w(\{x \in \mathbb{R}^n : \sup_{R>0} |S_R^{\delta(p)}(f)(x)| > \lambda\}) \le c_{p,w} \|f\|^p_{H^p_w}$$

for $f \in H^p_w \cap \mathcal{S}$, $w \in A_1$. (See [6] and, for the case $w = 1$, [10].)

Theorems 1 and 2 are immediate consequences of results for more general weights.

DEFINITION. Let w be a non-negative locally integrable function on $\mathbb{R}^n$ such that $M(w) < \infty$ a.e. Suppose $f \in L^1(\mathbb{R}^n)$. We say that $f \in \mathcal{H}_w$ if

$$\|f\|_{\mathcal{H}_w} = \|f\|_{L^1_w} + \sum_{j=1}^{n} \|R_j(f)\|_{L^1_w} < \infty,$$

where the operators R_j are the Riesz transforms: $(R_j(f))^\wedge(\xi) = i|\xi|^{-1}\xi_j \widehat{f}(\xi)$ ($f \in L^1 \cap L^2$).

We shall prove the following.

THEOREM 3. *We can find an $f \in \mathcal{H}_w$ such that*

$$\limsup_{m\to\infty} |S_m(f)(x)| = \infty \quad (m \in \mathbb{N}) \quad \text{almost everywhere.}$$

THEOREM 4. *There exists a $g \in \mathcal{H}_w$ such that $S_m(g)$ ($m \in \mathbb{N}$) diverges almost everywhere but*

$$\sup_{R>0} |S_R(g)(x)| < \infty \quad \text{almost everywhere.} \tag{1.2}$$

If $w \in A_1$, then $M(w) < \infty$ a.e. and we have the characterization of the space H^1_w in terms of the Riesz transforms (see [13] and also [12]); so Theorems 1 and 2 immediately follow from Theorems 3 and 4, respectively.

Let Q_k, for integers k, be cubes in $\mathbb{R}^n$ defined by $Q_k = [-2^k, 2^k)^n$. To prove Theorems 3 and 4, we shall use the following lemma.

LEMMA 1. *For $k, N \in \mathbb{N}$, we can find positive numbers t_k, $M_N^{(k)}$, L_N, functions $f_N^{(k)} \in \mathcal{H}_w \cap \mathcal{S}$ and measurable sets $E_N^{(k)}, F_N^{(k)} \subset Q_k$ such that*

(1) $M_N^{(k)}/8 \in \mathbb{N}$, $\lim_{N\to\infty} M_N^{(k)} = \infty$;
(2) $\lim_{N\to\infty} L_N = \infty$;
(3) $\|f_N^{(k)}\|_1 \le c2^{kn}$ *and* $\|f_N^{(k)}\|_{\mathcal{H}_w} \le ct_k 2^{kn}$;
(4) $\operatorname{supp}(\widehat{f}_N^{(k)}) \subset \{M_N^{(k)}/8 \le |\xi| \le 3M_N^{(k)}\}$;
(5) $\sup_{M_N^{(k)} \le m \le 2M_N^{(k)}} |S_m(f_N^{(k)})(x)| \ge \gamma_1 L_N$ ($m \in \mathbb{N}$) *for all* $x \in E_N^{(k)}$, *for some constant* $\gamma_1 > 0$;
(6) $\sup_{0<R\le 20M_N^{(k)}} |S_R(f_N^{(k)})(x)| \le \gamma_2 L_N$ *for all* $x \in F_N^{(k)}$, *for some constant* $\gamma_2 > 0$;
(7) $|Q_k \setminus E_N^{(k)}| \le c2^{kn}/L_N + 2^{-k}$ *and* $|Q_k \setminus F_N^{(k)}| \le c2^{kn}/L_N$.

Assuming Lemma 1, which will be proved in Sections 4–6, we shall prove Theorems 3 and 4 in Sections 2 and 3, respectively. To prove the principal part of Lemma 1, we shall use the techniques of Stein [9]; however, we need some modifications.

2. Proof of Theorem 3. Let $t_k, M_N^{(k)}, L_N$ and $f_N^{(k)}$ be as in Lemma 1. We select a sequence $\{N_k\}_{k=1}^{\infty}$ of positive integers satisfying the following conditions:

$$\sum_{k=1}^{\infty} J_k^{-1/2} 2^{kn}(t_k+1) < \infty, \quad \text{where} \quad J_k = L_{N_k}; \tag{2.1}$$

$$2^{kn}/J_k \le 2^{-k}; \tag{2.2}$$

$$B_{k+1}/8 > 3B_k, \quad \text{where} \quad B_k = M_{N_k}^{(k)}; \tag{2.3}$$

$$\sup_{R>B_{k+1}} \Big\| S_R\Big(\sum_{i=1}^{k} J_i^{-1/2} h_i\Big) - \sum_{i=1}^{k} J_i^{-1/2} h_i \Big\|_\infty \le 1, \quad \text{where} \quad h_i = f_{N_i}^{(i)}. \tag{2.4}$$

We note that (2.4) is feasible since $S_R(f)$ converges uniformly on $\mathbb{R}^n$ if $f \in \mathcal{S}$.

Put $f=\sum_{i=1}^{\infty} J_i^{-1/2}h_i$. Then by Lemma 1(3) and (2.1) we see that $f\in\mathcal{H}_w$. Put $I_k=[B_k,2B_k]\cap\mathbb{N}$. Then for $k\ge 2$ we have

$$\sup_{m\in I_k}|S_m(f)(x)|=\sup_{m\in I_k}\Big|S_m\Big(\sum_{i=1}^{k-1}J_i^{-1/2}h_i\Big)(x)+S_m(J_k^{-1/2}h_k)(x)\Big|,$$

since $S_m(\sum_{i=k+1}^{\infty}J_i^{-1/2}h_i)=0$ for $m\in I_k$ by Lemma 1(4) and (2.3). By (2.4) the right hand side is greater than or equal to

$$\sup_{m\in I_k}|S_m(J_k^{-1/2}h_k)(x)|-\Big|\sum_{i=1}^{k-1}J_i^{-1/2}h_i(x)\Big|-1.$$

Thus, by Lemma 1(5) we see that

$$\sup_{m\in I_k}|S_m(f)(x)|\ge\gamma_1J_k^{1/2}-\Big|\sum_{i=1}^{k-1}J_i^{-1/2}h_i(x)\Big|-1\quad\text{for } x\in E_{N_k}^{(k)}.$$

Note that there exists a measurable set $F\subset\mathbb{R}^n$ such that $|\mathbb{R}^n\setminus F|=0$ and

$$\sup_{k\ge2}\Big|\sum_{i=1}^{k-1}J_i^{-1/2}h_i(x)\Big|\le c_x<\infty\quad\text{for } x\in F.$$

Put $E=\limsup_{k\to\infty}E_{N_k}^{(k)}$. Then we have

$$\sup_{m\in I_k}|S_m(f)(x)|\ge\gamma_1J_k^{1/2}-c_x-1\quad\text{for } x\in E\cap F$$

for infinitely many values of k. This implies

$$\limsup_{m\to\infty}|S_m(f)(x)|=\infty\quad(m\in\mathbb{N})\quad\text{for } x\in E\cap F.$$

Thus, the proof of Theorem 3 will be finished if we prove $|\mathbb{R}^n\setminus E|=0$. Put $D=\liminf_{k\to\infty}E_{N_k}^{(k)}$. For the sake of the proof of Theorem 4, we prove a stronger assertion:

$$|\mathbb{R}^n\setminus D|=0. \tag{2.5}$$

To prove (2.5) it is sufficient to show $|Q_k\setminus D|=0$ for all k. Let $m\ge k$. By Lemma 1(7) and (2.2) we see that

$$\begin{aligned}|Q_k\setminus D|&=\Big|Q_k\setminus\bigcup_{j=1}^{\infty}\bigcap_{i=j}^{\infty}E_{N_i}^{(i)}\Big|=\Big|Q_k\cap\Big(\bigcap_{j=1}^{\infty}\bigcup_{i=j}^{\infty}\complement E_{N_i}^{(i)}\Big)\Big|\\&\le\Big|\bigcup_{i=m}^{\infty}\Big(Q_i\cap\complement E_{N_i}^{(i)}\Big)\Big|\le c\sum_{i=m}^{\infty}2^{-i}\le c2^{-m}.\end{aligned}$$

Letting $m\to\infty$, we have $|Q_k\setminus D|=0$. This completes the proof of Theorem 3.

3. Proof of Theorem 4. We choose a sequence $\{N_k\}_{k=1}^{\infty}$ of positive integers satisfying (2.2), (2.3) and

$$\sum_{k=1}^{\infty}J_k^{-1}2^{kn}(t_k+1)<\infty, \tag{3.1}$$

$$\sup_{R>B_{k+1}/8}\Big\|S_R\Big(\sum_{i=1}^{k}J_i^{-1}h_i\Big)-\sum_{i=1}^{k}J_i^{-1}h_i\Big\|_\infty\le\gamma_1/4, \tag{3.2}$$

where we have used the same notation as in §2.

Define $g=\sum_{i=1}^{\infty}J_i^{-1}h_i$. Then $g\in\mathcal{H}_w$ by Lemma 1(3) and (3.1). Put

$$F_1=\Big\{x\in\mathbb{R}^n:\sup_{k\ge1}\Big|\sum_{i=1}^{k}J_i^{-1}h_i(x)\Big|<\infty\Big\},$$

$$F_2=\{x\in\mathbb{R}^n:J_i^{-1}h_i(x)\to0\ (i\to\infty)\}$$

and $D=\liminf_{k\to\infty}F_{N_k}^{(k)}$, $E=\limsup_{k\to\infty}E_{N_k}^{(k)}$. We note that $|\mathbb{R}^n\setminus F_i|=0$ $(i=1,2)$, $|\mathbb{R}^n\setminus D|=0$ and $|\mathbb{R}^n\setminus E|=0$ (see the proof of (2.5)).

Let $\widehat{\eta}\in C_0^\infty$ be such that

$$\operatorname{supp}(\widehat{\eta})\subset\{|\xi|\le1/2\},\quad\widehat{\eta}(\xi)=(1-|\xi|^2)^{(n-1)/2}\quad\text{if } |\xi|\le1/4.$$

Then by Lemma 1(4) we have

$$S_R(J_k^{-1}h_k)=J_k^{-1}h_k*\eta_R\quad\text{for } R\ge20B_k. \tag{3.3}$$

Here we recall that $\eta_R(x)=R^n\eta(Rx)$.

If $x\in D\cap F_1$, by (4), (6) of Lemma 1, (2.3), (3.2) and (3.3) we see that

$$\begin{aligned}&\sup_{B_k\le R\le B_{k+1}}|S_R(g)(x)|\\&=\sup_{B_k\le R\le B_{k+1}}\Big|S_R\Big(\sum_{i=1}^{k-1}J_i^{-1}h_i\Big)(x)+S_R(J_k^{-1}h_k)(x)+S_R(J_{k+1}^{-1}h_{k+1})(x)\Big|\\&\le\gamma_1/4+\Big|\sum_{i=1}^{k-1}J_i^{-1}h_i(x)\Big|+\sup_{B_k\le R\le20B_k}|S_R(J_k^{-1}h_k)(x)|\\&\quad+cM(J_k^{-1}h_k)(x)+\sup_{B_{k+1}/8\le R\le B_{k+1}}|S_R(J_{k+1}^{-1}h_{k+1})(x)|\\&\le c_x+cM\Big(\sum_{i=1}^{\infty}J_i^{-1}|h_i|\Big)(x)\end{aligned}$$

for some $c_x > 0$ independent of $k \ge 2$, since x is contained in $F_{N_k}^{(k)}$ for all but a finite number of values of k. Therefore $\sup_{R>0} |S_R(g)(x)| < \infty$ a.e., since $\sum J_i^{-1}|h_i| \in L^1$ and the maximal operator M is of weak type $(1,1)$.

On the other hand, by (4), (5) of Lemma 1, (2.3) and (3.2), setting $I_k = [B_k, 2B_k] \cap \mathbb{N}$, for $x \in E \cap F_2$ we have

$$\begin{aligned}
\sup_{m\in I_k} &|S_m(g)(x) - S_{B_{k+1}/8}(g)(x)| \\
&\ge \sup_{m\in I_k} |S_m(J_k^{-1}h_k)(x)| \\
&\quad - \sup_{m\in I_k} \Big|S_m\Big(\sum_{i=1}^{k-1} J_i^{-1}h_i\Big)(x) - \sum_{i=1}^{k-1} J_i^{-1}h_i(x)\Big| \\
&\quad - J_k^{-1}|h_k(x)| - \Big|S_{B_{k+1}/8}\Big(\sum_{i=1}^{k} J_i^{-1}h_i\Big)(x) - \sum_{i=1}^{k} J_i^{-1}h_i(x)\Big| \\
&\ge \gamma_1 - \gamma_1/4 - J_k^{-1}|h_k(x)| - \gamma_1/4 \\
&= \gamma_1/2 - J_k^{-1}|h_k(x)|
\end{aligned}$$

for infinitely many values of k. From this we conclude that $S_m(g)(x)$ diverges for $x \in E \cap F_2$, since $J_k^{-1}|h_k(x)| \to 0$ $(k \to \infty)$ for $x \in F_2$. This completes the proof of Theorem 4.

4. Proof of Lemma 1 (part 1). In this section we construct the basic measure supported on Q_k and prove a key estimate for it (see (4.12)). Kronecker's theorem will be used in the proof (see [3], [8], [9]).

Decompose

$$[-2^k, 2^k) = \bigcup_{i=0}^{N-1} [-2^k + i2^{k+1}/N, -2^k + (i+1)2^{k+1}/N) = \bigcup_{i=0}^{N-1} I_i^{(k)}, \quad \text{say,}$$

and consider a partition:

$$Q_k = \bigcup_{(i_1,\dots,i_n)\in\{0,1,\dots,N-1\}^n} I_{i_1}^{(k)} \times \dots \times I_{i_n}^{(k)} = \bigcup_{i=1}^{N^n} Q_i^{(k)}, \tag{4.1}$$

where $\{Q_i^{(k)}\}_{i=1}^{N^n}$ is an enumeration of the family $\{I_{i_1}^{(k)} \times \dots \times I_{i_n}^{(k)}\}$ of cubes.

Let

$$F_k = \{x \in Q_k : M(w)(x) > t_k\}, \qquad G_k = Q_k \setminus F_k,$$

where $t_k > 0$ will be determined in the sequel.

If $x \in Q_k$ and $0 < 2s \le n^{1/2}2^k$, then we see that

$$\begin{aligned}
s^{-n} \int_{s<|x-y|<2s} \chi_{G_k}(y)\,dy
&= s^{-n} \int_{s<|x-y|<2s} \chi_{Q_k}(y)\,dy - s^{-n} \int_{s<|x-y|<2s} \chi_{F_k}(y)\,dy \\
&\ge c_1 - c_2 M(\chi_{F_k})(x),
\end{aligned} \tag{4.2}$$

where c_1, c_2 are positive constants depending only on the dimension.

For $\varepsilon \in (0,1)$, put

$$F_k^* = \{x \in Q_k : M(\chi_{F_k})(x) > \varepsilon\}, \qquad G_k^* = Q_k \setminus F_k^*.$$

If $x \in G_k^*$, $0 < 2s \le n^{1/2}2^k$ and if ε is small enough, then by (4.2) we have

$$s^{-n} \int_{s<|x-y|<2s} \chi_{G_k}(y)\,dy \ge c_1 - c_2\varepsilon \ge c_1/2. \tag{4.3}$$

Since $M(w) < \infty$ a.e., we can find t_k large enough so that

$$|F_k^*| \le c\varepsilon^{-1}|F_k| \le 2^{-k}. \tag{4.4}$$

Define a set of indices

$$\mathcal{I}_k = \{i : Q_i^{(k)} \cap G_k \ne \emptyset\}.$$

For each $i \in \mathcal{I}_k$, we take (and fix) $a_i \in Q_i^{(k)} \cap G_k$. Then we have

$$M(w)(a_i) \le t_k. \tag{4.5}$$

We can find a set $E_0 \subset Q_k$ such that $|Q_k \setminus E_0| = 0$ and for each $x \in E_0$ the numbers $|x - a_i|$ $(i \in \mathcal{I}_k)$ and 1 are linearly independent over the rationals (see [1] and [11, Chap. VII]).

We use Kronecker's theorem in the following form.

LEMMA 2. *Let real numbers $\theta_1, \dots, \theta_s, 1$ be linearly independent over the rationals. Let δ, ω be positive numbers. Then there exists a positive number M depending only on $\delta, \omega, \theta_1, \dots, \theta_s$ such that for any real numbers $\alpha_1, \dots, \alpha_s$ we can find integers $\ell, p_1, \dots, p_s$, depending on $\delta, \omega, \theta_1, \dots, \theta_s, \alpha_1, \dots, \alpha_s$, so that*

(1) $\omega < \ell \le M$;
(2) $|\ell\theta_j - p_j - \alpha_j| < \delta$ $(j = 1, \dots, s)$.

Here we show how we can take M independent of $\alpha_1, \dots, \alpha_s$; except for this assertion, Lemma 2 follows from Hardy–Wright [2, Theorem 442]. First, we can assume that $\alpha_j \in [0,1)$, $j = 1, \dots, s$. Let $Q = [0,1)^s$. Decompose $Q = \bigcup_i R_i$, where $R_i = Q_i^{(-1)} + (1/2, \dots, 1/2)$ with $1/N < \delta/2$ (see (4.1) with $n = s$). Take and fix $\alpha^{(i)} = (\alpha_1^{(i)}, \dots, \alpha_s^{(i)}) \in R_i$ for each i. For each $\alpha^{(i)}$, by [2, Theorem 442] we can find integers $\ell_i, p_1^{(i)}, \dots, p_s^{(i)}$ such that $\ell_i > \omega$

and $|\ell_i\theta_j - p_j^{(i)} - \alpha_j^{(i)}| < \delta/2 \quad (j=1,\dots,s)$. For any $\alpha \in (\alpha_1,\dots,\alpha_s) \in Q$, take R_{i_0} such that $\alpha \in R_{i_0}$. Then

$$|\ell_{i_0}\theta_j - p_j^{(i_0)} - \alpha_j| \le |\ell_{i_0}\theta_j - p_j^{(i_0)} - \alpha_j^{(i_0)}| + |\alpha_j^{(i_0)} - \alpha_j| < \delta/2 + 1/N < \delta.$$

Thus, we can take $M = \max_i \ell_i$.

Now we return to the proof of Lemma 1. Let $x \in E_0$. Then by Lemma 2 there exists an $M(x) > 0$ such that for any real numbers β_i $(i \in \mathcal{I}_k)$, we can find integers m and p_i $(i \in \mathcal{I}_k)$, depending on x and β_i $(i \in \mathcal{I}_k)$, so that

$$H_x < m \le M(x), \quad \text{where} \quad H_x = \sup_{i\in\mathcal{I}_k} |x-a_i|^{-1}; \tag{4.6}$$

$$|m|x-a_i| - p_i - \beta_i| < 10^{-10} \quad \text{for all } i \in \mathcal{I}_k. \tag{4.7}$$

We assume as we may that $M(x)$ is a measurable function on E_0. Take M_0 so that $M_0/8 \in \mathbb{N}$ and

$$|\{x \in E_0 : M(x) > M_0\}| \le 1/N. \tag{4.8}$$

Put $E_1 = \{x \in E_0 : M(x) \le M_0\}$.

Let $x \in E_1$. By the substitution $\beta_i = -M_0|x-a_i| + n/4$ in (4.7) we have

$$H_x < m \le M_0; \tag{4.9}$$

$$|(m+M_0)|x-a_i| - n/4 - p_i| < 10^{-10} \quad (i \in \mathcal{I}_k) \tag{4.10}$$

for some integers m, p_i. Define the measure μ by

$$\mu = 2^{kn}N^{-n}\sum_{i\in\mathcal{I}_k}\delta_{a_i},$$

where δ_{a_i} denotes the Dirac δ measure concentrated at a_i. Put

$$D_R(y) = |y|^{-n}\cos(2\pi R|y| - n\pi/2) \quad (y \in \mathbb{R}^n).$$

Then by (4.9) and (4.10) we have

$$\begin{aligned}
&\sup_{M_0+H_x\le m\le 2M_0,\, m\in\mathbb{N}} |D_m * \mu(x)| \\
&= \sup_{M_0+H_x\le m\le 2M_0,\, m\in\mathbb{N}} 2^{kn}N^{-n}\Big|\sum_{i\in\mathcal{I}_k}\cos(2\pi m|x-a_i| - n\pi/2)|x-a_i|^{-n}\Big| \\
&\ge 2^{-1}2^{kn}N^{-n}\sum_{i\in\mathcal{I}_k}|x-a_i|^{-n} = I, \quad \text{say.}
\end{aligned}$$

Suppose $x \in E_1 \cap G_k^*$. Let $\mathcal{I}_k(x)$ denote the set of those $i \in \mathcal{I}_k$ for which we have $Q_i^{(k)} \cap \{y : |y-x| > n^{1/2}2^{k+2}N^{-1}\} \ne \emptyset$. Note that if $i \in \mathcal{I}_k(x)$ and $y \in Q_i^{(k)}$, then $|x-a_i| \sim |x-y|$. Thus

$$\begin{aligned}
I &\ge 2^{-1}2^{kn}N^{-n}\sum_{i\in\mathcal{I}_k(x)}|x-a_i|^{-n} \\
&\ge c\sum_{i\in\mathcal{I}_k(x)}\int_{Q_i^{(k)}}|x-y|^{-n}\,dy = II, \quad \text{say.}
\end{aligned}$$

Next, note that $G_k \subset \bigcup_{i\in\mathcal{I}_k} Q_i^{(k)}$. So we have

$$G_k \cap \{y : |y-x| > n^{1/2}2^{k+2}N^{-1}\} \subset \bigcup_{i\in\mathcal{I}_k(x)} Q_i^{(k)}. \tag{4.11}$$

Put

$$A_\ell(x) = \{y : n^{1/2}2^{\ell+1}2^{k+2}N^{-1} > |y-x| > n^{1/2}2^\ell 2^{k+2}N^{-1}\} \quad (\ell \ge 0).$$

Then, if $N \ge 8$, by (4.3) and (4.11) we see that

$$\begin{aligned}
II &\ge c\int_{|y-x|>n^{1/2}2^{k+2}N^{-1}} \chi_{G_k}(y)|x-y|^{-n}\,dy \\
&\ge c\sum_{2^\ell\le N/8}\int_{A_\ell(x)}\chi_{G_k}(y)|x-y|^{-n}\,dy \\
&\ge c\sum_{2^\ell\le N/8} 2^{-\ell n}2^{-kn}N^n\int_{A_\ell(x)}\chi_{G_k}(y)\,dy \ge c\log N.
\end{aligned}$$

We have thus proved

$$\sup_{M_0\le m\le 2M_0,\, m\in\mathbb{N}} |D_m * \mu(x)| \ge c\log N \quad \text{if } x \in E_1 \cap G_k^* \text{ and } N \ge 8. \tag{4.12}$$

5. Proof of Lemma 1 (part 2). In this section we introduce the functions $f_N^{(k)}$ and we deal with Lemma 1(5).

Let $\widehat{\varphi}, \widehat{\psi} \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\operatorname{supp}(\widehat{\varphi}) \subset \{1/8 \le |\xi| \le 3\}, \quad \operatorname{supp}(\widehat{\psi}) \subset \{|\xi| \le 1/4\},$$
$$\widehat{\varphi}(\xi) + \widehat{\psi}(\xi) = 1 \quad \text{if } |\xi| \le 2.$$

Note that if $M \le R \le 2M$, then

$$\begin{aligned}
&(1 - R^{-2}|\xi|^2)_+^{(n-1)/2} \\
&\quad = (1 - R^{-2}|\xi|^2)_+^{(n-1)/2}\widehat{\varphi}(M^{-1}\xi) + (1 - R^{-2}|\xi|^2)_+^{(n-1)/2}\widehat{\psi}(M^{-1}\xi) \\
&\quad = (1 - R^{-2}|\xi|^2)_+^{(n-1)/2}\widehat{\varphi}(M^{-1}\xi) + \widehat{\eta}(R^{-1}\xi)\widehat{\psi}(M^{-1}\xi),
\end{aligned}$$

where η is as in §3. Consequently, we have

$$K_R(x) = K_R * \varphi_M(x) + \eta_R * \psi_M(x), \tag{5.1}$$

where $K(x) = \int(1-|\xi|^2)_+^{(n-1)/2}e^{2\pi i x\xi}\,d\xi$.

By a well-known property of the Bessel functions J_ν (see, e.g., [11]) we see that

$$(5.2)\quad K(x) = \pi^{-(n-1)/2}\Gamma((n+1)/2)|x|^{-n+1/2}J_{n-1/2}(2\pi|x|) = \pi^{-(n+1)/2}\Gamma((n+1)/2)|x|^{-n}\cos(2\pi|x| - n\pi/2) + r(x),$$

where $|r(x)| \le c(1+|x|)^{-n-1}$ if $|x| \ge 1$.

Let the measure μ and the function D_R be as in §4. Then, if $M \le R \le 2M$, by (5.1) and (5.2) we have

$$(5.3)\quad K_R * \varphi_M * \mu + \eta_R * \psi_M * \mu = \pi^{-(n+1)/2}\Gamma((n+1)/2)D_R * \mu + r_R * \mu.$$

Let the positive integer M_0 (depending on k, N and w) be as in §4 (see (4.8)). Put

$$f_N^{(k)}(x) = \varphi_{M_0} * \mu(x) = 2^{kn}N^{-n}\sum_{i\in\mathcal{I}_k}\varphi_{M_0}(x-a_i),$$
$$g_N^{(k)}(x) = \psi_{M_0} * \mu(x) = 2^{kn}N^{-n}\sum_{i\in\mathcal{I}_k}\psi_{M_0}(x-a_i).$$

Then by (4.5) we see that

$$\int |f_N^{(k)}(x)|w(x)\,dx \le 2^{kn}N^{-n}\sum_{i\in\mathcal{I}_k}\int|\varphi_{M_0}(x-a_i)|w(x)\,dx \le c2^{kn}N^{-n}\sum_{i\in\mathcal{I}_k}M(w)(a_i) \le c2^{kn}t_k.$$

Similarly we have

$$\int |R_j(f_N^{(k)})(x)|w(x)\,dx \le c2^{kn}t_k,$$

since

$$R_j(f_N^{(k)})(x) = 2^{kn}N^{-n}\sum_{i\in\mathcal{I}_k}\Phi_{M_0}^{(j)}(x-a_i)$$

for some $\Phi^{(j)} \in \mathcal{S}$. Therefore, we see that $f_N^{(k)} \in \mathcal{H}_w$ and $\|f_N^{(k)}\|_{\mathcal{H}_w} \le c2^{kn}t_k$.

Let the measurable set E_1 be as in §4. Then, if $M_0 \le R \le 2M_0$ and $x \in E_1$, by (5.3) we have

$$K_R * f_N^{(k)}(x) = \pi^{-(n+1)/2}\Gamma((n+1)/2)D_R * \mu(x) - \eta_R * g_N^{(k)}(x) + \zeta_R * \mu(x),$$

where $\zeta_R(x) = R^n b(Rx)(1+R|x|)^{-n-1}$ with some $b \in L^\infty$, since $R|x-a_i| \ge M_0|x-a_i| \ge 1$ for $x \in E_1$ (see (4.9)). Thus, for $x \in E_1$ we have

$$(5.4)\quad \sup_{M_0\le m\le 2M_0,\, m\in\mathbb{N}}|K_m * f_N^{(k)}(x)| \ge \pi^{-(n+1)/2}\Gamma((n+1)/2)\sup_{M_0\le m\le 2M_0,\, m\in\mathbb{N}}|D_m * \mu(x)| - cM(g_N^{(k)})(x) - cM(\mu)(x),$$

where $M(\mu)$ is the Hardy–Littlewood maximal function for a measure: $M(\mu)(x) = \sup_{R>0}R^{-n}\mu(B(x,R))$, $B(x,R) = \{y : |x-y| < R\}$.

Put

$$E_2 = E_1 \setminus (\{x : M(g_N^{(k)})(x) > \varepsilon\log N\} \cup \{x : M(\mu)(x) > \varepsilon\log N\}),$$

where ε is a small positive number which will be determined in a moment. By (4.8) and the weak type boundedness of M we have

$$(5.5)\quad |Q_k \setminus E_2| \le |Q_k \setminus E_1| + |E_1 \setminus E_2| \le 1/N + c(\|g_N^{(k)}\|_1 + \|\mu\|)/\log N \le 1/N + c2^{kn}/\log N \le c2^{kn}/\log N \quad (N \ge 2),$$

where $\|\mu\|$ denotes the total mass norm for a measure.

Put $E_N^{(k)} = E_2 \cap G_k^*$. If $x \in E_N^{(k)}$ and if ε is small enough, then by (4.12) and (5.4) we see that

$$(5.6)\quad \sup_{M_0\le m\le 2M_0,\, m\in\mathbb{N}}|K_m * f_N^{(k)}(x)| \ge c_0\log N - c\varepsilon\log N \ge (c_0/2)\log N$$

for $N \ge 8$ with some $c_0 > 0$. On the other hand, by (4.4) and (5.5) we have

$$(5.7)\quad |Q_k \setminus E_N^{(k)}| \le |Q_k \setminus E_2| + |Q_k \setminus G_k^*| \le c2^{kn}/\log N + 2^{-k} \quad (N \ge 2).$$

6. Proof of Lemma 1 (part 3). In this section we deal with Lemma 1(6), and then we complete the proof of Lemma 1.

Suppose $N \ge 2$. Put

$$F_N^{(k)} = \bigcap_{i\in\mathcal{I}_k}\{x \in Q_k : |x-a_i| \ge 2^kN^{-1}(\log N)^{-1/n}\}.$$

Then

$$(6.1)\quad |Q_k \setminus F_N^{(k)}| \le c2^{kn}/\log N.$$

We take M_0 large enough, keeping (4.8) and the property $M_0/8 \in \mathbb{N}$, and show that

$$(6.2)\quad \sup_{0<R\le 20M_0}|S_R(f_N^{(k)})(x)| \le c\log N \quad \text{for } x \in F_N^{(k)}.$$

Fix $x \in Q_{i_0}^{(k)} \cap F_N^{(k)}$. Put

$$\mathcal{I}_k^{(1)} = \{i \in \mathcal{I}_k : d(Q_i^{(k)}, Q_{i_0}^{(k)}) \le 2^k/N\},$$

where $d(A,B)$ denotes the distance between A and B. Note that the number of elements of $\mathcal{I}_k^{(1)}$ is less than a fixed number depending only on the dimension.

Decompose

$$\begin{aligned}
2^{kn}N^{-n}\int \varphi_{M_0}(y)K_R(x-a_i-y)\,dy
&= 2^{kn}N^{-n}\int_{|y|\le N^{-2}} \varphi_{M_0}(y)K_R(x-a_i-y)\,dy\\
&\quad + 2^{kn}N^{-n}\int_{|y|\ge N^{-2}} \varphi_{M_0}(y)K_R(x-a_i-y)\,dy\\
&= I_i + II_i, \quad \text{say.}
\end{aligned}$$

If $R\le 20M_0$, then, since $|K_R(x)|\le cR^n(1+R|x|)^{-n}$, we see that

$$\begin{aligned}
|II_i| &\le c2^{kn}N^{-n}\int_{|y|\ge N^{-2}} |\varphi_{M_0}(y)|R^n(1+R|x-a_i-y|)^{-n}\,dy\\
&\le c2^{kn}N^{-n}\int_{|y|\ge N^{-2}} M_0^{2n}(1+M_0|y|)^{-3n}\,dy\\
&\le c2^{kn}N^{-n}\int_{|y|\ge N^{-2}} M_0^{-n}|y|^{-3n}\,dy \le c2^{kn}M_0^{-n}N^{3n}.
\end{aligned}$$

If we take M_0 large enough so that $2^{kn}M_0^{-n}N^{4n}\le 1$, then we have

$$|II_i|\le cN^{-n} \quad (i\in\mathcal{I}_k). \tag{6.3}$$

Since $|x-a_i|\sim|x-a_i-y|$ if $|y|\le N^{-2}$ ($i\in\mathcal{I}_k$), we see that

$$\begin{aligned}
|I_i| &\le c2^{kn}N^{-n}\int_{|y|\le N^{-2}} |\varphi_{M_0}(y)|\cdot|x-a_i-y|^{-n}\,dy \qquad (6.4)\\
&\le c2^{kn}N^{-n}|x-a_i|^{-n}\le c\log N \quad (i\in\mathcal{I}_k^{(1)}).
\end{aligned}$$

Next, put $\mathcal{I}_k^{(2)}=\mathcal{I}_k\setminus\mathcal{I}_k^{(1)}$. Note that $|x-a_i-y|\sim|x-a_i|\sim|x-z|$ if $|y|\le N^{-2}$ and $z\in Q_i^{(k)}$ for $i\in\mathcal{I}_k^{(2)}$. Thus

$$|I_i|\le c\int_{Q_i^{(k)}} |x-y|^{-n}\,dy \quad (i\in\mathcal{I}_k^{(2)}). \tag{6.5}$$

If $R\le 20M_0$, then by (6.3)–(6.5) we have

$$\begin{aligned}
|S_R(f_N^{(k)})(x)| &= \Big|\sum_{i\in\mathcal{I}_k} 2^{kn}N^{-n}\int\varphi_{M_0}(y)K_R(x-a_i-y)\,dy\Big|\\
&\le \sum_{i\in\mathcal{I}_k^{(1)}}|I_i| + \sum_{i\in\mathcal{I}_k^{(2)}}|I_i| + \sum_{i\in\mathcal{I}_k}|II_i|
\end{aligned}$$

$$\begin{aligned}
&\le c\log N + \int_{2^kN^{-1}<|x-y|<n^{1/2}2^{k+1}} |x-y|^{-n}\,dy\\
&\le c\log N,
\end{aligned}$$

which proves (6.2).

To finish the proof of Lemma 1, we assume as we may that N is sufficiently large. Put $M_N^{(k)}=M_0$ ($M_0\ge N^4 2^k$) and $L_N=\log N$. Then, since we have already defined $t_k>0$, $f_N^{(k)}\in\mathcal{H}_w$, $E_N^{(k)}$ and $F_N^{(k)}$, collecting the results of Sections 4–6 (see (5.6), (5.7), (6.1), (6.2)), we conclude the proof of Lemma 1.

7. Comment on bounded divergence. We have the following result.

PROPOSITION. *Suppose $f\in L^1$. If $S_R(f)$ diverges almost everywhere, then $S_R(f)$ diverges unboundedly on a dense subset of $\mathbb{R}^n$.*

Therefore, we cannot drop the "almost" in (1.1) or (1.2).

The proof of the Proposition is completely analogous to that of Körner [3, §8, Theorem C].

LEMMA 3. *Let $\{f_R\}$ ($R>0$) be a family of continuous functions on $\mathbb{R}^n$. Let Q be a closed cube in $\mathbb{R}^n$. If $\sup_{R>0}|f_R(x)|<\infty$ for all $x\in Q$, then there exist a subcube $S\subset Q$ and an $M\ge 0$ such that $|f_R(x)|\le M$ for all $R>0$ and for all $x\in S$.*

Proof. Put $F_k=\{x\in Q:\sup_{R>0}|f_R(x)|\le k\}$. Then each F_k is closed in $\mathbb{R}^n$ and $Q=\bigcup_{k=1}^\infty F_k$. Thus the conclusion follows from Baire's category theorem.

LEMMA 4. *Suppose $f\in L^1$. If $\delta>(n-1)/2$, then*

$$S_R^\delta(f)(x)=b_\delta R^{-2\delta}\int_0^R S_r(f)(x)(R^2-r^2)^{\tau-1}r^n\,dr,$$

where $\tau=\delta-(n-1)/2$ and $b_\delta=2\Gamma(\delta+1)/(\Gamma((n+1)/2)\Gamma(\tau))$.

This can be proved as in [11, Chap. VII].

LEMMA 5. *Suppose $f\in L^1$. Let Q be a cube in $\mathbb{R}^n$. If $\sup_{R>0}|S_R(f)(x)|\le M$ for all $x\in Q$, then $|f(x)|\le M$ for almost every $x\in Q$.*

Proof. If $\delta>(n-1)/2$, by Lemma 4 we see that $\sup_{R>0}|S_R^\delta(f)(x)|\le M$ for $x\in Q$ since

$$b_\delta R^{-2\delta}\int_0^R (R^2-r^2)^{\tau-1}r^n\,dr = b_\delta\int_0^1(1-r^2)^{\tau-1}r^n\,dr = 1$$

(see [11, Chap. VII]). From this the conclusion follows as $\lim_{R\to\infty}S_R^\delta(f)(x)=f(x)$ a.e. for $\delta>(n-1)/2$.

LEMMA 6. *Suppose $f, g \in L^1$. If $f = g$ in a neighborhood of $x \in \mathbb{R}^n$, then*

$$\lim_{R\to\infty} |S_R(f)(x) - S_R(g)(x)| = 0.$$

This can be found in Bochner [1, Part II, Theorem III].

LEMMA 7. *If $f \in L^2$, then $\lim_{R\to\infty} S_R(f)(x) = f(x)$ a.e.*

This follows, for example, from [11, Chap. VII, Theorem 5.1] and a transference theorem.

Proof of Proposition. Suppose $f \in L^1$. If $\sup_{R>0} |S_R(f)(x)| < \infty$ in a non-empty open set, then by Lemma 3 there exist a cube Q and a non-negative number M such that $\sup_{R>0} |S_R(f)(x)| \le M$ for all $x \in Q$. Thus, by Lemma 5 we have $|f(x)| \le M$ for almost every $x \in Q$.

Define a bounded function with compact support by

$$g(x) = \begin{cases} f(x) & \text{if } x \in Q, \\ 0 & \text{otherwise.} \end{cases}$$

Then by Lemmas 6 and 7 we see that $\lim S_R(f)(x) = \lim S_R(g)(x) = f(x)$ for almost every $x \in Q$. Therefore, if $S_R(f)$ diverges a.e., there exists an x in every non-empty open set such that $\limsup_{R\to\infty} |S_R(f)(x)| = \infty$. This completes the proof.

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