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A characterization of probability measures by f-moments

by

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Abstract. Given a real-valued continuous function f on the half-line $[0,\infty)$ we denote by $\mathbf{P}^*(f)$ the set of all probability measures μ on $[0,\infty)$ with finite f-moments $\int_0^\infty f(x)\,\mu^{*n}(dx)$ $(n=1,2,\ldots)$. A function f is said to have the *identification property* if probability measures from $\mathbf{P}^*(f)$ are uniquely determined by their f-moments. A function f is said to be a *Bernstein function* if it is infinitely differentiable on the open half-line $(0,\infty)$ and $(-1)^n f^{(n+1)}(x)$ is completely monotone for some nonnegative integer n. The purpose of this paper is to give a necessary and sufficient condition in terms of the representing measures for Bernstein functions to have the identification property.

1. Preliminaries and notation. This paper generalizes the results of [11] where the identification property on $[0,\infty)$ was proved for the moment function $f(x)=x^p$ with p not being an integer. A related problem for the absolute moments and symmetric probability measures on $(-\infty,\infty)$ satisfying some additional conditions was studied by M. V. Neupokoeva [8] and M. Braverman [1]. In particular, M. Braverman, C. L. Mallows and L. A. Shepp showed in [2] that the function $f(x)=|x|$ does not have the identification property in the class of symmetric probability measures.

The paper is organized as follows. Section 1 collects together some basic facts and notation needed in the sequel. In particular, the notions of Bernstein functions and their representing measures are discussed. In Section 2 we describe the f-equivalence relation for Bernstein functions f in terms of their representing measures. The final section contains a description of Bernstein functions with the identification property. A necessary and sufficient condition is formulated in terms of representing measures and is related to a generalization of the celebrated Müntz Theorem on uniform approximation of continuous functions by polynomials with prescribed exponents (Müntz [7], Szász [10], Paley and Wiener [9], Kaczmarz and Steinhaus [5], Feller [3]).

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We denote by $\mathbf{P}$ the set of all probability measures defined on Borel subsets of the half-line $[0,\infty)$. Given $\mu,\nu\in\mathbf{P}$ we denote by $\mu*\nu$ the convolution of μ and ν. For $\mu\in\mathbf{P}$, $\widehat{\mu}$ denotes the Laplace transform of μ, i.e. $\widehat{\mu}(z)=\int_0^\infty e^{-zx}\,\mu(dx)$ ($\operatorname{Re} z\ge 0$). Given a real-valued continuous function f on $[0,\infty)$ we denote by $\mathbf{P}(f)$ the subset of $\mathbf{P}$ consisting of measures μ with finite $\int_0^\infty|f(x)|\,\mu(dx)$. For brevity of notation we put $m_k(\mu)=\int_0^\infty x^k\,\mu(dx)$ and $\mathbf{P}_k=\mathbf{P}(f_k)$, where $f_k(x)=x^k$ $(k=1,2,\ldots)$.

PROPOSITION 1.1. *Let $1\le k\le n$ and $\mu_1,\ldots,\mu_n\in\mathbf{P}_k$. Then for every s satisfying the condition $1\le s\le k$ the equality*

$$\sum_{r=1}^{n}(-1)^r\sum_{j_1,\ldots,j_r}m_s(\mu_{j_1}*\ldots*\mu_{j_r})=0$$

holds, where the summation $\sum_{j_1,\ldots,j_r}$ runs over all r-element subsets $\{j_1,\ldots,j_r\}$ of $\{1,\ldots,n\}$.

Proof. Since $\mu_j\in\mathbf{P}_k$, the Laplace transforms $\widehat{\mu}_j$ are k times differentiable at the origin. Consequently, the function

$$g(z)=\prod_{j=1}^{n}(1-\widehat{\mu}_j(z))$$

is k times differentiable at the origin and $g^{(s)}(0)=0$ for $s=1,\ldots,k$. Moreover, we have the formula

$$g(z)=1+\sum_{r=1}^{n}(-1)^r\sum_{j_1,\ldots,j_r}(\mu_{j_1}*\ldots*\mu_{j_r})^\wedge(z).$$

Taking the sth derivative at 0 of both sides of the above equation we get the desired assertion.

Given $1\le s\le k$ we denote by $A(k,s)$ the set of all s-tuples $\langle j_1,\ldots,j_s\rangle$ of nonnegative integers satisfying the conditions $j_s\ge 1$ and $j_1+2j_2+\ldots+sj_s=k$. For example, $A(k,1)=\{k\}$ and $A(k,k)=\{\langle 0,0,\ldots,0,1\rangle\}$. Define the s-tuple $\langle j_1^\circ,\ldots,j_s^\circ\rangle$ belonging to $A(k,s)$ by setting $j_1^\circ=k$ if $s=1$ and $j_1^\circ=k-s$, $j_s^\circ=1$, $j_r^\circ=0$ for $r\ne 1,s$ if $s>1$. Put $a(j_1,\ldots,j_s)=j_1+\ldots+j_s$. Observe that

$$a(j_1,\ldots,j_s)<k-s+1=a(j_1^\circ,\ldots,j_s^\circ)\tag{1.1}$$

for all s-tuples $\langle j_1,\ldots,j_s\rangle$ from $A(k,s)$ other than $\langle j_1^\circ,\ldots,j_s^\circ\rangle$. Given $1\le s\le k<n$, $\langle j_1,\ldots,j_s\rangle\in A(k,s)$ and $\mu\in\mathbf{P}_k$ we put

$$b(n,j_1,\ldots,j_s)=n(n-1)\ldots(n-a(j_1,\ldots,j_s)+1)$$

and

$$c(n,k,s,\mu)=\sum_{\langle j_1,\ldots,j_s\rangle\in A(k,s)}\frac{k!}{j_1!\ldots j_s!}\,b(n,j_1,\ldots,j_s)\prod_{r=1}^{s}\frac{m_r(\mu)^{j_r}}{(r!)^{j_r}}\,.$$

By a standard calculation we get the formula

$$m_k(\mu^{*n})=\sum_{s=1}^{k}c(n,k,s,\mu)\tag{1.2}$$

and, by (1.1),

$$\lim_{n\to\infty}b(n,j_1^\circ,\ldots,j_s^\circ)n^{s-k-1}=1,$$

while

$$\lim_{n\to\infty}b(n,j_1,\ldots,j_s)n^{s-k-1}=0$$

for $\langle j_1,\ldots,j_s\rangle\in A(k,s)$ other than $\langle j_1^\circ,\ldots,j_s^\circ\rangle$. Hence

$$\lim_{n\to\infty}c(n,k,s,\mu)n^{s-k-1}=\binom{k}{s}m_1(\mu)^{k-s}m_s(\mu),$$

which, by (1.2), yields the following asymptotic formula for $m_k(\mu^{*n})$.

PROPOSITION 1.2. *Let $1\le l\le k<n$ and $\mu\in\mathbf{P}_k$. Then there exists a function v of $l+2$ variables such that*

$$m_k(\mu^{*n})=v(n,k,l,m_1(\mu),\ldots,m_{l-1}(\mu))+\binom{k}{l}n^{k-l+1}m_1(\mu)^{k-l}m_l(\mu)+o(n^{k-l+1}).$$

Throughout this paper $\mathbf{M}$ will stand for the set of nonnegative Borel measures M on the half-line $[0,\infty)$ satisfying the conditions $M(\{0\})=0$, $M([1,\infty))<\infty$ and $\int_0^1x^{n+1}\,M(dx)<\infty$ for some nonnegative integer n. The least such n will be denoted by $q(M)$. Obviously, $\mathbf{M}$ is closed under addition and multiplication by positive numbers. In what follows $\operatorname{supp}M$ will stand for the support of the measure M.

Introduce the notation

$$e_n(z)=\sum_{j=0}^{n}\frac{(-z)^j}{j!}-e^{-z}\qquad(n=0,1,\ldots).$$

It is easy to verify that the functions $(-1)^ne_n$ are nonnegative and nondecreasing on $[0,\infty)$. Moreover, for $x,y\ge 0$,

$$(-1)^ne_n(x+y)\le 2^{n+1}(-1)^n(e_n(x)+e_n(y)).\tag{1.3}$$

Given $M\in\mathbf{M}$ we define the *generalized Bernstein transform* $\langle M\rangle$ on the half-plane $\operatorname{Re} z \geq 0$ by setting

$$\langle M\rangle(z) = \int_0^1 e_q(zx)\, M(dx) + \int_{1+}^{\infty} e_0(zx)\, M(dx), \tag{1.4}$$

where $q = q(M)$. The following statement is evident.

PROPOSITION 1.3. *For every $M \in \mathbf{M}$ the function $\langle M\rangle$ is continuous on the half-plane $\operatorname{Re} z \geq 0$ and analytic on $\operatorname{Re} z > 0$. Moreover,*

$$(-1)^n \langle M\rangle^{(n+1)}(z) = \int_0^{\infty} e^{-zx} x^{n+1}\, M(dx)$$

whenever $n \geq q(M)$.

In particular, it follows that the correspondence $M \Leftrightarrow \langle M\rangle$ is one-to-one. Now we give some examples of generalized Bernstein transforms.

EXAMPLE 1.1. $M(dx) = x^{-n-1}dx$ $(n = 1, 2, \ldots)$. Here we have $q(M) = n$ and

$$\langle M\rangle(z) = \frac{(-1)^n}{n!} z^n \log z + \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k!(n-k)!} z^k + \frac{(-1)^n}{n!}\Big(C - \sum_{k=1}^{n} \frac{1}{k}\Big) z^n,$$

where C is the Euler constant.

EXAMPLE 1.2. $M(dx) = x^{-p-1}dx$, where $n < p < n+1$ $(n = 0, 1, \ldots)$. Here we have $q(M) = n$ and

$$\langle M\rangle(z) = -\Gamma(-p) z^p + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{(p-k)k!} z^k.$$

EXAMPLE 1.3. $M(dx) = x^{-p-1}(1+x)^{-1}dx$, where $n < p < n+1$ $(n = 0, 1, \ldots)$. Here we have $q(M) = n$ and

$$\begin{aligned}\langle M\rangle(z) = &-\Gamma(-p)e^z\Gamma(p+1, z) - \frac{\pi}{\sin \pi p}\\ &+ \sum_{k=1}^{n} \frac{1}{k!}\Big((-1)^{k-1}\beta(p+1-k) - \frac{\pi}{\sin \pi p}\Big) z^k,\end{aligned}$$

where $\Gamma(p+1, z)$ is the incomplete gamma function and

$$\beta(z) = \int_0^{\infty} \frac{e^{-zx}}{1 - e^{-x}}\, dx.$$

EXAMPLE 1.4. $M(dx) = x^{-1}(e^x - 1)^{-1}dx$. Here we have $q(M) = 1$ and

$$\langle M\rangle(z) = -\log \Gamma(z+1) + (1 - C - \log(e-1))z,$$

where C is the Euler constant.

By standard calculations we get the following statements.

PROPOSITION 1.4. *Let $M \in \mathbf{M}$ and $q(M) = q$. Then*

$$\lim_{x\to\infty} x^{-q-1}\langle M\rangle(x) = 0, \tag{1.5}$$

$$\lim_{x\to\infty} \langle M\rangle(x) = M((0,\infty)) \quad \text{if } q = 0, \tag{1.6}$$

$$\lim_{x\to\infty} x^{-q}\langle M\rangle(x) = \infty \quad \text{if } q \geq 1. \tag{1.7}$$

Observe that, by (1.3) and (1.4), the inequality

$$|\langle M\rangle(x+y)| \leq b(|\langle M\rangle(x)| + |\langle M\rangle(y)| + 1)$$

holds for all $x, y \geq 0$ and some positive constant b. This yields the following assertion.

PROPOSITION 1.5. *The set $\mathbf{P}(\langle M\rangle)$ is closed under convolution.*

Moreover, by (1.6) and (1.7), we have the following simple description of $\mathbf{P}(\langle M\rangle)$.

PROPOSITION 1.6. *Let $M \in \mathbf{M}$ and $q(M) = q$. Then $\mu \in \mathbf{P}(\langle M\rangle)$ if and only if $\mu \in \mathbf{P}_q$ and the function*

$$\sum_{j=0}^{q} \frac{(-x)^j}{j!} m_j(\mu) - \widehat{\mu}(x)$$

is M-integrable on the interval $[0, 1]$.

Here we use the notation $\mathbf{P}_0 = \mathbf{P}$.

PROPOSITION 1.7. *Let $M \in \mathbf{M}$ and $\mu, \nu \in \mathbf{P}(\langle M\rangle)$. If $\widehat{\mu} = \widehat{\nu}$ M-almost everywhere on $[0,\infty)$, then $m_k(\mu) = m_k(\nu)$ for $k = 1, \ldots, q(M)$.*

Proof. Of course it suffices to prove our assertion for $q(M) = q \geq 1$. Then, by the definition of $q(M)$,

$$\int_0^1 x^q\, M(dx) = \infty. \tag{1.8}$$

We conclude, by Proposition 1.6, that the function

$$\sum_{j=0}^{q} \frac{(-x)^j}{j!} (m_j(\mu) - m_j(\nu))$$

is M-integrable on $[0, 1]$. By (1.8) this is possible in the case $m_k(\mu) = m_k(\nu)$ for $k = 1, \ldots, q$ only, which completes the proof.

Denote by $\mathbf{A}$ the set of all polynomials with real coefficients considered on the half-line $[0,\infty)$. Given $g \in \mathbf{A}$ we denote by $d(g)$ the degree of g. The following statement is evident.

PROPOSITION 1.8. *Let $g \in \mathbf{A}$ and $d(g) = k$. Then $\mathbf{P}(g) = \mathbf{P}_k$ and, consequently, $\mathbf{P}(g)$ is closed under convolution.*

A real-valued continuous function f on $[0,\infty)$ is said to be a *Bernstein function* if it is infinitely differentiable on $(0,\infty)$ and

$$(1.9)\qquad (-1)^n f^{(n+1)}(x) \ge 0$$

for sufficiently large n and all $x \in (0,\infty)$. The set of all Bernstein functions will be denoted by $\mathbf{B}$. It is clear that $\mathbf{A} \subset \mathbf{B}$ and, by Proposition 1.3, $\langle M\rangle \in \mathbf{B}$ whenever $M \in \mathbf{M}$. Moreover, $\mathbf{B}$ is closed under addition and multiplication by positive numbers. Consequently, functions of the form $\langle M\rangle + g$ with $M \in \mathbf{M}$ and $g \in \mathbf{A}$ are Bernstein functions. The converse implication is also true.

PROPOSITION 1.9. *Every Bernstein function f has a unique representation $f = \langle M\rangle + g$ with $M \in \mathbf{M}$ and $g \in \mathbf{A}$. Moreover,*

$$(1.10)\qquad \mathbf{P}(f) = \mathbf{P}(\langle M\rangle) \cap \mathbf{P}(g).$$

Proof. Suppose that (1.9) holds for $n \ge k \ge 0$. Then the function $(-1)^k f^{(k+1)}$ is completely positive on $(0,\infty)$ and, consequently, has an integral representation

$$(1.11)\qquad (-1)^k f^{(k+1)}(z) = \int_0^\infty e^{-zx} x^{k+1}\, M(dx) + a$$

with some real constant a and nonnegative measure M satisfying the conditions $\int_0^1 x^{k+1}\, M(dx) < \infty$ and $M(\{0\}) = 0$. Denote by q the least nonnegative integer satisfying $\int_0^1 x^{q+1} M(dx) < \infty$. Obviously, $q \le k$, the functions

$$h_1(z) = \int_0^1 e_q(zx)\, M(dx), \qquad h_2(z) = -\int_{1+}^\infty e^{-zx}\, M(dx)$$

are analytic on the half-plane $\operatorname{Re} z > 0$ and

$$(-1)^k (h_1(z) + h_2(z))^{(k+1)} = \int_0^\infty e^{-zx} x^{k+1}\, M(dx).$$

Comparing this with (1.11) we conclude that $f = h_1 + h_2 + g_0$ for some $g_0 \in \mathbf{A}$ with $d(g_0) \le k+1$. Since f, h_1 and g_0 are continuous at the origin, so is h_2. Consequently, $M((1,\infty)) < \infty$, which yields $M \in \mathbf{M}$. Now applying Proposition 1.3 and formulae (1.4) and (1.11) we get a representation $f = \langle M\rangle + g$ for some $g \in \mathbf{A}$. Observe that, by (1.11), the measure M is uniquely determined by f, which proves the uniqueness of the above representation.

It remains to prove formula (1.10). The inclusion $\mathbf{P}(f) \supset \mathbf{P}(\langle M\rangle) \cap \mathbf{P}(g)$ is obvious. To prove the converse inclusion it suffices to show that $\mathbf{P}(f) \subset \mathbf{P}(g)$ or equivalently, by Proposition 1.8, $\mathbf{P}(f) \subset \mathbf{P}_s$, where $s = d(g)$. Observe that $q(M) = q$. If $q < s$, then, by (1.5),

$$\lim_{x\to\infty} x^{-s} f(x) = \lim_{x\to\infty} x^{-s} g(x) \ne 0,$$

which yields the inclusion $\mathbf{P}(f) \subset \mathbf{P}_s$. If $q \ge s$ and $q \ge 1$, then, by (1.7),

$$\lim_{x\to\infty} x^{-q} f(x) = \infty,$$

which yields the inclusion $\mathbf{P}(f) \subset \mathbf{P}_q \subset \mathbf{P}_s$. In the remaining case $q \ge s$ and $q = 0$ we have $\mathbf{P}_s = \mathbf{P}_0 = \mathbf{P}$ and, consequently, $\mathbf{P}(f) \subset \mathbf{P}_s$. This completes the proof.

Given a Bernstein function f we define the *order* $r(f)$ by the formula $r(f) = \max(q(M), d(g))$, where $f = \langle M\rangle + g$ with $M \in \mathbf{M}$ and $g \in \mathbf{A}$. The measure M is said to be the *representing measure* for f. We illustrate these notions by some examples of Bernstein functions induced by Examples 1.1–1.4 of generalized Bernstein transforms.

EXAMPLE 1.5. (a) The Bernstein function $(-1)^n z^n \log z$ of order n has representing measure $n! x^{-n-1}\, dx$ $(n = 1, 2, \ldots)$.

(b) The Bernstein function $(-1)^n z^p$ $(n < p < n+1)$ of order n has representing measure $|\Gamma(-p)|^{-1} x^{-p-1}\, dx$ $(n = 0, 1, \ldots)$.

(c) The Bernstein function $(-1)^n e^z \Gamma(p+1, z)$ $(n < p < n+1)$ of order n has representing measure $|\Gamma(-p)|^{-1} x^{-p-1}(1+x)^{-1}\, dx$ $(n = 0, 1, \ldots)$.

(d) The Bernstein function $-\log\Gamma(z+1)$ of order 1 has representing measure $x^{-1}(e^x - 1)^{-1}\, dx$.

(e) Let ζ denote the zeta-function. Then for every $a > 1$ the Bernstein function $-\zeta(z+a)$ of order 0 has representing measure $\sum_{n=2}^\infty n^{-a} \delta_{\log n}$, where δ_c denotes the probability measure concentrated at the point c.

As an immediate consequence of (1.10) and of Propositions 1.5, 1.6 and 1.8 we get the following statements.

PROPOSITION 1.10. *For every Bernstein function f the set $\mathbf{P}(f)$ is closed under convolution.*

PROPOSITION 1.11. *Let $M \in \mathbf{M}, q(M) = q$, $g(x) = \sum_{j=0}^k a_j x^j$ and $f = \langle M\rangle + g$. Then for every $\mu \in \mathbf{P}(f)$,*

$$\int_0^\infty f(x)\, \mu(dx) = \int_0^\infty \Big(\sum_{j=0}^q \frac{(-x)^j}{j!} m_j(\mu) - \widehat{\mu}(x) \Big) M(dx) + \sum_{j=0}^k a_j m_j(\mu).$$

PROPOSITION 1.12. *Let $f = \langle M\rangle + g$ with $M \in \mathbf{M}$ and $g \in \mathbf{A}$, $n > r(f)$ and $\mu_1, \ldots, \mu_n \in \mathbf{P}(f)$. Then*

$$\int_0^\infty \prod_{j=1}^n (1 - \widehat{\mu}_j(y))\, M(dy) = -f(0) - \sum_{r=1}^n (-1)^r \sum_{j_1,\ldots,j_r} \int_0^\infty f(x)\, \mu_{j_1} * \ldots * \mu_{j_r}(dx),$$

where the summation $\sum_{j_1,\ldots,j_r}$ runs over all r-element subsets $\{j_1, \ldots, j_r\}$ of $\{1, \ldots, n\}$.

Proof. Put $q = q(M)$, $k = d(g)$ and $g(x) = f(0) + \sum_{j=1}^{k} a_j x^j$. Applying Proposition 1.11 for $\mu = \mu_{j_1} * \ldots * \mu_{j_r}$ belonging, by Proposition 1.10, to $\mathbf{P}(f)$ we have

$$\int_0^\infty f(x)\,\mu_{j_1} * \ldots * \mu_{j_r}(dx)$$
$$= \int_0^\infty \Big(\sum_{s=0}^{q} \frac{(-x)^s}{s!} m_s(\mu_{j_1} * \ldots * \mu_{j_r}) - (\mu_{j_1} * \ldots * \mu_{j_r})^\wedge(x) \Big) M(dx)$$
$$+ \sum_{s=1}^{k} a_s\, m_s(\mu_{j_1} * \ldots * \mu_{j_r}) + f(0),$$

which, by Proposition 1.1, yields the formula

$$\sum_{r=1}^{n} (-1)^r \sum_{j_1,\ldots,j_r} \int_0^\infty f(x)\,\mu_{j_1} * \ldots * \mu_{j_r}(dx)$$
$$= \sum_{r=1}^{n} (-1)^r \sum_{j_1,\ldots,j_r} \int_0^\infty (1 - (\mu_{j_1} * \ldots * \mu_{j_r})^\wedge(x))\, M(dx) - f(0)$$
$$- \int_0^\infty \prod_{j=1}^{n} (1 - \widehat{\mu}_j(y))\, M(dy) - f(0),$$

which completes the proof.

2. f-equivalence relation. Let f be a Bernstein function. Two measures μ and ν from $\mathbf{P}(f)$ are said to be *f-equivalent*, in symbols $\mu \underset{f}{\sim} \nu$, if

$$\int_0^\infty f(x)\,\mu^{*n}(dx) = \int_0^\infty f(x)\,\nu^{*n}(dx)$$

for all positive integers n. Here the powers of measures are taken in the sense of convolution. Since, by Proposition 1.10, the set $\mathbf{P}(f)$ is closed under convolution the integrals appearing in the above definition are finite.

PROPOSITION 2.1. *Let $g \in \mathbf{A}$ and $\mu, \nu \in \mathbf{P}(g)$. Then $\mu \underset{g}{\sim} \nu$ if and only if $m_j(\mu) = m_j(\nu)$ for $j = 1, \ldots, d(g)$.*

Proof. For $d(g) = 0$ our statement is evident. Suppose that $d(g) = k \ge 1$ and $g(x) = \sum_{j=0}^{k} a_j x^j$ with $a_k \ne 0$. The sufficiency of the condition $m_j(\mu) = m_j(\nu)$ for $j = 1, \ldots, k$ is also evident. To prove its necessity suppose that $\mu \underset{g}{\sim} \nu$. Thus

$$\sum_{j=1}^{k} a_j (m_j(\mu^{*n}) - m_j(\nu^{*n})) = 0 \quad (n = 1, 2, \ldots). \tag{2.1}$$

By Proposition 1.2 for $l = 1$ and $k < n$ the left-hand side of the above equality can be written in the form

$$c_k k n^k (m_1(\mu)^k - m_1(\nu)^k) + o(n^k),$$

which yields the equality $m_1(\mu) = m_1(\nu)$. Proving our assertion by induction assume that $1 < l \le k$ and $m_s(\mu) = m_s(\nu)$ for $s = 1, \ldots, l-1$. Then, by Proposition 1.2, the left-hand side of (2.1) is of the form

$$c_k \binom{k}{l} n^{k-l+1} (m_1(\mu)^{k-l} m_l(\mu) - m_1(\nu)^{k-l} m_l(\nu)) + o(n^{k-l+1}),$$

which yields the equality $m_l(\mu) = m_l(\nu)$. This completes the proof.

In order to study the f-equivalence relation for an arbitrary Bernstein function f we introduce some auxiliary spaces. In the sequel $\mathbf{C}$ will denote the space of all real-valued functions continuous on the compactified half-line $[0, \infty]$ and vanishing at the origin with the norm

$$\|F\| = \max\{|F(t)| : t \in [0, \infty]\}.$$

Given $M \in \mathbf{M}$, $\mathbf{L}^2(M)$ will stand for the space of all real-valued Borel functions defined on $[0, \infty)$ with finite norm

$$\|F\|_{2,M} = \Big(\int_0^\infty F(t)^2\, M(dt) \Big)^{1/2}$$

and the inner product

$$(F, G)_{2,M} = \int_0^\infty F(t) G(t)\, M(dt).$$

Put $\mathbf{C}(M) = \mathbf{C} \cap \mathbf{L}^2(M)$. The space $\mathbf{C}(M)$ is equipped with the norm

$$\|F\|_M = \|F\| + \|F\|_{2,M}.$$

Observe that $\mathbf{C}(M)$ is a Banach algebra under pointwise multiplication and

$$\|FG\|_M \le \|F\| \cdot \|G\|_M \tag{2.2}$$

for $F, G \in \mathbf{C}(M)$. Identifying functions equal M-almost everywhere we get a natural quotient mapping τ from $\mathbf{C}(M)$ into $\mathbf{L}^2(M)$. It is clear that the set $\tau\mathbf{C}(M)$ is dense in $\mathbf{L}^2(M)$ in the $\|\ \|_{2,M}$-topology.

By a *subalgebra* of $\mathbf{C}(M)$ we mean a subset of $\mathbf{C}(M)$ closed under linear combinations and multiplication. We say that a subset $\mathbf{A}$ of $\mathbf{C}(M)$ *separates points* if for every pair of distinct points $a, b \in [0, \infty)$ there exists a function

$F \in \mathbf{A}$ such that $F(a) \neq F(b)$. In the sequel $\operatorname{lin} \mathbf{A}$ will denote the linear span of $\mathbf{A}$. A sequence $\{E_n\}$ of functions from $\mathbf{C}(M)$ is called an *approximate unit* if $\lim_{n\to\infty} \|F - FE_n\|_M = 0$ for every $F \in \mathbf{C}(M)$.

LEMMA 2.1. *Let G be a decreasing positive function such that $(1-G)^k \in \mathbf{C}(M)$ for some positive integer k. Then the set*

$$\mathbf{A}_k(G) = \operatorname{lin}\Big\{\prod_{j=1}^{k}(1-G^{n_j}) : n_j = 1,2,\ldots;\ j = 1,\ldots,k\Big\}$$

is a subalgebra of $\mathbf{C}(M)$ which separates points and contains an approximate unit.

Proof. Since functions from $\mathbf{C}(M)$ vanish at the origin, we have $G(0) = 1$ and, consequently, $0 < G(t) \le 1$ for $t \in [0,\infty)$. Thus

$$\prod_{j=1}^{k}(1-G^{n_j}) \le \sum_{j=1}^{k} n_j^k (1-G)^k,$$

which yields the inclusion $\mathbf{A}_k(G) \subset \mathbf{C}(M)$. Further, from the equality

$$\begin{aligned}(2.3)\qquad \prod_{j=1}^{k}(1-G^{n_j})\prod_{j=1}^{k}(1-G^{m_j}) \\ = \sum \prod_{j\in I}(1-G^{n_j})\prod_{j\in J}(1-G^{m_j})\prod_{j\in K}(G^{n_j+m_j}-1),\end{aligned}$$

where the summation runs over all partitions of $\{1,\ldots,k\}$ into disjoint subsets I, J and K, it follows that $\mathbf{A}_k(G)$ is closed under multiplication and, consequently, is a subalgebra of $\mathbf{C}(M)$. Since the function $(1-G)^k$ is increasing on $[0,\infty]$ we conclude that $\mathbf{A}_k(G)$ separates points. Finally, setting $E_n = (1-G^n)^k$ we have $E_n \in \mathbf{A}_k(G)$,

$$(2.4)\qquad \|E_n\| \le 1 \quad (n = 1,2,\ldots)$$

and

$$(2.5)\qquad \lim_{n\to\infty} E_n(u) = 1$$

for every $u \in (0,\infty)$. Since the function $1-E_n$ is decreasing, we have

$$\|F - FE_n\| \le \|1-E_n\| \max\{|F(t)| : t \in [0,u]\} + \|F\|(1-E_n(u))$$

for $F \in \mathbf{C}(M)$ and $u \in (0,\infty)$. Observe that $\lim_{u\to 0} F(u) = 0$. Combining this with (2.4) and (2.5) we get

$$\lim_{n\to\infty} \|F - FE_n\| = 0.$$

On the other hand, by the bounded convergence theorem we derive from (2.4) and (2.5),

$$\|F - FE_n\|_{2,M}^2 = \int_0^\infty (1-E_n(t))^2 F(t)^2\, M(dt) \to 0$$

as $n \to \infty$, which shows that $\{E_n\}$ is an approximate unit. The lemma is thus proved.

LEMMA 2.2. *Let $\mathbf{A}$ be a subalgebra of $\mathbf{C}(M)$ separating points and containing an approximate unit. Then $\mathbf{A}$ is dense in $\mathbf{C}(M)$ in the $\|\ \|_M$-topology.*

Proof. Let $\{E_n\}$ be an approximate unit belonging to $\mathbf{A}$. Put

$$\mathbf{A}_0 = \{FE_n : F \in \mathbf{C}(M),\ n = 1,2,\ldots\}.$$

It is clear that $\mathbf{A}_0$ is dense in $\mathbf{C}(M)$ in the $\|\ \|_M$-topology. Since $\mathbf{A}$ separates points we conclude, by the Stone–Weierstrass Theorem ([6], Theorem 4E), that $\mathbf{A}$ is dense in $\mathbf{C}(M)$ in the $\|\ \|$-topology. Consequently, for every $F \in \mathbf{C}(M)$ we can find a sequence $F_n \in \mathbf{A}$ $(n = 1,2,\ldots)$ such that $\|F - F_n\| \to 0$ as $n \to \infty$. Since $F_nE_k \in \mathbf{A}$ for every k and, by (2.2), $\|FE_k - F_nE_k\|_M \le \|E_k\|_M\|F - F_n\|$ $(n = 1,2,\ldots)$ we infer that $\mathbf{A}$ is dense in $\mathbf{A}_0$ in the $\|\ \|_M$-topology, which completes the proof.

LEMMA 2.3. *Let $\mathbf{A}$ be a subalgebra of $\mathbf{C}(M)$ separating points and containing an approximate unit. Let U_0 be a linear and multiplicative $\|\ \|_{2,M}$-isometry from $\mathbf{A}$ into $\tau\mathbf{C}(M)$. Then U_0 can be extended to a linear $\|\ \|_{2,M}$-isometry U from $\mathbf{C}(M)$ into $\mathbf{L}^2(M)$ and there exists a Borel mapping φ from $[0,\infty)$ into itself such that*

$$(2.6)\qquad (UF)(t) = \tau F(\varphi(t))$$

for every $F \in \mathbf{C}(M)$ and M-almost every $t \in [0,\infty)$.

Proof. From the inequality $\|\ \|_{2,M} \le \|\ \|_M$ it follows that U_0 is continuous from $\mathbf{A}$ equipped with the $\|\ \|_M$-topology into $\mathbf{L}^2(M)$ with the $\|\ \|_{2,M}$-topology. Since, by Lemma 2.2, $\mathbf{A}$ is dense in $\mathbf{C}(M)$ in the $\|\ \|_M$-topology, U_0 can be extended to a linear and multiplicative $\|\ \|_{2,M}$-isometry U from $\mathbf{C}(M)$ into $\mathbf{L}^2(M)$. In particular, we have

$$(2.7)\qquad U(FG) = U(F)U(G)$$

for $F, G \in C(M)$. Put $s = q(M) + 1$ and $G(t) = \exp(-t^s)$. It is clear that G and $k = 1$ satisfy the conditions of Lemma 2.1, which together with Lemma 2.2 shows that $\mathbf{A}_1(G)$ is dense in $\mathbf{C}(M)$ in the $\|\ \|_M$-topology. Consequently, to prove the lemma it suffices to show that formula (2.6) holds for $F \in \mathbf{A}_1(G)$ and a suitably chosen function φ.

Setting $H_n = 1 - G^n$ $(n = 1, 2, \ldots)$ we have $\mathbf{A}_1(G) = \operatorname{lin}\{H_n : n = 1, 2, \ldots\}$. Introduce the notation

$$UH_n = 1 - W_n \quad (n = 1, 2, \ldots). \tag{2.8}$$

By (2.7) we have the equality

$$U(H_n H_m) = (1 - W_n)(1 - W_m) \quad (n, m = 1, 2, \ldots).$$

On the other hand, using the formula $H_n H_m = H_n + H_m - H_{n+m}$ we have

$$U(H_n H_m) = 1 - W_n - W_m + W_{n+m} \quad (n, m = 1, 2, \ldots),$$

which yields the equality $W_{n,m} = W_n W_m$ M-almost everywhere $(n, m = 1, 2, \ldots)$. Thus

$$W_n = W_1^n \quad (n = 1, 2, \ldots) \tag{2.9}$$

M-almost everywhere. Again by (2.7), $U(H_n^r) = (1 - W_n)^r = (1 - W_1^n)^r$ $(n, r = 1, 2, \ldots)$. Since U is a $\| \|_{2,M}$-isometry, we have

$$\int_0^\infty (1 - \exp(-nt^s))^{2r}\, M(dt) = \|H_n^r\|_{2,M}^2 = \|U(H_n^r)\|_{2,M}^2$$

$$= \int_0^\infty (1 - W_1^n(t))^{2r}\, M(dt).$$

Observe that for every n the left-hand side of the above equality tends to 0 as $r \to \infty$. This yields the inequality $0 < W_1(t) \le 1$ for M-almost every $t \in [0, \infty)$. Changing if necessary the function W_1 on a set of M-measure zero we may assume without loss of generality that W_1 is a Borel function satisfying the inequality $0 < W_1(t) \le 1$ for all $t \in [0, \infty)$. Then the function

$$\varphi(t) = (-\log W_1(t))^{1/s}$$

maps the half-line $[0, \infty)$ into itself and, by (2.8) and (2.9),

$$(UH_n)(t) = \tau H_n(\varphi(t))$$

for $n = 1, 2, \ldots$ and M-almost every $t \in [0, \infty)$. This completes the proof.

LEMMA 2.4. *Let f be a Bernstein function with the representing measure M and $\mu, \nu \in \mathbf{P}(f)$. If $\mu \underset{f}{\sim} \nu$, then $\widehat{\mu}(t) = \widehat{\nu}(t)$ for $t \in \operatorname{supp} M$.*

Proof. By Proposition 1.12 the relation $\mu \underset{f}{\sim} \nu$ yields the formula

$$\int_0^\infty \prod_{j=1}^n (1 - \widehat{\mu}^{k_j}(x))\, M(dx) = \int_0^\infty \prod_{j=1}^n (1 - \widehat{\nu}^{k_j}(x))\, M(dx) < \infty \tag{2.10}$$

for $n > r(f)$ and every n-tuple $k_1, \ldots, k_n$ of positive integers. If at least one of the measures μ, ν is concentrated at the origin, say $\mu = \delta_0$, then $\widehat{\mu}(x) = 1$ for all $x \in [0, \infty)$ and, by (2.10), $\widehat{\nu}(x) = 1$ M-almost everywhere. Taking into account the continuity of the Laplace transform we get $\widehat{\nu}(x) = 1$ for $x \in \operatorname{supp} M$, which yields our assertion.

Suppose now that neither μ nor ν is concentrated at the origin. Without loss of generality we may assume that

$$\widehat{\mu}(\infty) \le \widehat{\nu}(\infty) < 1. \tag{2.11}$$

Setting $k = r(f) + 1$ and $G = \widehat{\mu}$ or $\widehat{\nu}$ we conclude, by (2.10), that $(1 - G)^k$ belongs to $\mathbf{C}(M)$. Since both $\widehat{\mu}$ and $\widehat{\nu}$ are decreasing, we infer that G and k satisfy the conditions of Lemma 2.1, which together with Lemma 2.2 shows that $\mathbf{A}_k(G)$ is a dense subset of $\mathbf{C}(M)$ in the $\| \|_M$-topology. Introduce the notation

$$H(n_1, \ldots, n_k; \lambda) = \prod_{j=1}^k (1 - \widehat{\lambda}^{n_j})$$

for $\lambda \in \mathbf{P}$. It is clear that

$$\mathbf{A}_k(\widehat{\lambda}) = \operatorname{lin}\{H(n_1, \ldots, n_k; \lambda) : n_j = 1, 2, \ldots;\ j = 1, \ldots, k\}.$$

Moreover, by (2.10),

$$\begin{aligned}(H(n_1, \ldots, n_k; \mu), H(r_1, \ldots, r_k; \mu))_{2,M} \\ = (H(n_1, \ldots, n_k; \nu), H(r_1, \ldots, r_k; \nu))_{2,M},\end{aligned} \tag{2.12}$$

which yields the following implication: if a linear combination

$$\sum c(n_1, \ldots, n_k) H(n_1, \ldots, n_k; \mu)$$

vanishes M-almost everywhere, then so does

$$\sum c(n_1, \ldots, n_k) H(n_1, \ldots, n_k; \nu).$$

Using this property we can extend the mapping defined by the formula

$$U_0 H(n_1, \ldots, n_k; \mu) = \tau H(n_1, \ldots, n_k; \nu) \tag{2.13}$$

for every k-tuple $n_1, \ldots, n_k$ of positive integers to a linear mapping U_0 from $\mathbf{A}_k(\widehat{\mu})$ into $\tau \mathbf{A}_k(\widehat{\nu})$. By (2.3) the mapping U_0 is multiplicative. Moreover, by (2.12) it is a $\| \|_{2,M}$-isometry. Now applying Lemma 2.3 we conclude that U_0 has an extension to a linear $\| \|_{2,M}$-isometry U from $\mathbf{C}(M)$ into $\mathbf{L}^2(M)$ of the form

$$(UF)(t) = \tau F(\varphi(t)) \tag{2.14}$$

for every $F \in \mathbf{C}(M)$ and M-almost every $t \in [0, \infty)$, where φ is a nonnegative Borel function defined on $[0, \infty)$. In particular, setting $n_1 = \ldots = n_k = 1$ in (2.13) and $F = H(1, \ldots, 1; \mu)$ in (2.14) we get the formula $(1 - \widehat{\mu}(t))^k = (1 - \widehat{\mu}(\varphi(t)))^k$ for M-almost all $t \in [0, \infty)$, which yields the equality

$$\widehat{\nu}(t) = \widehat{\mu}(\varphi(t)) \tag{2.15}$$

M-almost everywhere. Observe that the inverse function $\widehat{\mu}^{-1}$ is decreasing and maps $(\widehat{\mu}(\infty),1]$ onto $[0,\infty)$. Consequently, by (2.11), the superposition $\varphi_0(t)=\widehat{\mu}^{-1}(\widehat{\nu}(t))$ is well defined for $t\in[0,\infty)$, increasing and infinitely differentiable on $(0,\infty)$. Moreover, by (2.15), $\varphi(t)=\varphi_0(t)$ M-almost everywhere, which, by the continuity of $\widehat{\mu},\widehat{\nu}$, yields

$$\widehat{\nu}(t)=\widehat{\mu}(\varphi_0(t)) \quad \text{for } t\in \operatorname{supp} M. \tag{2.16}$$

Further, by (2.14),

$$(UF)(t)=\tau F(\varphi_0(t)) \tag{2.17}$$

for every $F\in \mathbf{C}(M)$ and M-almost every $t\in[0,\infty)$.

Given $u>0$ and $n>u^{-1}$ we put $F_n(u,t)=0$ if $t\in[0,u-n^{-1}]$, $F_n(u,t)=n(t-u)+1$ if $t\in(u-n^{-1},u)$ and $F_n(u,t)=1$ if $t\in[u,\infty)$. It is clear that $F_n(u,\cdot)\in\mathbf{C}(M)$, $F_n(u,\varphi_0(\cdot))\in\mathbf{C}(M)$ and, consequently, by (2.17),

$$\|F_n(u,\cdot)\|_{2,M}=\|F_n(u,\varphi_0(\cdot))\|_{2,M}.$$

Moreover, setting $F(u,t)=0$ if $t\in[0,u)$ and $F(u,t)=1$ otherwise, we have

$$\lim_{n\to\infty}F_n(u,t)=F(u,t)$$

for every $t\in[0,\infty)$, which, by the bounded convergence theorem, yields

$$\|F(u,\cdot)\|_{2,M}=\|F(u,\varphi_0(\cdot))\|_{2,M}. \tag{2.18}$$

Denote by ψ the inverse function of φ_0. The function ψ is increasing, continuous and maps $[0,\varphi_0(\infty))$ onto $[0,\infty)$. Observe that $F(u,\varphi_0(t))=0$ if $u\ge\varphi_0(\infty)$, $t\ge0$, and $F(u,\varphi_0(t))=F(\psi(u),t)$ if $u\in(0,\varphi_0(\infty))$, $t\ge0$. Inserting the above expressions into (2.18) we get

$$M([u,\infty))=0 \quad \text{if } u\ge\varphi_0(\infty) \tag{2.19}$$

and

$$M([u,\infty))=M([\psi(u),\infty)) \quad \text{if } u\in(0,\varphi_0(\infty)). \tag{2.20}$$

Given $u\in(0,\varphi_0(\infty))$ we introduce the notation

$$\alpha(u)=\min(u,\psi(u)),\qquad \beta(u)=\max(u,\psi(u)).$$

Both α and β are increasing and continuous. Moreover,

$$u\in\{\alpha(u),\beta(u)\} \tag{2.21}$$

and, by (2.20),

$$M([\alpha(u),\beta(u)))=0. \tag{2.22}$$

Suppose that $t\in(0,\varphi_0(\infty))$ and $\alpha(t)<\beta(t)$. Taking into account the continuity of α and β we can find a pair u_1,u_2 satisfying the conditions $0<u_1<t<u_2<\varphi_0(\infty)$, $\alpha(t)<\beta(u_1)$ and $\alpha(u_2)<\beta(t)$. Since α and β are increasing, we also have $\alpha(u_1)<\alpha(t)$ and $\beta(t)<\beta(u_2)$. Hence and from (2.21) and (2.22) it follows that the set $(\alpha(u_1),\beta(u_1))\cup(\alpha(u_2),\beta(u_2))$ is a neighbourhood of t of M-measure zero. Thus $t\notin\operatorname{supp}M$. It follows that for every $t\in(0,\varphi_0(\infty))\cap\operatorname{supp}M$ we have $\alpha(t)=\beta(t)$ or, equivalently, $\varphi_0(t)=t$. By the continuity of φ_0, the formula $M(\{0\})=0$ and (2.19), the above equality holds for all $t\in\operatorname{supp}M$. Comparing this with (2.16) we get the assertion of lemma.

Now we are in a position to prove the main result of this section.

THEOREM 2.1. *Let f be a Bernstein function with representing measure M and $\mu,\nu\in\mathbf{P}(f)$. Then $\mu\underset{f}{\sim}\nu$ if and only if $\widehat{\mu}(t)=\widehat{\nu}(t)$ for $t\in\operatorname{supp}M$ and $m_k(\mu)=m_k(\nu)$ for $k=1,\ldots,r(f)$.*

Proof. *Sufficiency*. Suppose that $\mu,\nu\in\mathbf{P}(f)$, $\widehat{\mu}(t)=\widehat{\nu}(t)$ for $t\in\operatorname{supp}M$ and $m_k(\mu)=m_k(\nu)$ for $k=1,\ldots,r(f)$. It is clear that $(\mu^{*n})^\wedge(t)=(\nu^{*n})^\wedge(t)$ for $t\in\operatorname{supp}M$ and, by (1.2), $m_k(\mu^{*n})=m_k(\nu^{*n})$ for $k=1,\ldots,r(f)$ and $n=1,2,\ldots$ Applying Proposition 1.11 we conclude that $\int_0^\infty f(x)\,\mu^{*n}(dx)=\int_0^\infty f(x)\,\nu^{*n}(dx)$ for $n=1,2,\ldots,$ which completes the proof of the sufficiency of our conditions.

Necessity. Suppose that $\mu,\nu\in\mathbf{P}(f)$ and $\mu\underset{f}{\sim}\nu$. By Proposition 1.9 the function f has a representation

$$f=\langle M\rangle+g \tag{2.23}$$

with $M\in\mathbf{M}$ and $g\in\mathbf{A}$. Moreover, by (1.10),

$$\mu,\nu\in\mathbf{P}(\langle M\rangle)\cap\mathbf{P}(g). \tag{2.24}$$

Further, by Lemma 2.4, $\widehat{\mu}(t)=\widehat{\nu}(t)$ for $t\in\operatorname{supp}M$, which, by (2.24) and Proposition 1.7, yields $m_k(\mu)=m_k(\nu)$ for $k=1,\ldots,q(M)$. Consequently, $(\mu^{*n})^\wedge(t)=(\nu^{*n})^\wedge(t)$ for $t\in\operatorname{supp}M$ and, by (1.2), $m_k(\mu^{*n})=m_k(\nu^{*n})$ for $k=1,\ldots,q(M)$ and $n=1,2,\ldots$ Now applying Proposition 1.11 we have

$$\int_0^\infty\langle M\rangle(x)\,\mu^{*n}(dx)=\int_0^\infty\langle M\rangle(x)\,\nu^{*n}(dx)\qquad(n=1,2,\ldots)$$

or, equivalently, $\mu\underset{\langle M\rangle}{\sim}\nu$. Comparing this with (2.23) we get $\mu\underset{g}{\sim}\nu$, which, by (2.24) and Proposition 2.1, yields $m_k(\mu)=m_k(\nu)$ for $k=1,\ldots,d(g)$. Observe that $r(f)=\max(q(M),d(g))$, which completes the proof of the necessity of our conditions.

3. An identification problem.

A Bernstein function f is said to have the *identification property* if for $\mu,\nu\in\mathbf{P}(f)$ the f-equivalence relation $\mu\underset{f}{\sim}\nu$ yields $\mu=\nu$. The aim of this section is to characterize the Bernstein functions with the identification property in terms of their representing measures.

We denote by $\mathbf{D}$ the set of all functions F analytic in the right half-plane $\operatorname{Re} z > 0$, real-valued on the half-line $(0, \infty)$ and for every nonnegative integer n satisfying the condition

$$|F^{(n)}(z)| \le a_n \max(|z|^{k_n}, |z|^{-k_n})$$

for $\operatorname{Re} z > 0$, some positive number a_n and some nonnegative integer k_n. It is clear that $\mathbf{D}$ is closed under differentiation, multiplication and linear combinations with real coefficients. Setting $(JF)(z) = F(z^{-1})$ for $\operatorname{Re} z > 0$, we have

$$(JF)^{(n)}(z) = z^{-2n}(JF^{(n)})(z) + \sum_{k=1}^{n-1} \binom{n-1}{k}\binom{n}{k} k!\, z^{k-2n}(JF^{(n-k)})(z)$$

for $n = 1, 2, \ldots$, which shows that $\mathbf{D}$ is invariant under J.

Let $\mathbf{K}$ denote the set of all functions F analytic in the right half-plane $\operatorname{Re} z > 0$, real-valued on $(0, \infty)$ and for every pair k, n of nonnegative integers satisfying the condition

$$|F^{(n)}(z)| \le b_{k,n} \min(|z|^k, |z|^{-k})$$

for $\operatorname{Re} z > 0$ and some positive number $b_{k,n}$. It is clear that $\mathbf{K}$ is closed under differentiation, linear combinations with real coefficients and multiplication by functions from $\mathbf{D}$. Notice that $\mathbf{K} \subset \mathbf{D}$ and $\mathbf{K}$ is nontrivial, i.e. contains a function not vanishing identically. To prove this, set

$$h_k(x) = (-1)^k (\sin 2\pi \log x) \exp(-\log^2 x + k \log x) \quad (k = 0, 1, \ldots)$$

for $x > 0$ and $h_k(0) = 0$. It is easy to verify that h_k is infinitely differentiable on $[0, \infty)$ with integrable derivatives $h_k^{(r)}$ and $h_k^{(r)}(0) = 0$ $(r = 0, 1, \ldots)$. Setting $H_{k,r}(z) = \widehat{h}_k^{(r)}(z)$ $(k, r = 0, 1, \ldots)$ we have the formulae

$$zH_{k,r}(z) = H_{k,r+1}(z), \quad H_{k,r}^{(n)}(z) = H_{k+n,r}(z) \quad \text{and} \quad H_{k,r}^{(n)}(0) = 0.$$

It follows that for every pair k, n of nonnegative integers the functions $z^k H_{n,0}(z)$ and $z^{-k} H_{n,0}(z)$ are bounded in the half-plane $\operatorname{Re} z > 0$. Thus setting $G(z) = H_{0,0}(z)$ we have $G^{(n)}(z) = H_{n,0}(z)$, which yields the inequality

$$|G^{(n)}(z)| \le c_{k,n} \min(|z|^k, |z|^{-k})$$

for $\operatorname{Re} z > 0$ and some positive number $c_{k,n}$. It is clear that G is analytic in the half-plane $\operatorname{Re} z > 0$, real-valued on $[0, \infty)$ and does not vanish identically. This shows that $G \in \mathbf{K}$.

In what follows $\widehat{g}$ will denote the Laplace transform of g.

PROPOSITION 3.1. *Every function from $\mathbf{K}$ is of the form $\widehat{g}$, where g is a real-valued function on $[0, \infty)$ satisfying the conditions $\int_0^\infty x^n |g(x)|\, dx < \infty$ and $\int_0^\infty x^n g(x)\, dx = 0$ $(n = 0, 1, \ldots)$.*

Proof. Put $V(z) = \min(|z|, |z|^{-1})$. By standard calculations we conclude that the function $\int_{-\infty}^{\infty} V(x+iy)^2\, dy$ is bounded for $x \in (0, \infty)$. Let $F \in \mathbf{K}$. Then for every nonnegative integer n the function $F^{(n)}$ is analytic in the right half-plane $\operatorname{Re} z > 0$ and $|F^{(n)}(z)| \le c_n V(z)$ for $\operatorname{Re} z > 0$ and some positive number c_n. Hence $\int_{-\infty}^{\infty} |F^{(n)}(x+iy)|^2\, dy$ is bounded for $x \in (0, \infty)$. In other words, $F^{(n)}$ belongs to the Hardy space $\mathbf{H}^2$ on the right half-plane $\operatorname{Re} z > 0$. By the classical theorem of Paley and Wiener ([4], Chapter 8) there then exists a function g_n on $(0, \infty)$ satisfying the condition

$$\int_0^\infty |g_n(x)|^2\, dx < \infty \tag{3.1}$$

such that

$$F^{(n)} = \widehat{g}_n \quad \text{for } \operatorname{Re} z > 0. \tag{3.2}$$

Since $F^{(n)}$ is real-valued on $(0, \infty)$, we conclude that g_n is real-valued almost everywhere on $(0, \infty)$. Of course without loss of generality we may assume that g_n is real-valued. Moreover,

$$g_n(x) = (-1)^n x^n g_0(x) \quad (n = 0, 1, \ldots) \tag{3.3}$$

almost everywhere on $(0, \infty)$, which, by (3.1), yields

$$\int_0^\infty x^{2n} g_0(x)^2\, dx < \infty.$$

Hence, by the Schwarz inequality

$$\Big(\int_1^\infty x^n |g_0(x)|\, dx\Big)^2 = \Big(\int_1^\infty x^{-1} x^{n+1} |g_0(x)|\, dx\Big)^2 \le \int_1^\infty x^{2(n+1)} g_0(x)^2\, dx$$

we get

$$\int_1^\infty x^n |g_0(x)|\, dx < \infty \quad (n = 0, 1, \ldots). \tag{3.4}$$

Since $\lim_{z \to 0} F^{(n)}(z) = 0$ for $F \in \mathbf{K}$ and $n = 0, 1, \ldots$, we conclude, by (3.2)–(3.4), that

$$\int_0^\infty x^n g_0(x)\, dx = 0 \quad (n = 0, 1, \ldots).$$

Thus the representation $F = \widehat{g}_0$ has the required properties, which completes the proof.

We define the function Λ on subsets of $[0, \infty)$ by setting $\Lambda(\emptyset) = 0$ and

$$\Lambda(A) = \sup \sum_{b \in B} \min(b, b^{-1})$$

for nonempty subsets A, where the supremum runs over all nonempty finite subsets B of A.

The following results concerning the determination of the Laplace transform by its values on a subset of $[0,\infty)$ will play a crucial role in our considerations. They can be regarded as an extension of Feller's version of Müntz' Theorem given in [3].

PROPOSITION 3.2. *For every subset A of $[0,\infty)$ with $\Lambda(A)<\infty$ there exists a pair of distinct measures $\mu,\nu\in\bigcap_{k=1}^{\infty}\mathbf{P}_k$ such that $m_k(\mu)=m_k(\nu)$ for $k=1,2,\ldots$ and $\widehat{\mu}(t)=\widehat{\nu}(t)$ for $t\in A$.*

Proof. From the assumption $\Lambda(A)<\infty$ it follows that there exists a sequence $1\le a_1<a_2<\ldots$ of real numbers tending to ∞ such that $\sum_{n=1}^{\infty}a_n^{-1}<\infty$ and

$$A\subset\{0\}\cup\{a_n:n=1,2,\ldots\}\cup\{a_n^{-1}:n=1,2,\ldots\}. \tag{3.5}$$

W. Feller proved in [2], Lemma 1, that there exists a real-valued function g on $[0,\infty)$ such that

$$0<\int_0^\infty e^x|g(x)|\,dx<\infty \tag{3.6}$$

and

$$\widehat{g}(a_n)=0\quad(n=1,2,\ldots). \tag{3.7}$$

It is clear that $\widehat{g}$ is analytic in the half-plane $\operatorname{Re}z>0$, real-valued on $(0,\infty)$ and, by (3.6), all its derivatives $\widehat{g}^{(n)}$ are bounded in the half-plane $\operatorname{Re}z>0$. Consequently, $\widehat{g}$ and $J\widehat{g}$ belong to $\mathbf{D}$. Moreover, by (3.7),

$$(J\widehat{g})(a_n^{-1})=0\quad(n=1,2,\ldots). \tag{3.8}$$

Taking a function H from $\mathbf{K}$ not vanishing identically we put $F=H\widehat{g}(J\widehat{g})$. Obviously F does not vanish identically, belongs to $\mathbf{K}$ and, by (3.7) and (3.8),

$$F(a_n)=F(a_n^{-1})=0\quad(n=1,2,\ldots). \tag{3.9}$$

Proposition 3.1 shows that F is of the form $F=\widehat{h}$, where h is a real-valued function on $[0,\infty)$ satisfying

$$0<\int_0^\infty x^n|h(x)|\,dx<\infty\quad(n=0,1,\ldots) \tag{3.10}$$

and

$$\int_0^\infty x^n h(x)\,dx=0\quad(n=0,1,\ldots). \tag{3.11}$$

Set $H_+(x)=\max(h(x),0)$ and $H_-(x)=\max(-h(x),0)$. By (3.10) and (3.11) for $n=0$ we have

$$c=\int_0^\infty h_+(x)\,dx=\int_0^\infty h_-(x)\,dx>0.$$

We define two probability measures on $[0,\infty)$ by $\mu(dx)=c^{-1}h_+(x)\,dx$ and $\nu(dx)=c^{-1}h_-(x)\,dx$. Of course $\mu\ne\nu$. Moreover, by (3.10) and (3.11), $\mu,\nu\in\bigcap_{k=1}^{\infty}\mathbf{P}_k$ and $m_k(\mu)=m_k(\nu)$ $(k=1,2,\ldots)$. Since $\widehat{\mu}(t)-\widehat{\nu}(t)=c^{-1}F(t)$, we have, by (3.9), $\widehat{\mu}(a_n)=\widehat{\nu}(a_n)$ and $\widehat{\mu}(a_n^{-1})$ $(n=1,2,\ldots)$. Taking into account (3.5) and the obvious formula $\widehat{\mu}(0)=\widehat{\nu}(0)=1$ we get the equality $\widehat{\mu}(t)=\widehat{\nu}(t)$ for $t\in A$, which completes the proof.

Taking into account the formulation of Müntz' Problem in terms of Laplace transforms in [3] and the inequality

$$\Lambda(A)\le 2\sum_{a\in A}\frac{a}{1+a^2}$$

we get, by Szász' version of Müntz' Theorem ([9], Theorem XV), the following statement.

PROPOSITION 3.3. *Let A be a subset of $[0,\infty)$ with $\Lambda(A)=\infty$, and $\mu,\nu\in\mathbf{P}$. Then the equality $\widehat{\mu}(t)=\widehat{\nu}(t)$ for $t\in A$ yields $\mu=\nu$.*

As an immediate consequence of Theorem 2.1 and Propositions 3.2 and 3.3 we get the following description of Bernstein functions with the identification property.

PROPOSITION 3.4. *A Bernstein function with representing measure M has the identification property if and only if $\Lambda(\operatorname{supp}M)=\infty$.*

Applying the above proposition to functions appearing in Example 1.5 we conclude that the functions $(-1)^n z^n\log z$ $(n=1,2,\ldots)$, $(-1)^k z^p$, $(-1)^k e^z\Gamma(p+1,z)$ $(k<p<k+1,\ k=0,1,\ldots)$, $-\log\Gamma(z+1)$ and $-\zeta(z+a)$ $(a>1)$ have the identification property.

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