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Extension of operators from weak*-closed subspaces of ℓ_1 into $C(K)$ spaces

by

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Abstract. It is proved that every operator from a weak*-closed subspace of ℓ_1 into a space $C(K)$ of continuous functions on a compact Hausdorff space K can be extended to an operator from ℓ_1 to $C(K)$.

1. Introduction. This work is part of an effort to characterize those subspaces E of a Banach space X for which the pair (E, X) has the following

EXTENSION PROPERTY (E.P., in short). *Every (bounded, linear) operator T from E into any $C(K)$ space Y has an extension* $\mathbf{T}: X \to Y$.

There is a quantitative version of the E.P.: for any $\lambda \geq 1$ we say that the pair (E, X) has the *λ-E.P.* if for every $T: E \to Y$ there is an extension $\mathbf{T}: X \to Y$ with $\|\mathbf{T}\| \leq \lambda \|T\|$. It is easy to see that if (E, X) has the E.P., then it has the λ-E.P. for some λ.

It is known [Zip] that for each $1 < p < \infty$ and every subspace E of ℓ_p, (E, ℓ_p) has the 1-E.P., while for $F \subset c_0$, (F, c_0) has the $(1+\varepsilon)$-E.P. for every $\varepsilon > 0$ [LP]. However, there is a subspace F of c_0 for which (F, c_0) does not have the 1-E.P. [JZ2]. If E itself is a $C(K)$ space then, clearly, (E, X) has the E.P. if and only if E is complemented in X. It follows from [Ami] that $C(K)$ has a subspace E for which $(E, C(K))$ does not have the E.P. if K is any compact metric space whose ωth derived set is nonempty (which is equivalent [BePe] to saying that $C(K)$ is not isomorphic to c_0).

Since every separable Banach space is a quotient of ℓ_1, the following fact demonstrates the important rôle of the space ℓ_1 in extension problems.

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PROPOSITION 1.1. *Let E be a subspace of a Banach space X and let Q be an operator from Z onto X so that $\|Q\| = 1$ and $Q\,\mathrm{Ball}\,Z \supset \delta\,\mathrm{Ball}\,X$. If $(Q^{-1}E, Z)$ has the λ-E.P. then (E, X) has the λ/δ-E.P.*

Proof. Let T be an operator from E into any $C(K)$ space Y. Consider the operator $S = TQ : Q^{-1}E \to Z$. If $\mathbf{S} : Z \to Y$ extends S then since $\mathbf{S}$ vanishes on $\ker Q$, $\mathbf{S}$ induces an operator $\widetilde{S}$ from $X \sim Z/\ker Q$ into Y so that $\widetilde{S}Q = \mathbf{S}$ and $\|\widetilde{S}\| \le \|\mathbf{S}\|/\delta$. ■

An immediate consequence of Proposition 1.1 is that ℓ_1 contains a subspace F for which (F, ℓ_1) does not have the E.P. Indeed, if E denotes an uncomplemented subspace of $C[0,1]$ which is isomorphic to $C[0,1]$ ([Ami]) and if $Q : \ell_1 \to C[0,1]$ is a quotient map and $F = Q^{-1}E$, then (F, ℓ_1) does not have the E.P. The main purpose of this paper is to prove the following.

THEOREM. *Let $\{X_n\}_{n=1}^\infty$ be finite-dimensional and let E be a weak*-closed subspace of $X = (\sum X_n)_1$, regarded as the dual of $X_* = (\sum X_n^*)_{c_0}$. Then (E, X) has the E.P. Moreover, if E has the approximation property, then (E, X) has the $(1+\varepsilon)$-E.P. for every $\varepsilon > 0$.*

Remark. Under the hypotheses of the Theorem, we do not know whether (E, X) has the $(1+\varepsilon)$-E.P. for every $\varepsilon > 0$ when E fails the approximation property. The proof we give yields only that (E, X) has the $(3+\varepsilon)$-E.P. for all $\varepsilon > 0$.

We know very little about the extension problem for general pairs (E, X). However, the Theorem makes the following small contribution in the general case.

COROLLARY 1.1. *Let E be a subspace of the separable space X. Assume that there is a weak*-closed subspace F of ℓ_1 such that X/E is isomorphic to ℓ_1/F. Then (E, X) has the E.P.*

Proof. Let $Q : \ell_1 \to X$ and $S : X \to X/E$ be quotient maps. Theorem 2 of [LR] implies that there is an automorphism of ℓ_1 which maps $Q^{-1}E = \ker(SQ)$ onto F. Since (F, ℓ_1) has the E.P. by our Theorem, so does the pair $(Q^{-1}E, \ell_1)$. It follows from Proposition 1.1 that (E, X) has the E.P. ■

We use standard Banach space theory notation and terminology, as may be found in [LT1], [LT2].

2. Preliminaries.

Let E be a subspace of X, $\lambda \ge 1$, and $0 < \varepsilon < 1$. Given an operator $S : E \to Y$ we say that the operator $T : X \to Y$ is a *(λ, ε)-approximate extension* of S if $\|T\| \le \lambda\|S\|$ and

$$\|S - T|_E\| \le \varepsilon\|S\|.$$

Our first observation is that the existence of approximate extensions implies the existence of extensions.

LEMMA 2.1. *Let E be a subspace of X and assume that each operator $S : E \to Y$ has a (λ, ε)-approximate extension. Then the pair (E, X) has the μ-E.P. with $\mu \le \lambda(1-\varepsilon)^{-1}$.*

Proof. Put $S_1 = S$ and let T_1 be a (λ, ε)-approximate extension of S_1. Then $\|T_1\| \le \lambda\|S_1\| = \lambda\|S\|$ and $\|S_1 - T_1|_E\| \le \varepsilon\|S\|$. Construct by induction sequences of operators $\{S_n\}_{n=1}^\infty$ from E into Y and $\{T_n\}_{n=1}^\infty$ from X into Y such that for each $n \ge 1$, $S_{n+1} = S_n - T_n|_E$ and T_{n+1} is a (λ, ε)-approximate extension of S_{n+1}. Then, by definition, $\|T_n\| \le \lambda\|S_n\|$ and $\|S_{n+1}\| \le \varepsilon\|S_n\|$ for every $n \ge 1$. It follows that $\|S - \sum_{i=1}^n T_i|_E\| \le \varepsilon^n\|S\|$ and $\|T_n\| \le \lambda\varepsilon^{n-1}\|S\|$ for all $n \ge 1$. Hence the operator $T = \sum_{i=1}^\infty T_i$ extends S and $\|T\| \le \lambda(1-\varepsilon)^{-1}\|S\|$. ■

Given a finite-dimensional decomposition (FDD, in short) $\{Z_n\}_{n=1}^\infty$ of a space Z, we will be interested in subspaces of Z with FDD's which are particularly well-positioned with respect to $\{Z_n\}_{n=1}^\infty$.

DEFINITION. Let $F \subset Z$ and let $\{F_n\}_{n=1}^\infty$ be an FDD for F. We say that $\{F_n\}_{n=1}^\infty$ is *alternately disjointly supported* with respect to $\{Z_n\}_{n=1}^\infty$ if there exist integers $1 = k(1) < k(2) < \ldots$ such that for each $n \ge 1$, $F_n \subset Z_{k(n)} + Z_{k(n)+1} + \ldots + Z_{k(n+2)-1}$.

An important property of an alternatively disjointly supported FDD is that if $\{n(j)\}_{j=1}^\infty$ is any increasing sequence of integers and if we drop $\{F_{n(j)}\}_{j=1}^\infty$, then the remaining F_n's can be grouped into blocks

$$\widetilde{F}_j = \sum_{i=n(j)+1}^{n(j+1)-1} F_i$$

which form an FDD that is disjointly supported on the $\{Z_n\}_{n=1}^\infty$; more precisely, with the above notation,

$$\widetilde{F}_j \subset \sum_{m=k(n(j)+1)}^{k(n(j+1)+1)-1} Z_m \quad \text{for all } j \ge 1.$$

We will show that for certain subspaces of a dual space with an FDD, a given FDD can be replaced by one which is alternately disjointly supported.

We first need the following main tool:

PROPOSITION 2.1. *Let $\{X_n\}_{n=1}^\infty$ be a shrinking FDD for X, let Q be a quotient mapping of X onto Y and suppose that $\{\widetilde{E}_n\}_{n=1}^\infty$ is an FDD for Y. Then there are a blocking $\{E'_n\}_{n=1}^\infty$ of $\{\widetilde{E}_n\}_{n=1}^\infty$, an FDD $\{W_n\}_{n=1}^\infty$ of X which is equivalent to $\{X_n\}_{n=1}^\infty$, and $1 = k(1) < k(2) < \ldots$ so that for each*

n and each $k(n) \le j < k(n+1)$, $QW_j \subset E'_n + E'_{n+1}$. Moreover, given $\varepsilon > 0$, $\{E'_n\}_{n=1}^\infty$ and $\{W_n\}_{n=1}^\infty$ can be chosen so that there is an automorphism T on X with $\|I - T\| < \varepsilon$ and $TX_n = W_n$ for all n.

Proof. In order to avoid complicated notation we shall prove the statement for the case where, for every $n \ge 1$, X_n (and hence also W_n) is one-dimensional. The same arguments, with only obvious modifications, yield the FDD case. (Actually, in the proof of the Theorem, only the basis case of Proposition 2.1 is needed. Indeed, in Step 3 of the proof of the Theorem, one can replace E by $E_1 \equiv E \oplus_1 (\sum G_n)_1$ and X by $X_1 = X \oplus_1 (\sum G_n)_1$, where $\{G_n\}_{n=1}^\infty$ is a sequence which is dense in the sense of the Banach–Mazur distance in the set of all finite-dimensional spaces, and use the fact [JRZ], [Peł] that E_1 has a basis. In fact, this trick is used in a different way for the proof of the "moreover" statement in the Theorem.)

So assume that X has a normalized shrinking basis $\{x_n\}_{n=1}^\infty$ with biorthogonal functionals $\{f_n\}_{n=1}^\infty$; we are looking for an equivalent basis $\{w_n\}_{n=1}^\infty$ of X for which the statement holds. First we perturb the basis for X to get another basis whose images under Q are supported on finitely many of the $\widetilde{E}_n$'s. This step does not require the hypothesis that $\{x_n\}_{n=1}^\infty$ be shrinking.

For each $n \ge 1$ let $\widetilde{Q}_n$ be the FDD's natural projection from Y onto $\widetilde{E}_1 + \ldots + \widetilde{E}_n$. Let $1 > \varepsilon > 0$ and set $C = \sup_n \|f_n\|$. Choose $p_1 < p_2 < \ldots$ so that for each n, $\|Qx_n - \widetilde{Q}_{p_n} Qx_n\| < \varepsilon C^{-1} 2^{-n}$. Since Q is a quotient mapping, there is for each n a vector z_n in X with $\|z_n\| < \varepsilon C^{-1} 2^{-n}$ and $Qz_n = Qx_n - \widetilde{Q}_{p_n} Qx_n$. Let $y_n = x_n - z_n$, so that Qy_n is in $\widetilde{E}_1 + \ldots + \widetilde{E}_{p_n}$. It is standard to check that $\{y_n\}_{n=1}^\infty$ is equivalent to $\{x_n\}_{n=1}^\infty$. Indeed, define an operator S on X by $Sx = \sum_{n=1}^\infty f_n(x) z_n$. Then $\|S\| < \varepsilon$ and $Sx_n = z_n$, so $I - S$ is an isomorphism from X onto X which maps x_n to y_n.

Define a blocking $\{E_n\}_{n=1}^\infty$ of $\{\widetilde{E}_n\}_{n=1}^\infty$ by $E_n = \widetilde{E}_{p_{n-1}+1} + \ldots + \widetilde{E}_{p_n}$ (where $p_0 \equiv 0$). Then for each n, Qy_n is in $E_1 + \ldots + E_n$.

Let Q_n be the basis projection from Y onto $E_1 + \ldots + E_n$, P_n the basis projection from X onto $\operatorname{span}\{y_1, \ldots, y_n\}$, and set $C_1 = \sup_n \|P_n\|$. Since $\{y_n\}_{n=1}^\infty$ is shrinking, $\lim_{m\to\infty} \|Q_n Q(I - P_m)\| = 0$ for each n. Since Q is a quotient mapping, for each n there exists a mapping T_n from $E_1 + \ldots + E_n$ into X so that QT_n is the identity on $E_1 + \ldots + E_n$. Set $M_n = \|T_n\|$, let $1 > \varepsilon > 0$, and recursively choose $0 = k(0) < 1 = k(1) < k(2) < \ldots$ so that for each n, $\|Q_{k(n)} Q(I - P_{k(n+1)-1})\| < (2C_1 M_{k(n)})^{-1} 2^{-n} \varepsilon$. Setting $w_j = y_j - T_{k(n)} Q_{k(n)} Q y_j$ for $k(n+1) \le j < k(n+2)$, we see that Qw_j is in $E_{k(n)+1} + \ldots + E_{k(n+2)}$ when $k(n+1) \le j < k(n+2)$.

The desired blocking of $\{\widetilde{E}_n\}_{n=1}^\infty$ is defined by $E'_n = E_{k(n-1)+1} + E_{k(n-1)+2} + \ldots + E_{k(n)}$, but it remains to be seen that $\{w_n\}_{n=1}^\infty$ is a suitably small perturbation of $\{y_n\}_{n=1}^\infty$. The inequality $\|Q_{k(n)} Q(I - P_{k(n+1)-1})\| < (2C_1 M_{k(n)})^{-1} 2^{-n} \varepsilon$ implies, by composing on the right with $P_{k(n+2)-1}$, that $\|Q_{k(n)} Q(P_{k(n+2)-1} - P_{k(n+1)-1})\| < (2M_{k(n)})^{-1} 2^{-n} \varepsilon$. Thus if we define an operator V on X by $Vx = \sum_{n=0}^\infty T_{k(n)} Q_{k(n)} Q(P_{k(n+2)-1} - P_{k(n+1)-1})x$, we see that $\|V\| < \varepsilon$ and hence $T \equiv I - V$ is invertible. But for $k(n+1) \le j < k(n+2)$, $Vy_j = T_{k(n)} Q_{k(n)} Q x_j$; that is, $Ty_j = w_j$. ∎

Using a duality argument we get from Proposition 2.1 the following.

COROLLARY 2.1. *Let $\{Z_n\}_{n=1}^\infty$ be an ℓ_1-FDD for a space Z. Regard Z as the dual of the space $Z_* = (\sum Z_n^*)_{c_0}$ and let F be a weak*-closed subspace of Z with an FDD. Then Z and F have ℓ_1-FDD's $\{V_n\}_{n=1}^\infty$ and $\{U_n\}_{n=1}^\infty$, respectively, so that $\{U_n\}_{n=1}^\infty$ is alternately disjointly supported with respect to $\{V_n\}_{n=1}^\infty$. Moreover, given $\varepsilon > 0$, $\{V_n\}_{n=1}^\infty$ can be chosen so that for some blocking $\{Z'_n\}_{n=1}^\infty$ of $\{Z_n\}_{n=1}^\infty$, there is an automorphism T of Z_* with $\|I - T\| < \varepsilon$ and $TZ'_n = V_n$ for all $n \ge 1$.*

Proof. Being weak*-closed, F has a predual $F_* = Z_*/F_\perp$ which is a quotient space of Z_*. By [JRZ], F_* has a shrinking FDD and consequently, by Theorem 1 of [JZ1], F_* has a shrinking c_0-FDD $\{\widetilde{E}_n\}_{n=1}^\infty$. Let $Q : Z_* \to F_*$ be the quotient mapping. By Proposition 2.1 there are a blocking $\{E'_n\}_{n=1}^\infty$ of $\{\widetilde{E}_n\}_{n=1}^\infty$, an FDD $\{W_n\}_{n=1}^\infty$ of Z_* which is equivalent to $\{Z_n^*\}_{n=1}^\infty$, even the image of $\{Z_n^*\}_{n=1}^\infty$ under some automorphism on Z_* which is arbitrarily close to I_{Z_*}, and $1 = k(1) < k(2) < \ldots$ so that for each n and $k(n) \le j < k(n+1)$, $QW_j \subset E'_n + E'_{n+1}$. The equivalence implies that $\{W_n\}_{n=1}^\infty$ is a c_0-FDD and, being a blocking of a c_0-FDD, $\{E'_n\}_{n=1}^\infty$ is a c_0-FDD. Let $\{V_n\}_{n=1}^\infty$ (resp. $\{U_n\}_{n=1}^\infty$) be the dual FDD of $\{W_n\}_{n=1}^\infty$ (resp. $\{E'_n\}_{n=1}^\infty$) for Z (resp. F). Then $\{V_n\}_{n=1}^\infty$ is an ℓ_1-FDD for Z and $\{U_n\}_{n=1}^\infty$ is an ℓ_1-FDD for F. Moreover, suppose that u is in U_n and w_j is in W_j, where either $j < k(n)$ or $j \ge k(n+2)$. Let m be the integer for which $k(m) \le j < k(m+1)$. Then either $m < n$ or $m > n+1$ hence $n \ne m$ and $n \ne m+1$. Then $Qw_j \in E'_m + E'_{m+1}$, hence $u(w_j) = \langle u, Qw_j \rangle = 0$. This proves that U_n is supported on $\sum_{j=k(n)}^{k(n+2)-1} V_j$. ∎

3. Proof of the Theorem.

The proof consists of four parts, the first three of which are essentially simple special cases of the Theorem.

STEP 1. *E has an FDD $\{E_n\}_{n=1}^\infty$ with $E_n \subset X_n$ for all n.*

Proof. Let $Y = C(K)$ and let $S : E \to Y$ be any operator. Using the $\mathcal{L}_{\infty,1+\varepsilon}$-property of Y (or see Theorem 6.1 of [Lin]), one sees that the finite rank operator $S|_{E_n}$ has an extension $S_n : X_n \to Y$ with $\|S_n\| \le (1+\varepsilon)\|S_n\|$. Define the extension $\mathbf{S}$ of S by $\mathbf{S}(\sum_{n=1}^\infty x_n) = \sum_{n=1}^\infty S_n x_n$. Since $\{X_n\}_{n=1}^\infty$ is an exact ℓ_1-decomposition, it follows that $\|\mathbf{S}\| \le (1+\varepsilon)\|S\|$.

STEP 2. *E has an ℓ_1-FDD $\{E_n\}_{n=1}^\infty$ which is alternately disjointly supported with respect to $\{X_n\}_{n=1}^\infty$.*

Proof. Given $\delta > 0$, let $1 < (1+\varepsilon)(1-\varepsilon)^{-1} < 1+\delta$ and choose an integer $N > (2+\varepsilon)M\varepsilon^{-1}$ where M is the constant of the ℓ_1-FDD $\{E_n\}_{n=1}^\infty$, that is, the constant of equivalence of $\{E_n\}_{n=1}^\infty$ to the natural ℓ_1-FDD for $(\sum E_n)_1$. Let $Y = C(K)$ and let $S : E \to Y$ be an operator with $\|S\| = 1$. For each $1 \le j \le N$ let

$$Z_j = \overline{\operatorname{span}}\{E_i : i \ne kN + j,\ k = 0, 1, 2, \ldots\}.$$

Each subspace Z_j has a natural ℓ_1-FDD which is disjointly supported with respect to $\{X_n\}_{n=1}^\infty$ because $\{E_n\}_{n=1}^\infty$ is alternately disjointly supported with respect to $\{X_n\}_{n=1}^\infty$. By Step 1, $S|_{Z_j}$ has an extension $T_j : X \to Y$ with

$$\|T_j\| \le (1+\varepsilon)\|S_j\| \le (1+\varepsilon)\|S\| = 1+\varepsilon.$$

Define $T : Z \to Y$ by $T = N^{-1}\sum_{j=1}^N T_j$. Then $\|T\| \le (1+\varepsilon)\|S\| = 1+\varepsilon$. Moreover, if $e \in E_i$ and $i = kN + h$ for some $1 \le h \le N$, then $T_j e = S_j e = Se$ for all $j \ne h$, hence T is "almost" an extension of S. Indeed, $\|Te - Se\| = N^{-1}\|T_h e - Se\| \le (2+\varepsilon)N^{-1}\|e\|$ whenever $e \in E_i$ for some i. Recalling that the ℓ_1-FDD $\{E_n\}_{n=1}^\infty$ has constant M, we have

$$\|T|_E - S\| \le M \sup_n \|T|_{E_n} - S|_{E_n}\| \le M(2+\varepsilon)N^{-1} < \varepsilon.$$

This proves that T is a $(1+\varepsilon, \varepsilon)$-approximate extension of S and therefore, by Lemma 2.1, (E, Z) has the $(1+\varepsilon)(1-\varepsilon)^{-1}$-E.P.

Step 3. *E has an FDD.*

Proof. By Corollary 2.1 we see that X and E have ℓ_1-FDD's $\{Z_n\}_{n=1}^\infty$ and $\{E_n\}_{n=1}^\infty$, respectively, where $\{E_n\}_{n=1}^\infty$ is alternately disjointly supported with respect to $\{Z_n\}_{n=1}^\infty$, and, by Remark 2.1, $\{Z_n\}_{n=1}^\infty$ has constant of equivalence to $(\sum Z_n)_1$ arbitrarily close to one. Hence, by Step 2, (E, X) has the $(1+\delta)$-E.P. for every $\delta > 0$.

This gives the "moreover" statement when E has an FDD. When E just has the approximation property, we enlarge X to $X_1 \equiv X \oplus_1 C_1$, where $C_1 = (\sum G_n)_1$ and $\{G_n\}_{n=1}^\infty$ is a sequence of finite-dimensional spaces which is dense (in the sense of the Banach–Mazur distance) in the set of all finite-dimensional spaces; and we enlarge E to $E_1 \equiv E \oplus_1 C_1$. Again, X_1 is an exact ℓ_1-sum of finite-dimensional spaces and E_1 is weak*-closed in X_1. Moreover, since E is a separable dual space which has the approximation property, E has the metric approximation property [LT1], and hence by [Joh], E_1 is a π-space, whence, since E_1 is a dual space, E_1 has an FDD by [JRZ]. Thus by Step 3, (E_1, X_1) has the $(1+\delta)$-E.P. for each $\delta > 0$, and, therefore, so does (E, X).

Step 4. *The general case.*

We start with a lemma.

Lemma 3.1. *Let Z be a Banach space and let E be a subspace of Z. Suppose that E has a subspace F such that (F, Z) has the λ-E.P. and $(E/F, Z/F)$ has the μ-E.P. Then (E, Z) has the $(\lambda + \mu(1+\lambda))$-E.P.*

Proof. Let $Y = C(K)$ and let $S : E \to Y$ be any operator. Let $S_1 : Z \to Y$ be an extension of $S|_F$ with $\|S_1\| \le \lambda\|S\|$. The operator $W = S - S_1|_E$ from E into Y vanishes on F and so induces an operator $\widetilde{W} : E/F \to Y$ in the usual way, and $\|\widetilde{W}\| = \|W\| \le \|S\| + \|S_1\| \le (1+\lambda)\|S\|$. By our assumptions, $\widetilde{W}$ extends to an operator $W_1 : Z/F \to Y$ with $\|W_1\| \le \mu\|\widetilde{W}\| \le \mu(1+\lambda)\|S\|$. Let $Q : Z \to Z/F$ denote the quotient map. Then $T = S_1 + W_1 Q$ is the desired extension of S. Indeed, for every $e \in E$,

$$Te = S_1 e + W_1 Qe = S_1 e + We = S_1 e + (S - S_1)e = Se$$

and $\|T\| \le \|S_1\| + \|W_1\| \le (\lambda + \mu(1+\lambda))\|S\|$. ∎

Let us now return to the proof of the general case. Being a weak*-closed subspace of ℓ_1, E is the dual of the quotient space $E_* = (\sum X_n^*)_{c_0}/E_\perp$. Our main tool in this part of the proof is Theorem IV.4 of [JR] and its proof. This theorem states that E_* has a subspace V so that both V and E_*/V have shrinking FDD's. Under these circumstances, Theorem 1 of [JZ1] implies that both V and E_*/V have c_0-FDD's. In order to prove our Theorem it suffices, in view of Lemma 3.1, to show that both pairs $(V^\perp, X)$ and $(E/V^\perp, X/V^\perp)$ have the E.P. Now, $(V^\perp, X)$ has the $(1+\delta)$-E.P. for all $\delta > 0$ by Step 3, so it remains to discuss the pair $(E/V^\perp, X/V^\perp)$. This discussion requires some preparation and some minor modification in the proof of Theorem IV.4 of [JR]. We first need a known perturbation lemma:

Lemma 3.2. *Suppose E, F are subspaces of X^* with F norm dense in X^* and X^* is separable. Then for each $\varepsilon > 0$ there is an automorphism T on X so that $\|I - T\| < \varepsilon$ and $T^*E \cap F$ is norm dense in T^*E.*

Proof. Let (x_n, x_n^*) be a biorthogonal sequence in $X \times E$ with $\overline{\operatorname{span}}\, x_n^* = E$ (see, e.g., [Mac]) and take $y_n^* \in F$ so that $\sum \|x_n^* - y_n^*\|\,\|x_n\| < \varepsilon$. Define $T : X \to X$ by

$$Tx = x - \sum_{n=1}^\infty \langle x_n^* - y_n^*, x\rangle x_n. \ ∎$$

Returning to the proof of the Theorem, we may assume, in view of Lemma 3.2, that $E \cap \operatorname{span}\bigcup_{n=1}^\infty X_n$ is norm dense in E. The standard back-and-forth technique [Mac] for producing biorthogonal sequences yields a biorthogonal sequence $\{(x_n, x_n^*)\}_{n=1}^\infty \subset X_* \times E$ with $\operatorname{span}\{Qx_n\}_{n=1}^\infty = \operatorname{span}\bigcup_{n=1}^\infty QX_n^*$, $\operatorname{span}\{x_n^*\}_{n=1}^\infty = E \cap \operatorname{span}\bigcup_{n=1}^\infty X_n$, and where Q is the quotient mapping from the predual $X_* = (\sum X_n^*)_{c_0}$ of X onto the predual E_* of E. This means that for any N, x_j^* is in $\operatorname{span}\bigcup_{n=N}^\infty X_n$ if j is sufficiently large.

We now refer to the construction in Theorem IV.4 of [JR] and the finite sets $\Delta_1 \subset \Delta_2 \subset \ldots$ of natural numbers defined there. From that construction, it is clear that, having defined Δ_n, the smallest element, $k(n)$, in $\Delta_{n+1} \setminus \Delta_n$ can be as large as we desire. In particular, if $\{x_j^*\}_{j=1}^{\max \Delta_n}$ is a subset of span $\bigcup_{i=1}^{m(n)} X_i$, then we choose $k(n)$ large enough so that for $j \geq k(n)$, x_j^* is in span $\bigcup_{i=m(n)+1}^{\infty} X_i$. Thus setting

$$Z_n = \operatorname{span}\{x_j^* : j \in \Delta_n \setminus \Delta_{n+1}\}$$

(where $\Delta_0 \equiv \emptyset$), we infer that $\{Z_n\}_{n=1}^{\infty}$ is disjointly supported relative to $\{X_n\}_{n=1}^{\infty}$. In the notation above and setting $m(0) = 0$, we have, for each n,

$$(*) \qquad Z_n \subset \operatorname{span}\{X_j\}_{j=m(n-1)+1}^{m(n)}.$$

The subspace V of E_* is defined to be the annihilator of $\{x_j^* : j \in \bigcup_{n=1}^{\infty} \Delta_n\}$ and, as mentioned earlier, it follows from [JR] and [JZ1] that V has a c_0-FDD and thus $V^* = E/V^{\perp}$ has an ℓ_1-FDD. It is also proved in [JR], but is obvious from the "extra" we have added here, that $\overline{\operatorname{span}}\{Z_j\}_{j=1}^{\infty}$ is weak*-closed and hence equals $V^{\perp}$. It is also obvious from $(*)$ that $X/V^{\perp}$ has an ℓ_1-FDD. Therefore, by Step 3, $(E_*/V^{\perp}, X/V^{\perp})$ has the E.P. ∎

The Extension Property is concerned with extension of operators into $C(K)$ spaces. However, in the proof of the Theorem, the only place where the fact was used that the range of the mapping is a $C(K)$ space was in Step 1, where we needed to extend an operator from a finite-dimensional subspace. This uses only the $\mathcal{L}_\infty$-property of $C(K)$ spaces, so we can state a formally stronger version of the Theorem:

COROLLARY 3.1. *Let $\{X_n\}_{n=1}^{\infty}$ be finite-dimensional and let E be a weak*-closed subspace of $X = (\sum X_n)_1$, regarded as the dual of $X_* = (\sum X_n^*)_{c_0}$. Let T be an operator from E into an $\mathcal{L}_{\infty,\lambda}$ space Y. Then there is an extension of T to an operator $\mathbf{T}$ from X into Y. Moreover, if E has the approximation property, then for any $\varepsilon > 0$, $\mathbf{T}$ can be chosen so that $\|\mathbf{T}\| \leq (\lambda + \varepsilon)\|T\|$.*

4. Concluding remarks and problems. Very little is known about the Extension Property, so there is no shortage of problems.

PROBLEM 4.1. *If E is a subspace of X and X is reflexive, does (E, X) have the E.P.? What if X is superreflexive? What if X is L_p, $1 < p \neq 2 < \infty$?*

PROBLEM 4.2. *If E is a reflexive subspace of the separable space X, does (E, X) have the E.P.? What if E is just isomorphic to a conjugate space? In the latter case, what if, in addition, X is ℓ_1?*

In Problem 4.2 it is necessary to restrict attention to separable X to avoid known counterexamples. (If E is an infinite-dimensional reflexive subspace of ℓ_∞, then no isomorphism from E into $C[0,1]$ can extend to an operator from ℓ_∞ into $C[0,1]$.)

If E is a subspace of c_0, then (E, c_0) has the $(1+\varepsilon)$-E.P. for every $\varepsilon > 0$ [LP] but need not have the 1-E.P. [JZ2]. We do not know if this phenomenon can occur in the setting of "nice" spaces:

PROBLEM 4.3. *If X is a reflexive smooth space and (E, X) has the $(1+\varepsilon)$-E.P. for every $\varepsilon > 0$, does (E, X) have the 1-E.P.?*

The following observation gives an affirmative answer to Problem 4.3 in a special case.

PROPOSITION 4.1. *If X is uniformly smooth and (E, X) has the $(1+\varepsilon)$-E.P. for every $\varepsilon > 0$, then (E, X) has the 1-E.P.*

Proof. In preparation for the proof, we recall Proposition 2 of [Zip], which says:

(E, X) has the λ-E.P. if and only if there exists a weak-continuous extension mapping from* Ball E^* *to* λ Ball X^*, *that is, a continuous mapping* $\phi : (\operatorname{Ball} E^*, \text{weak}^*) \to (\lambda \operatorname{Ball} X^*, \text{weak}^*)$ *for which* $(\phi e^*)|_E = e^*$ *for every* e^* *in* Ball E^*.

Since X is uniformly smooth, given $\varepsilon > 0$ there exists $\delta > 0$ so that if x^*, y^* in X^* and x in X satisfy $\|x^*\| = \|x\| = 1 = \langle x^*, x\rangle = \langle y^*, x\rangle$ with $\|y^*\| < 1 + \delta$, then $\|x^* - y^*\| < \varepsilon$. Letting $\phi_n : \operatorname{Ball} E^* \to (1 + n^{-1}) \operatorname{Ball} X^*$ be a weakly continuous extension mapping and letting $f : \operatorname{Sphere} E^* \to \operatorname{Sphere} X^*$ be the (uniquely defined, by smoothness) Hahn–Banach extension mapping, we conclude that

$$\lim_{n\to\infty} \sup\{\|\phi_n(x^*) - f(x^*)\| : x^* \in \operatorname{Sphere} E^*\} = 0.$$

That is, $\{\phi_n|_{\operatorname{Sphere} E^*}\}_{n=1}^{\infty}$ is uniformly convergent to $f|_{\operatorname{Sphere} E^*}$. Since each ϕ_n is weakly continuous, so is $f|_{\operatorname{Sphere} E^*}$.

If E is finite-dimensional, then clearly the positively homogeneous extension of f to a mapping from Ball E^* into Ball X^* is a weakly continuous extension mapping. So assume that E has infinite dimension. But then Sphere E^* is weakly dense in Ball E^*, so by the weak continuity of the ϕ_n's and the weak lower semicontinuity of the norm, we have

$$\begin{aligned}\sup\{\|\phi_n(x^*) - \phi_m(x^*)\| : x^* \in \operatorname{Ball} E^*\} \\ = \sup\{\|\phi_n(x^*) - \phi_m(x^*)\| : x^* \in \operatorname{Sphere} E^*\},\end{aligned}$$

which we saw tends to zero as n, m tend to infinity. That is, $\{\phi_n\}_{n=1}^{\infty}$ is a uniformly Cauchy sequence of weakly continuous functions and hence its limit is also weakly continuous. ∎

It is apparent from the proof of Proposition 4.1 that the 1-E.P. is fairly easy to study in a smooth reflexive space X because every extension mapping from $\operatorname{Ball} E^*$ to $\operatorname{Ball} X^*$ is, on the unit sphere of E^*, the unique Hahn–Banach extension mapping. Let us examine this situation a bit more in the general case. Suppose E is a subspace of X and let $A(E)$ be the collection of all norm one functionals in E^* which attain their norm at a point of $\operatorname{Ball} E$. The Bishop–Phelps theorem [BP], [Die] says that $A(E)$ is norm dense in $\operatorname{Sphere} E^*$, hence, if E has infinite dimension, $A(E)$ is weak*-dense in $\operatorname{Ball} E^*$. Therefore (E, X) has the 1-E.P. if and only if there is a weak*-continuous Hahn–Banach selection mapping $\phi : A(E) \to \operatorname{Ball} X^*$ which has a weak*-continuous extension to a mapping ϕ from $\overline{A(E)}^{w^*} = \operatorname{Ball} E^*$ to $\operatorname{Ball} X^*$, since clearly ϕ will then be an extension mapping. The existence of ϕ is equivalent to saying that whenever $\{x_\alpha^*\}$ is a net in $A(E)$ which weak* converges in E^*, then $\{\phi x_\alpha^*\}$ weak* converges in X^* (see, for example, [Bou, I.8.5]). Now, when X is smooth, there is only one mapping ϕ to consider, and in this case the above discussion yields the next proposition when $\dim E = \infty$ (when $\dim E < \infty$ one extends from $\operatorname{Sphere} E^* = \overline{A(E)}^{w^*}$ to $\operatorname{Ball} E^*$ by homogeneity).

PROPOSITION 4.2. *Let E be a subspace of the smooth space X. The pair (E, X) fails the 1-E.P. if and only if there are nets $\{x_\alpha^*\}$, $\{y_\alpha^*\}$ of functionals in* $\operatorname{Sphere} X^*$ *which attain their norm at points of* $\operatorname{Sphere} E$ *and which weak* converge to distinct points x^* and y^*, respectively, which satisfy $x^*|_E = y^*|_E$.*

An immediate, but surprising to us, corollary to Proposition 4.2 is:

COROLLARY 4.1. *Let E be a subspace of the smooth space X. If the pair (E, X) fails the 1-E.P., then there is a subspace F of X of codimension one which contains E so that (F, X) fails the 1-E.P.*

Proof. Get x^*, y^* from Proposition 4.2 and set $F = \operatorname{span} E \cup (\ker x^* \cap \ker y^*)$. ∎

PROBLEM 4.4. *Is Corollary 4.1 true for a general space X?*

COROLLARY 4.2. *For $1 < p \neq 2 < \infty$, L_p has a subspace E for which (E, L_p) fails the 1-E.P.*

Proof. We regard L_p as $L_p(0,2)$ and make the identifications $L_p^* = L_q = L_q(0,2)$, where $q = p/(p-1)$ is the conjugate index to p. Let

$$f = \mathbf{1}_{(0,1/2)} - \mathbf{1}_{(1/2,1)}, \quad g = -2 \cdot \mathbf{1}_{(1/2,1)} - \mathbf{1}_{(1,2)},$$

regarded as elements of L_q, and define

$$E = (f-g)^\perp = \Big\{x \in L_p(0,2) : \int_0^2 x = 0\Big\}.$$

Notice that $|f|^{q-1} \operatorname{sign} f$ is in E, which implies that $1 = \|f\|_q = \|f\|_{L_p^*} = \|f|_E\|_{E^*}$. So f and g induce the same linear functional on E (we write $f|_E = g|_E$), and f is the unique Hahn–Banach extension of this functional to a functional in $L_p^* = L_q$.

CLAIM. *There exists h in L_q supported on $[0,1/2]$ so that $\int_0^2 h = 0 = \int_0^2 |g+h|^{q-1} \operatorname{sign}(g+h)$.*

Assume the claim. Set $\lambda = \|g+h\|_q$ and let $\{h_n\}_{n=1}^\infty$ be a sequence of functions which have the same distribution as h, are supported on $[0,1/2]$, and are probabilistically independent as random variables on $[0,1/2]$ with normalized Lebesgue measure. Then $g_n \equiv \lambda^{-1}(g+h_n)$ defines a sequence on the unit sphere of $L_q(0,2)$ which converges weakly to $\lambda^{-1} g$. Moreover, $|g_n|^{q-1} \operatorname{sign} g_n$ is in E, which means that as a linear functional on L_p, g_n attains its norm at a point on the unit sphere of E. In view of Proposition 4.2, to complete the proof it suffices to find a sequence $\{f_n\}_{n=1}^\infty$ on the unit sphere of L_q which converges weakly in L_q to $\lambda^{-1} f$ so that $|f_n|^{q-1} \operatorname{sign} f_n$ is in E. This is easy: take w supported on $[1,2]$ so that

$$\int_0^2 w = 0 = \int_0^2 |w|^{q-1} \operatorname{sign} w \quad \Big(= \int_0^2 |f+w|^{q-1} \operatorname{sign}(f+w)\Big)$$

and $\|f+w\|_q^q = 1 = 1 + \|w\|_q^q = \lambda^q$ (so w can be a multiple of $\mathbf{1}_{(1,3/2)} - \mathbf{1}_{(3/2,2)}$). Let $\{w_n\}_{n=1}^\infty$ be a sequence of functions which have the same distribution as w, are supported on $[1,2]$, and are probabilistically independent as random variables on $[1,2]$. Now set $f_n = \lambda^{-1}(f + w_n)$.

We turn to the proof of the claim. Fix any $0 < \varepsilon < 1/4$. For appropriate d, the choice

$$h = d(4\varepsilon \mathbf{1}_{(0,1/4)} - \mathbf{1}_{(1/2-\varepsilon,1/2)})$$

works. Indeed, $\int_0^2 h = 0$ no matter what d is, and $gh = 0$, so we need choose d to satisfy

$$(*) \qquad -\int_0^2 |g|^{q-1} \operatorname{sign} g = \int_0^2 |h|^{q-1} \operatorname{sign} h.$$

The left side of $(*)$ is $2^{q-1} + 1 > 0$, while the right side is $|d|^{q-1} \operatorname{sign} d \varepsilon^{q-1} \times [(1/4)^{2-q} - \varepsilon^{2-q}]$, so such a choice of d is possible for $p \neq 2$. ∎

PROBLEM 4.5. *If E is a weak*-closed subspace of ℓ_1, does (E, ℓ_1) have the $(1+\varepsilon)$-E.P. for every $\varepsilon > 0$?*

A negative answer to Problem 4.5 would be particularly interesting, because it would justify the weird approach we used to prove the Theorem.

Samet [Sam1], [Sam2] proved that if E is a finite-dimensional subspace of ℓ_1 then (E, ℓ_1) has the 1-E.P. Our final proposition shows that for most weak*-closed hyperplanes E in ℓ_1, (E, ℓ_1) does not have the 1-E.P.

PROPOSITION 4.3. *Let $x = \{a_n\}_{n=1}^{\infty}$ be a norm one vector in c_0 with $a_n \neq 0$ for infinitely many n, and let $E = x^{\perp}$ in $\ell_1 = c_0^*$. Then (E, ℓ_1) does not have the* 1-*E.P.*

Proof. By using an onto isometry of c_0, we can assume, without loss of generality, that $a_1 = 1$ and a_{2n-1} is positive for each n. Assuming for contradiction that (E, ℓ_1) has the 1-E.P., we get from Proposition 2 of [Zip] a weak*-continuous extension mapping ϕ from $\operatorname{Ball} E^*$ to $\operatorname{Ball} \ell_\infty = \operatorname{Ball} \ell_1^*$. For $n = 1, 2, \ldots$, define a vector $x(n)$ in c_0 by having the first $2n$ coordinates agree with those of x, the $(2n+1)$th coordinate be minus one, and other coordinates be zero. Regarding the $x(n)$'s as linear functionals on ℓ_1, we have $x(n)|_E$ is in $\operatorname{Ball} E^*$ and $x(n)|_E \to 0$ weak* in E^*. We can write $\phi(x(n)|_E) = x(n) + b_n x$ and $\phi(-x(n)|_E) = -x(n) + c_n x$; by weak* continuity of ϕ, these two sequences must converge weak* in ℓ_1^* to the same functional, namely, to $\phi(0)$. Since ϕ maps into the unit ball, $|1 + b_n|$, $|-1 + b_n a_{2n+1}|$, $|-1 + c_n|$, and $|1 + c_n a_{2n+1}|$ are all at most one. Hence since $a_{2n+1} > 0$, $b_n = c_n = 0$. So $\phi(x(n)|_E) = x(n) \to x$ and $\phi(-x(n)|_E) = -x(n) \to -x$ weak* in ℓ_∞, which is a contradiction. ■

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