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An index formula for chains

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Abstract. We derive a formula for the index of Fredholm chains on normed spaces.

0. Introduction. The theory of the "index" of a Fredholm complex of bounded linear operators between Banach spaces seems to be complicated, involving ([2], [16]) "gaps" between subspaces, in contrast to the situation in Hilbert space, where with the help of the adjoint we can reduce the discussion to single operators ([6], [14], [15]). Following an idea of Putinar [12], Harte [10] showed how to some extent a "generalized inverse" could be used to the same effect on Banach spaces. We continue this discussion here, obtaining an expression for the "Euler number" of a Fredholm chain of bounded operators.

If X and Y are normed spaces we shall write $\mathrm{BL}(X,Y)$ for the set of all bounded linear operators from X to Y, and BL for the category of all bounded linear operators between normed spaces. Many of our definitions and results remain valid in more general additive categories; two important examples would be a single Banach algebra with identity, and the Calkin category obtained by quotienting out the compact operators $\mathrm{KL}(X,Y)$ from each $\mathrm{BL}(X,Y)$. If $T\in\mathrm{BL}(X,Y)$ and $S\in\mathrm{BL}(Y,Z)$ satisfy

$$ST=0 \tag{0.1}$$

we shall call the pair (S,T) a *chain*, and write

$$(S,T)\in\mathrm{BL}(X,Y,Z); \tag{0.2}$$

in particular, the pair (S,T) is *compatible* in the sense that the product ST is defined at all. More generally, a *chain* is a sequence $T_j : X_j\to X_{j+1}$ ($j=0,1,\dots,n$) for which

$$T_jT_{j-1}=0 \quad (j=1,\dots,n), \tag{0.3}$$

written $(T_n,\dots,T_1,T_0)\in\mathrm{BL}(X_0,X_1,\dots,X_{n+1})$. Whether or not the chain condition (0.1) is satisfied, the compatible pair (S,T) will be called *invertible*

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if there are $T' \in \mathrm{BL}(Y, X)$ and $S' \in \mathrm{BL}(Z, Y)$ for which

$$S'S + TT' = I; \tag{0.4}$$

a longer sequence $(T_0, T_1, \ldots, T_n)$ will be called *invertible* if each pair (T_j, T_{j-1}) satisfies the condition (0.4). We shall call the pair (S, T) *weakly invertible* if there is implication, for arbitrary compatible U and V in BL,

$$UT = SV = 0 \Rightarrow UV = 0, \tag{0.5}$$

with the corresponding extension to longer sequences. Necessary and sufficient for the chain condition (0.1) is the inclusion $T(X) \subseteq S^{-1}(0)$, or equivalently

$$\mathrm{cl}\, T(X) \subseteq S^{-1}(0); \tag{0.6}$$

for the weak invertibility (0.5) (with or without (0.1)) the condition is

$$S^{-1}(0) \subseteq \mathrm{cl}\, T(X). \tag{0.7}$$

We shall call the pair (S, T) *regular* if each operator is, so that there is a pair (T', S') for which

$$S = SS'S \quad \text{and} \quad T = TT'T. \tag{0.8}$$

It is familiar that if $T \in \mathrm{BL}(X, Y)$ has a generalized inverse then it also has a *normalized* generalized inverse, T', for which

$$T = TT'T \quad \text{and} \quad T' = T'TT'; \tag{0.9}$$

if (0.8) holds with $T' = U$ then (0.9) holds with $T' = UTU$. What is less familiar is that if a chain (S, T) is regular in the sense of (0.8) then (see [10], Theorem 2) (0.8) holds with another chain $(T', S') \in \mathrm{BL}(Z, Y, X)$: indeed, if (0.8) is satisfied with $(T', S') = (U, V)$ then (0.8) is also satisfied with the chain $(T', S') = (UTU, (I - TU)V)$. If (T', S') is a generalized inverse for the chain (S, T) then the projections $S'S$ and TT' satisfy $S'STT' = 0$ and hence (see [9], Theorem 2.5.4) $(I - TT')(I - S'S)$ is another projection, with

$$S^{-1}(0) = T(X) \oplus (I - TT')(I - S'S)(Y). \tag{0.10}$$

It is of course clear ([8], Theorem 1.6; [9], Theorem 10.3.3; [10], Theorem 1) that a chain is invertible iff it is both regular and weakly invertible; also ([10], Theorem 2), an invertible chain always has at least one inverse which is also a chain (for the same reason as a generalized inverse). Notice too ([10], Theorem 2) that the invertibility of a regular chain can be tested by that of a single operator:

1. THEOREM. *If (T', S') is a generalized inverse for the chain (S, T) then*

$$(S, T) \text{ invertible} \Leftrightarrow S'S + TT' \text{ invertible}. \tag{1.1}$$

Proof. If (T', S') is a generalized inverse for a weakly invertible (S, T) then (0.1), (0.5) and (0.8) give

$$S'S + TT' = I + TT'S'S = (I - TT'S'S)^{-1}, \tag{1.2}$$

so that $S'S + TT'$ is invertible; conversely, if

$$J(S'S + TT') = I = (S'S + TT')J \tag{1.3}$$

then

$$(I - TT'J)(I - JS'S) = S'SJ^2TT', \tag{1.4}$$

so that for the invertibility of (S, T) it is sufficient that

$$S'SJ^2TT' = 0. \tag{1.5}$$

But now (T', S') is a generalized inverse for (S, T); thus

$$SJ^2T = S(S'SJ)(JTT')T = S(I - TT'J)(I - JS'S)T = 0. \blacksquare$$

Notice that the generalized inverse (T', S') was not required to be normalized, nor to be a chain, for (1.1); if it is a chain then (1.2) gives $S'S + TT' = I$. Whether or not (T', S') is even a generalized inverse, if (T', S') is a chain for which $S'S + TT' = J^{-1}$ is invertible, then $S'S$ commutes with TT' and hence also with J, so that (1.5) again holds and (S, T) is invertible. Harte ([10], (2.8)) asked whether the condition (1.5) could be dispensed with altogether; the following rather simple example shows not:

2. EXAMPLE. *If $u : E \to E$ and $v : E \to E$ satisfy $vu = 1 \neq uv$ and*

$$S = \begin{pmatrix} 0 & 1 - uv \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} E \\ E \end{pmatrix} \to \begin{pmatrix} E \\ E \end{pmatrix} \tag{2.1}$$
$$T = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} : \begin{pmatrix} E \\ E \end{pmatrix} \to \begin{pmatrix} E \\ E \end{pmatrix}$$

then

$$S + T \text{ is invertible and } (S, T) \text{ is not invertible}. \tag{2.2}$$

Proof. An inverse for the operator $S + T$ is given by

$$\begin{pmatrix} v & 0 \\ 1 - uv & u \end{pmatrix},$$

but the condition (0.5) fails: indeed, with

$$U = \begin{pmatrix} 1 - uv & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 \\ u \end{pmatrix}$$

we have

$$SV = UT = 0 \neq UV. \blacksquare$$

If for example u and v are the unilateral shifts on one of the sequence spaces E then $S + T$ is essentially the bilateral shift.

Theorem 1 extends to chains of three or more operators ([10], Theorem 3): if the chain (R,S,T) has a generalized inverse (T',S',R') then

$$(2.3)\qquad (R,S,T) \text{ invertible} \Leftrightarrow \begin{pmatrix} S'S+TT' & 0 \\ 0 & R'R+SS' \end{pmatrix} \text{ invertible.}$$

We should remark that if the chain (R,S,T) is (generalized) invertible, so that (S,T) and (R,S) have (generalized) inverse chains (T',S') and (S'',R'') respectively, then ([10], Theorem 3) we can arrange $S''=S'$:

$$(2.4)\qquad S'S+TT' = I = R''R+SS'' \Rightarrow S'S+TT' = I = (I-SS')R''R+SS'.$$

Chains which begin and end in zeroes are special: for example ([10], Theorem 5), in the situation of (2.3),

$$(2.5)\qquad (0,R,S,T,0) \text{ invertible} \Leftrightarrow \begin{pmatrix} T & S' \\ 0 & R \end{pmatrix} \text{ and } \begin{pmatrix} T' & 0 \\ S & R' \end{pmatrix} \text{ invertible;}$$

if, in particular, (R',S',T') is a normalized generalized inverse chain then also, since the matrices are mutually generalized inverse,

$$(2.6)\qquad \begin{pmatrix} T & S' \\ 0 & R \end{pmatrix} \text{ invertible} \Leftrightarrow \begin{pmatrix} T' & 0 \\ S & R' \end{pmatrix} \text{ invertible,}$$

strengthening (2.5).

Theorem 1 and its relatives have immediate Fredholm analogues, obtained by repeating verbatim the arguments in the Calkin category, or alternatively the "finite Calkin category", in which the finite rank operators $\mathrm{KL}_0(X,Y)$ are quotiented out of each $\mathrm{BL}(X,Y)$. We recall ([9], Theorem 10.6.2; [10], Theorem 7) that a chain (S,T) is *Fredholm* if and only if it is regular and satisfies

$$(2.7)\qquad \dim S^{-1}(0)/\mathrm{cl}\,T(X) < \infty;$$

by itself the condition (2.6) is the analogue of condition (0.5) ("weakly Fredholm"). As before, a longer chain $(T_n,\ldots,T_1,T_0)$ will be called *(weakly) Fredholm* iff each pair (T_j,T_{j-1}) is. The *Euler number* of a weakly Fredholm chain will be defined ([9], (10.6.3.2); [10], (7.5)) as

$$(2.8)\qquad \mathrm{Euler}(S,T) = \dim S^{-1}(0)/\mathrm{cl}\,T(X),$$

and is extended to longer chains by setting

$$(2.9)\qquad \mathrm{Euler}(T_n,\ldots,T_1,T_0) = \sum_{k=1}^{n}(-1)^{k-1}\,\mathrm{Euler}(T_k,T_{k-1}).$$

Thus for example the *index* of a Fredholm operator is given by the Euler number of the induced chain:

$$(2.10)\qquad \mathrm{index}(T) = \mathrm{Euler}(0,T,0).$$

3. THEOREM. *If (S,T) is a regular chain, with generalized inverse (T',S'), then*

$$(3.1)\qquad S'S+TT' \textit{ Fredholm} \Rightarrow (S,T) \textit{ Fredholm} \Rightarrow S'S+TT' \textit{ Fredholm of index zero.}$$

Proof. If $S'S+TT'$ is Fredholm, so that there is J for which, analogously to (1.3), $I-J(S'S+TT')$ and $I-(S'S+TT')J$ are in KL_0, then also, analogously to (1.4), $(I-TT'J)(I-JS'S)$ is finite rank and hence (S,T) is Fredholm. Conversely, if (S,T) is weakly Fredholm then, analogously to (1.2), $S'S+TT' \in (I-TT'S'S)^{-1}+\mathrm{KL}_0$ is a finite rank perturbation of an invertible operator, therefore ([9], Theorem 6.5.2; [5], Exercise I.2.5; [4]) Fredholm of index zero. ∎

When (T',S') is a normalized generalized inverse chain for (S,T) then, by (0.10),

$$(3.2)\qquad \mathrm{Euler}(S,T) = \dim(S'S+TT')^{-1}(0).$$

Theorem 3 has extensions to chains (R,S,T) and $(0,R,S,T,0)$, and longer; however, recall [7] that an arbitrary chain of the form $(0,T_n,\ldots,T_1,0)$ can be represented by a single operator: for example the invertibility of a chain $(0,R,S,T,0)$ is equivalent to the invertibility of the chain

$$(3.3)\qquad \left(\begin{pmatrix} 0&0&0&0\\ T&0&0&0\\ 0&S&0&0\\ 0&0&R&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\ T&0&0&0\\ 0&S&0&0\\ 0&0&R&0 \end{pmatrix}\right).$$

Naturally, the same construction also tests for Fredholmness, but is ([7], (5.6)) unable to assist in the proof of the spectral mapping theorem, and also gets the Euler number wrong: for example the Euler number of the chain (3.3) is

$$(3.4)\qquad \dim(T'T)^{-1}(0)+\dim(S'S+TT')^{-1}(0) + \dim(R'R+SS')^{-1}(0)+\dim(RR')^{-1}(0),$$

losing the alternating sign. For that we need ([2], [3]) a "symmetrical chain", that is, a chain of the form (S,T,S). Thus the chain $(0,S,T,0)$ is represented by the chain

$$\left(\begin{pmatrix} 0\\ S \end{pmatrix}, \begin{pmatrix} T & 0 \end{pmatrix}, \begin{pmatrix} 0\\ S \end{pmatrix}\right);$$

4. THEOREM. *If (R,S,T) is a chain then*

$$(4.1)\qquad (0,R,S,T,0) \textit{ non-singular} \Leftrightarrow \left(\begin{pmatrix} 0&0\\ S&0 \end{pmatrix}, \begin{pmatrix} T&0\\ 0&R \end{pmatrix}, \begin{pmatrix} 0&0\\ S&0 \end{pmatrix}\right) \textit{ non-singular,}$$

where "non-singular" means "regular", "invertible" or "Fredholm". When the chains are both Fredholm then

$$\text{(4.2)}\qquad \operatorname{Euler}(0,R,S,T,0)=\operatorname{Euler}\left(\begin{pmatrix}0&0\\S&0\end{pmatrix},\begin{pmatrix}T&0\\0&R\end{pmatrix},\begin{pmatrix}0&0\\S&0\end{pmatrix}\right).$$

Proof. If $(0,T',S',R',0)$ is either a generalized inverse, or an inverse, or an essential inverse for $(0,R,S,T,0)$, then so is

$$\left(\begin{pmatrix}0&0\\S'&0\end{pmatrix},\begin{pmatrix}T'&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\S'&0\end{pmatrix}\right)$$

for the matrix chain in (4.1); conversely, any (generalized/essential) inverse for the matrix chain induces one for the chain $(0,R,S,T,0)$. To identify the Euler numbers use (3.2): if (T',S',R') is a normalized generalized inverse chain for (R,S,T) then

$$\dim\begin{pmatrix}T'T&0\\0&R'R+SS'\end{pmatrix}^{-1}\begin{pmatrix}0\\0\end{pmatrix}-\dim\begin{pmatrix}S'S+TT'&0\\0&RR'\end{pmatrix}^{-1}\begin{pmatrix}0\\0\end{pmatrix}$$

is the Euler number of the matrix chain. ∎

Analogously to (2.4), if a symmetric chain (S,T,S) has a (generalized) inverse then it has one which is a symmetric chain:

$$\text{(4.3)}\qquad S'S+TT'=I=SS''+T''T$$
$$\Rightarrow (S''SS')S+T(T'TT'')=I=S(S''SS')+(T'TT'')T.$$

The case of a symmetric chain (S,T,S) is at once a specialization of (2.3) and a generalization of (4.1):

5. Theorem. *If (S,T,S) is a regular chain, with normalized generalized inverse chain (S',T',S'), then*

$$\text{(5.1)}\qquad (S,T,S)\ \textit{invertible} \Leftrightarrow T+S'\ \textit{invertible} \Leftrightarrow S+T'\ \textit{invertible},$$

and

$$\text{(5.2)}\qquad (S,T,S)\ \textit{Fredholm} \Leftrightarrow T+S'\ \textit{Fredholm} \Leftrightarrow S+T'\ \textit{Fredholm}.$$

If (S',T',S') is an essential inverse chain for (S,T,S) then

$$\text{(5.3)}\qquad \operatorname{Euler}(S,T,S)=\operatorname{index}(T+S')=-\operatorname{index}(S+T').$$

Proof. Whether or not T' and S' are normalized, (2.3) gives, with (S,T,S) in place of (R,S,T),

$$\text{(5.4)}\qquad (S,T,S)\ \text{invertible} \Leftrightarrow T'T+SS'\ \text{and}\ S'S+TT'\ \text{invertible};$$

but since all the necessary products are zero,

$$\text{(5.5)}\qquad T'T+SS'=(S+T')(T+S')\quad\text{and}\quad S'S+TT'=(T+S')(S+T'),$$

so that $T'T+SS'$ and $TT'+S'T$ can be replaced in (5.4) by $T+S'$ and $S+T'$. Finally, these two operators are mutually generalized inverse: if $T'=T'TT'$ and $S'=S'SS'$ then

$$\text{(5.6)}\qquad \begin{aligned}T+S'&=(T+S')(S+T')(T+S'),\\ S+T'&=(S+T')(T+S')(S+T'),\end{aligned}$$

so that $T+S'$ is invertible iff $S+T'$ is. This establishes (5.1), and hence, translating into the finite Calkin category, (5.2). To compute the Euler number observe, by (0.10), that

$$\text{(5.7)}\qquad \begin{aligned}T^{-1}(0)/S(Y)&\cong(T'T+SS')^{-1}(0),\\ S^{-1}(0)/T(X)&\cong(S'S+TT')^{-1}(0),\end{aligned}$$

which with (5.5) and (5.6) gives (5.3). ∎

Theorem 4 ([10], Theorem 8) is a special case of Theorem 5: if (S,T,S) is replaced by

$$\left(\begin{pmatrix}0&0\\S&0\end{pmatrix},\begin{pmatrix}T&0\\0&R\end{pmatrix},\begin{pmatrix}0&0\\S&0\end{pmatrix}\right)$$

then $T+S'$ and $S+T'$ are replaced, respectively, by

$$\begin{pmatrix}T&S'\\0&R\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}T'&0\\S&R'\end{pmatrix}.$$

In the space of chains, invertible and symmetric chains form open sets, and the Euler number is continuous, that is, locally constant:

6. Theorem. *If (S,T,S) is an (essentially) invertible chain, with (essential) inverse chain (S',T',S'), and if $(S+H,T+K,S+H)$ is a chain so close to (S,T,S) that*

$$\text{(6.1)}\qquad I+S'H=(H')^{-1},\quad I+KT'=(K')^{-1}\quad\textit{and}$$
$$I+(K+(I-H')S')(T'+S)\ \textit{are invertible}$$

then $(S+H,T+K,S+H)$ is (essentially) invertible, with

$$\text{(6.2)}\qquad \operatorname{Euler}(S+H,T+K,S+H)=\operatorname{Euler}(S,T,S).$$

Proof. To see that invertible chains (S,T) form an open set ([10], Theorem 6) suppose (6.1) and remember the identity ([9], Theorem 10.3.4)

$$\text{(6.3)}\qquad S'(S+H)(I+KT')+(I+S'H)(T+K)T'$$
$$-(I+S'H)(I+KT')=S'S+TT'-I,$$

valid whenever $ST=0=(S+H)(T+K)$; now if the right hand side vanishes and we pre- and post-multiply by H' and K' we get

$$\text{(6.4)}\qquad (H'S')(S+H)+(T+K)(T'K')=I.$$

The inverse pair $(T'K', H'S')$ is also a chain: (6.1) implies also ([9], Theorem 3.1.3)

$$I + T'K = (K'')^{-1} \quad \text{and} \quad I + HS' = (H'')^{-1} \tag{6.5}$$

with $K'' = I + T'K'K$ and $H'' = I + HS'H'$, so that

$$K''T' = T'K' \quad \text{and} \quad S'H'' = H'S', \tag{6.6}$$

giving $(T'K')(H'S') = K''(T'S')H'' = 0$. This argument now extends [10] to invertible chains (R, S, T), and then in particular to symmetric chains (S, T, S); then taking cosets converts to the argument for Fredholm chains (S, T, S). Finally, the local constancy of the Euler number reduces to that of the index:

$$\begin{aligned} \mathrm{Euler}(S, T, S) &= \mathrm{index}(T + S') = \mathrm{index}(T + K + H'S') \\ &= \mathrm{Euler}(S + H, T + K, S + H). \end{aligned} \tag{6.7}$$

The middle equality is the third part of (6.1) together with Theorem 6.5.5 of [9]. ■

For Banach spaces X, Y and Z, all three parts of (6.1) can be arranged by making $\|H\|$ and $\|K\|$ sufficiently small.

Theorem 5 applies in particular to the symmetric chain derived from the *Koszul complex* ([13], [7], [9]) associated with a commuting system of Banach space operators: if $a = (a_1, \ldots, a_n)$ is a commuting n-tuple of operators acting on a Banach space E then its *Koszul complex* is the chain

$$(0, \Lambda_n(a), \ldots, \Lambda_1(a), 0) \tag{6.8}$$

induced by the operator

$$\Lambda(a) : x_0 + \sum_{k=1}^{n} \sum_{|j|=k} x_j dz_j \mapsto \sum_{i=1}^{n} \Big(a_i x_0 dz_i + \sum_{k=1}^{n} \sum_{|j|=k} a_i x_j dz_i \wedge dz_j \Big), \tag{6.9}$$

acting on the space of exterior differential forms in n complex variables with coefficients in the space E. In effect, $\Lambda(a)$ is the tensor product of exterior differentiation with the action of the operators a_j; each $\Lambda_j(a) : \Lambda_{j-1}(E) \to \Lambda_j(E)$ is the restriction of $\Lambda(a)$ to forms homogeneous of degree $j-1$, and takes them to forms homogeneous of degree j. The chain $(\Lambda(a), \Lambda(a))$ is derived from the chain (6.8) as in (3.3); the symmetric chain of (4.1) can be written

$$(\Lambda^{\mathrm{even}}(a), \Lambda^{\mathrm{odd}}(a), \Lambda^{\mathrm{even}}(a)), \tag{6.10}$$

where the operators $\Lambda^{\mathrm{even}}(a)$ and $\Lambda^{\mathrm{odd}}(a)$ are the restrictions of $\Lambda(a)$ to forms of even and of odd degree:

$$\Lambda(a) \cong \begin{pmatrix} 0 & \Lambda^{\mathrm{even}}(a) \\ \Lambda^{\mathrm{odd}}(a) & 0 \end{pmatrix} : \begin{pmatrix} \Lambda^{\mathrm{odd}}(E) \\ \Lambda^{\mathrm{even}}(E) \end{pmatrix} \to \begin{pmatrix} \Lambda^{\mathrm{odd}}(E) \\ \Lambda^{\mathrm{even}}(E) \end{pmatrix}. \tag{6.11}$$

We define the commuting n-tuple $a = (a_1, \ldots, a_n)$ to be *Taylor invertible* (or "splitting Taylor exact") iff the chain (6.8) is invertible in the sense of (0.4), or equivalently if the symmetric pair (6.10) is invertible in the sense of (0.4), and *Fredholm* if the chain (6.8), equivalently the chain (6.10), is Fredholm. When the commuting system a is Taylor Fredholm we define

$$\mathrm{index}(a) = \mathrm{Euler}(\Lambda^{\mathrm{even}}(a), \Lambda^{\mathrm{odd}}(a), \Lambda^{\mathrm{even}}(a)), \tag{6.12}$$

or equivalently the Euler number of the chain (6.8).

It is instructive to look at the case $n = 2$: the Koszul complex of a commuting pair (a, b) of operators on a Banach space E is given by

$$(0, S, T, 0) = \left(0, (-b \ \ a), \begin{pmatrix} a \\ b \end{pmatrix}, 0\right). \tag{6.13}$$

The regularity of S and T neither implies nor is implied by the regularity of a and b. For example if $b = 1$ then for arbitrary a the chain (6.13) is invertible and hence both S and T are regular. In the other direction, we have

7. Example. *If $w : F \to F$ is one-one and dense but not onto, and*

$$a = \begin{pmatrix} 0 & 1 \\ 0 & w \end{pmatrix}, \quad b = \begin{pmatrix} w & -1 \\ 0 & 0 \end{pmatrix}, \tag{7.1}$$

then

$$a \text{ and } b \text{ are both regular,} \tag{7.2}$$

while

$$\text{neither } S \text{ nor } T \text{ is regular.} \tag{7.3}$$

Proof. If

$$a' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -b' \tag{7.4}$$

then

$$a = aa'a \quad \text{and} \quad b = bb'b \quad \text{with } ba = 0 = ab \text{ and } a'b' = 0 = b'a', \tag{7.5}$$

while

$$T(X) \neq S^{-1}(0) = \mathrm{cl}\, T(X) \quad \text{and} \quad S(Y) \neq Z = \mathrm{cl}\, S(Y). \ ■ \tag{7.6}$$

Whether or not the operators a and b are regular, it is rather easy to see that the behaviour of the complex $(0, S, T, 0)$ is unchanged when we interchange a and b.

If both the operators a and b and the Koszul operators S and T are regular then we can hope for an expression for the generalized inverses of S and T in terms of those of a and b:

8. LEMMA. *If (a,b) is a commuting pair of bounded regular operators on a Banach space E which is Taylor Fredholm then we can find generalized inverse pairs (a',b') and (a'',b'') for which*

$$(8.1)\qquad (b'b)(a'a) = (a'a)(b'b) \quad \textit{and} \quad (bb'')(aa'') = (aa'')(bb'').$$

Proof. Note that if an element $a = aa^\wedge a$ is regular and if projections $p = p^2$ and $q = q^2$ satisfy

$$(8.2)\qquad a^{-1}(0) = p^{-1}(0) \quad \text{and} \quad \mathrm{cl}\, a(E) = q(E)$$

then we can find a generalized inverse a' for a with $p = a'a$ and $q = aa'$: simply take

$$(8.3)\qquad a' = pa^\wedge q.$$

Now if the pair (a,b) is middle Taylor exact then ([11], Theorem 4) there are equalities

$$(8.4)\qquad a(E)\cap b(E) = (ab)(E) \quad \text{and} \quad a^{-1}(0)+b^{-1}(0) = (ba)^{-1}(0),$$

and also ([11], Theorem 3) the product ba is regular. This holds to within finite dimensions in the Fredholm situation; thus $a(E)\cap b(E)$ and $a^{-1}(0)+b^{-1}(0)$ are also complemented, and we may write

$$(8.5)\qquad E = E_0\oplus E_1\oplus E_2\oplus E_3 = E_0'\oplus E_1'\oplus E_2'\oplus E_3',$$

with

$$(8.6)\qquad \begin{aligned} a(E) &= E_0\oplus E_1, & b(E) &= E_0\oplus E_2,\\ a^{-1}(0) &= E_0'\oplus E_1', & b^{-1}(0) &= E_0'\oplus E_2'. \end{aligned}$$

This means we can write

$$(8.7)\qquad e_0+e_1+e_2+e_3 = 1 = e_0'+e_1'+e_2'+e_3' \quad \text{with } e_ie_j = \delta_{ij}e_i \text{ and } e_i'e_j' = \delta_{ij}e_i'.$$

Thus by (8.3) we can find $a = aa'a$ and $b = bb'b$ with $a'a = e_2'+e_3'$ and $b'b = e_1'+e_3'$, and obtain the first part of (8.1); also, we can find $a = aa''a$ and $b = bb''b$ with $aa'' = e_0+e_1$ and $bb'' = e_0+e_2$, and obtain the second part. ∎

With such a choice of generalized inverses for a and b, we get formulae for the index of the Taylor Fredholm pair (a,b):

9. THEOREM. *If (a,b) is a commuting pair of regular operators on a Banach space E, and if the pair (a,b) is Taylor Fredholm, then we can arrange that*

$$(9.1)\qquad a = aa''a,\quad b = bb''b,\quad \mathrm{index}(a,b) = \mathrm{index}\begin{pmatrix} a & -b'' \\ b & a''(1-bb'') \end{pmatrix},$$

and also that

$$(9.2)\qquad a = aa'a,\quad b = bb'b,\quad \mathrm{index}(a,b) = \mathrm{index}\begin{pmatrix} a' & (1-a'a)b' \\ -b & a \end{pmatrix}.$$

Proof. Choosing (a',b') and (a'',b'') as in (8.1) gives

$$(9.3)\qquad \begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} a' & b' \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix},\\ \begin{pmatrix} -b & a \end{pmatrix} &= \begin{pmatrix} -b & a \end{pmatrix}\begin{pmatrix} -b'' \\ a'' \end{pmatrix}\begin{pmatrix} -b & a \end{pmatrix}. \end{aligned}$$

Now (6.12) and (5.3) give (9.1) and (9.2). ∎

We can sometimes deduce that the index is zero:

10. THEOREM. *If (a,b) is a commuting Taylor Fredholm pair for which either $(a(E)+b(E))/a(E)$ or $(a(E)+b(E))/b(E)$ is finite-dimensional then* $\mathrm{index}(a,b) = 0$.

Proof. If (a'',b'') is a generalized inverse pair for (a,b) such that aa'' and bb'' commute, and if $(a(E)+b(E))/b(E)$ is finite-dimensional, then the operator $aa''(1-bb'')$ is finite rank; thus

$$\mathrm{index}\begin{pmatrix} a & -b'' \\ b & a''(1-bb'') \end{pmatrix} = \mathrm{index}\begin{pmatrix} a & -b'' \\ b & 0 \end{pmatrix}$$

and hence

$$\mathrm{index}(a,b) = \mathrm{index}\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -b'' \end{pmatrix} = \mathrm{index}(b)+\mathrm{index}(b'') = 0.$$

If $(a(E)+b(E))/a(E)$ is finite-dimensional then work instead with a' and b'. Alternatively, the same argument shows that (b,a) has index zero, and hence again (a,b). ∎

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A remark on non-existence of an algebra norm for the algebra of continuous functions on a topological space admitting an unbounded continuous function

by

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Abstract. Let X be any topological space, and let $C(X)$ be the algebra of all continuous complex-valued functions on X. We prove a conjecture of Yood (1994) to the effect that if there exists an unbounded element of $C(X)$ then $C(X)$ cannot be made into a normed algebra.

Throughout, $C(X)$ denotes the algebra of all continuous complex-valued functions on a topological space X.

THEOREM. *Let X be a topological space such that $C(X)$ has an unbounded element. Then there is no normed algebra norm on $C(X)$.*

As Yood [4] notes, it is easy to see that there is no Banach algebra norm on $C(X)$, so that the main content of the Theorem is the non-existence of a norm under which $C(X)$ would be an incomplete normed algebra. Our Theorem was conjectured by Yood [4] who showed that it does hold under the additional assumption that X is a locally compact Hausdorff space such that every character of $C(X)$ is a point-evaluation. Yood also showed that this condition on the characters is implied by the existence of a function $h \in C(X)$ such that $\{x : h(x) = \alpha\}$ is compact for every $\alpha \in \mathbb{C}$.

To prove our Theorem, we recall a result due to Kaplansky [2, Thm. 6.2] and also used by Yood [4] in his work. By $C_0(X)$ we mean the algebra of continuous complex-valued functions on X vanishing at infinity, where X is a topological space and $C_0(X)$ is always equipped with the supremum norm. For more information on interesting issues related to the following Proposition, see [3, pp. 244 and 576–579].

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