STUDIA MATHEMATICA 115 (3) (1995)

Abel means of operator-valued processes

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Abstract. Let (X_j) be a sequence of independent identically distributed random operators on a Banach space. We obtain necessary and sufficient conditions for the Abel means of $X_n \ldots X_2 X_1$ to belong to Hardy and Lipschitz spaces a.s. We also obtain necessary and sufficient conditions on the Fourier coefficients of random Taylor series with bounded martingale coefficients to belong to Lipschitz and Bergman spaces.

Introduction. In percolation theory it is of interest to analyse the asymptotic behaviour of products $Y_n = X_n X_{n-1} \ldots X_1$ of random linear operators. It is known as a consequence of the subadditive ergodic theorem [11, p. 893] that if X_j are a sequence of independent identically distributed random operators on a Banach space E whose norms satisfy the integrability condition $\mathbb{E}|\log \|X_j\|_{B(E)}| \le C$, then the random variables

$$n^{-1} \log \|X_n X_{n-1} \ldots X_1\|_{B(E)}$$

converge almost surely to a constant $\gamma \le \mathbb{E} \log \|X_j\|_{B(E)}$. When $\gamma \le 0$ the operator-valued functions

$$H(z) = (1-z) \sum_{n=0}^{\infty} z^n X_n X_{n-1} \ldots X_1 \tag{1}$$

are well defined on the disc $D = \{|z| < 1\}$ almost surely and define the *Abel means* of the stochastic process $X_n X_{n-1} \ldots X_1$. In this paper we consider necessary and sufficient conditions involving the mean operator $T = \mathbb{E}X_j$ and the random operators $A_k = X_k - T$ for $H(z)f$ $(f \in E)$ to belong to various Hardy and Lipschitz spaces almost surely. The geometrical structure of the underlying Banach space E plays an important role in the results.

In Section 1 we consider the case in which X_k are independent copies of a random matrix X with entries $[X]_{ij}$. The matrices are regarded as operators

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1991 *Mathematics Subject Classification*: Primary 47B80; Secondary 60G46.

Key words and phrases: Abel means, martingale transforms, subadditive ergodic theory.

between Lebesgue spaces $L^p(\pi)$ of functions $f(i)$ defined on the integers. We obtain a sufficient condition under which $\mathbb{E}\|H(r)f\|_{L^s} \to 0$ as $r \to 1-$.

In the following section we show that if T is a contraction on a B-convex Banach space E, then $H(z)f$ belongs to the Hardy space H^1E with $\mathbb{E}\|H(z)f\|_{H^1E} < \infty$ only if $(T^n - T^{n+1})f \to 0$ in norm. In Section 3 we obtain an upper estimate on $r \mapsto \mathbb{E}\|H(re^{i\theta})f\|_{L^p_\theta E}$ where E is a ζ-convex space.

In Section 4 we obtain a sufficient condition for the matrix coefficients $\langle H(re^{i\theta})f, f'\rangle$ to belong to the Lipschitz space $\mathcal{A}_\gamma$ almost surely. The hypotheses involve stringent compactness assumptions on T. In Section 5 we show that when T is a self-adjoint Hilbert–Schmidt operator on Hilbert space a special argument leads to an analogous result involving a modified Lipschitz space.

The proofs of Theorems 4 and 5 use Proposition 4, which concerns the Lipschitz norm of a random Taylor series with martingale coefficients. A converse result is given in Section 6. The main results use a factorization lemma which we state after introducing a little notation.

NOTATION. Let $(\Omega, \mathbb{P})$ be a probability space, E a complex Banach space. We take X to be a random operator on a Banach space E so that $\Omega \ni \omega \mapsto X(\omega) \in B(E)$ is strongly measurable and $\mathbb{E}\|X\|_{B(E)} < \infty$. We let X_n be a sequence of independent copies of X. We put $Y_n = X_n X_{n-1} \ldots X_1$ for $n \geq 1$, and $Y_0 = I = Y_{-1}$. We also introduce $\mathbb{E}X_j = T$, the mean operator, and write $X_j = T + A_j$ where $A_0 = I$. The σ-algebra generated by A_j with $j \leq n$ will be denoted by $\mathcal{F}_n$. We denote by T^t the transpose of T.

C, c are positive constants taking possibly different values in successive equations.

We denote by $L^p_\theta E$ the Bochner–Lebesgue space of strongly measurable functions $h(\theta)$ taking values in E for which $\|h(\theta)\|^p_E$ is integrable. Mixed norms will be denoted by

$$\|F(re^{i\theta})\|_{L^p_r L^q_\theta E} = \bigg(\int_0^1 \bigg(\int_{\mathbb{T}} \|F(re^{i\theta})\|^q_E \frac{d\theta}{2\pi}\bigg)^{p/q} dr\bigg)^{1/p}$$

The Hardy space formed by taking the closure in $L^p_\theta E$ of the analytic trigonometric polynomials with coefficients in E will be denoted by H^pE.

DEFINITION. A complex Banach space E is said to be *B-convex* if there is a $q > 1$ and a constant C for which

$$\mathrm{Ave}_{\mp}\Big\|\sum_k \mp f_k\Big\|_E \leq C\Big(\sum_k \|f_k\|^q_E\Big)^{1/q} \tag{2}$$

for any finite sequence (f_k) of elements of E. Such a q is called a *type* of E.

A Banach space E is said to be a *UMD space* or be *ζ-convex* if there is a constant C_E such that

$$\mathbb{E}\Big\|\sum_k \varepsilon_k x_k\Big\|^2_E \leq C_E \mathbb{E}\Big\|\sum_k x_k\Big\|^2_E \tag{3}$$

for any choice of signs $\varepsilon_k = \mp 1$ whenever (x_k) is a finite sequence of martingale differences taking values in E.

Examples of *UMD* spaces include the Lebesgue spaces L^q for $1 < q < \infty$ and many of the reflexive spaces which arise in classical analysis. See [6, pp. 271–273] and [5] for a discussion of their properties. Any *UMD* space is B-convex.

We shall repeatedly use the following contraction principle [8, p. 691].

THEOREM. *Let (v_n) be a martingale difference sequence with $|v_n| \leq K$ almost surely and let (r_n) be the Rademacher functions. If $\Phi : \mathbb{R}^m \to \mathbb{R}_+$ is any convex function then*

$$\mathbb{E}\Phi(v_1, v_2, \ldots, v_m) \leq \mathbb{E}\Phi(Kr_1, Kr_2, \ldots, Kr_m). \tag{4}$$

The starting point for the analysis is the following algebraic lemma (cf. [13]).

FACTORIZATION LEMMA. *Let T be a contraction on a Banach space E. The Abel mean operator $H(z) = (1-z)\sum_{n=0}^\infty z^n Y_n$ admits a factorization $H(z) = F(z)G(z)$ where*

$$F(z) = (1-z)(I - zT)^{-1} \quad \textit{and} \quad G(z) = \sum_{n=0}^\infty z^n A_n Y_{n-1}. \tag{5}$$

The non-random operator function $F(z)$ is determined by the mean operator T, whereas $G(z)$ is a random operator function whose coefficients form a martingale difference sequence with respect to $(\mathcal{F}_n)$.

Proof. An induction argument on n establishes that

$$Y_n = \sum_{k=0}^n T^{n-k} A_k Y_{k-1} \tag{6}$$

where we put $A_0 = I$ and $Y_{-1} = I$. Changing the index of summation gives

$$\begin{aligned}\sum_{n=0}^\infty z^n Y_n &= \sum_{n=0}^\infty \sum_{k=0}^n z^n T^{n-k} A_k Y_{k-1} = \sum_{k=0}^\infty z^k \Big(\sum_{n=k}^\infty z^{n-k} T^{n-k}\Big) A_k Y_{k-1} \\ &= (I - zT)^{-1} \sum_{k=0}^\infty z^k A_k Y_{k-1} = (1 - zT)^{-1} G(z).\end{aligned} \tag{7}$$

1. Processes with reversible mean matrix. Let us consider function spaces on the integers, and random operators defined on them. Suppose that

X_k are independent copies of a matrix X whose entries will be denoted by $[X]_{ij}$. We further suppose that the mean matrix T is stochastic so that $\sum_j [T]_{ij} = 1$ for each i and that its entries are positive. We suppose that there is a sequence of positive numbers π_j for which $\pi_i[T]_{ij} = \pi_j[T]_{ji}$. Such a T is said to be *reversible* and the choice of (π_j) is unique up to a scalar multiple. We let $L^p(\pi)$ denote the space of sequences pth power summable with respect to the measure π defined by $\pi(\{j\}) = \pi_j$. Then the matrix T defines an operator on $L^p(\pi)$ by $Tf(j) = \sum_i [T]_{ji} f(i)$ which is a contraction for $1 \le p \le \infty$. We shall prove that if the size of the entries of T^n decays as $n \to \infty$ then the components of the vector $H(r)f$ decay in size as r increases to unity. The arguments are based upon those of [14, p. 220].

THEOREM 1. *Suppose that the mean matrix satisfies*

(a) $\|\pi_j^{-1}[T^n]_{ij}\|_{L_i^\infty L_j^\infty} \le Cn^{-\nu}$ *for some* $\nu > 2$.

If some $f \in L^2(\pi)$ *satisfies*

(b) $\mathbb{E}\|A_k Y_{k-1} f\|_{L^2}^2 = o(1)$ *as* $k \to \infty$,

then $\mathbb{E}\|H(r)f\|_{L^s} = o((1-r)^{1/2})$ *as* $r \to 1-$ *where* $1/s = 1/2 - 1/\nu$.

LEMMA 1. *Suppose that the mean matrix satisfies* (a). *Then the operator* $\Gamma = \sum_{n=0}^\infty T^n$ *is bounded as an operator from* L^2 *into* L^s *where* $1/s = 1/2 - 1/\nu$.

Proof. We prove this by considering a decomposition of the Green's function Γ. Set

$$\Gamma_N = \sum_{n=0}^{N-1} T^n, \quad \Gamma^N = \sum_{n=N}^{\infty} T^n. \tag{8}$$

Take $f \in L^p$ where $p < \nu$. Letting m denote π-measure we have the Marcinkiewicz decomposition

$$m[|\Gamma f| \ge 2t] \le m[|\Gamma_N f| \ge t] + m[|\Gamma^N f| \ge t]. \tag{9}$$

We begin by considering the second summand in (9). An easy application of Hölder's inequality gives

$$\|T^n f\|_{L^\infty} \le \|\pi_j^{-1}[T^n]_{ij}\|_{L_i^\infty L_j^q} \|f\|_{L^p} \tag{10}$$

where q is the exponent conjugate to p. Since $\|\pi_j^{-1}[T^n]_{ij}\|_{L_i^\infty L_j^\infty} = O(n^{-\nu})$ and T is stochastic, an interpolation argument shows that

$$\|\pi_j^{-1}[T^n]_{ij}\|_{L_i^\infty L_j^q} = O(n^{-\nu/p}). \tag{11}$$

It follows that

$$\|\Gamma^N f\|_{L^\infty} \le C\Big(\sum_{n=N}^\infty n^{-\nu/p}\Big)\|f\|_{L^p} \le Cp(\nu-p)^{-1}N^{(p-\nu)/p}\|f\|_{L^p}. \tag{12}$$

Consequently, the second summand in (9) is zero when $t > p(\nu-p)^{-1} \times CN^{(p-\nu)/p}\|f\|_{L^p}$. Since $p < \nu$ we can take N to be the first integer greater than $(p(\nu-p)^{-1}t^{-1}\|f\|_{L^p})^{p/(\nu-p)}$ in order to satisfy this condition.

We now consider the first summand in (9). The elementary estimate $\|\Gamma_N f\|_{L^p} \le N\|f\|_{L^p}$ leads at once to $t^p m[|\Gamma_N f| \ge t] \le N^p \|f\|_{L^p}^p$. For our choice of N we arrive at the desired estimate

$$m[|\Gamma f| \ge 2t] \le m[|\Gamma_N f| \ge t] \le Ct^{p\nu/(p-\nu)}\|f\|_{L^p}^{\nu p/(\nu-p)} = Ct^{-w}\|f\|_{L^p}^w \tag{13}$$

where $1/w = 1/p - 1/\nu$. Hence Γ is weak (p, w) bounded for $p < \nu$ and our ν has $\nu > 2$. By the Marcinkiewicz interpolation theorem the operator Γ is bounded $L^2 \to L^s$ where $1/s = 1/2 - 1/\nu$.

Proof of Theorem 1. Suppose that the hypotheses of the theorem are satisfied. By the Factorization Lemma and Lemma 1 we have

$$\|H(r)f\|_{L^s} \le \|F(r)\|_{L^2 \to L^s}\|G(r)f\|_{L^2} \le C(1-r)\|G(r)f\|_{L^2}. \tag{14}$$

Squaring and taking expectations gives

$$\mathbb{E}\|H(r)f\|_{L^s}^2 \le C(1-r)^2\mathbb{E}\Big\|\sum_{k=0}^\infty r^k A_k Y_{k-1} f\Big\|_{L^2}^2. \tag{15}$$

The last factor in (15) may be evaluated by using the parallelogram law and orthogonality of martingale differences in L^2. We have

$$\begin{aligned}\mathbb{E}\|G(r)f\|_{L^2}^2 &= \sum_{k=0}^\infty r^{2k}\mathbb{E}\langle A_k Y_{k-1} f, A_k Y_{k-1} f\rangle \\ &\le C\sum_{k=0}^\infty r^{2k}\mathbb{E}\|A_k Y_{k-1} f\|_{L^2}^2.\end{aligned} \tag{16}$$

By the assumption (b) on the martingale differences this is $o((1-r)^{-1})$ as $r \to 1-$. Hence

$$\mathbb{E}\|H(r)f\|_{L^s}^2 = o((1-r)^2(1-r)^{-1}) = o((1-r)) \quad \text{as } r \to 1-. \tag{17}$$

2. Lower estimates. To obtain converse inequalities one considers the norm of the Abel mean $H(re^{i\theta})f$ in the vector valued Hardy space H^1E. The following inequalities are counterparts of Hardy's Inequality. A detailed discussion of such inequalities is contained in [3].

PROPOSITION 2. *Let E be a B-convex Banach space or let $E = L^1$. Then there is a constant $C_E > 0$ for which*

$$\mathbb{E}\lim_{r\to 1-}\|H(re^{i\theta})f\|_{L_\theta^1 E} \ge C_E\sum_{k=2}^\infty k^{-1}\mathbb{E}\|A_k Y_{k-1} f\|_E. \tag{18}$$

Also the following inequality holds:

$$\mathbb{E}\lim_{r\to 1-}\|H(re^{i\theta})f\|_{L^1_\theta E}\ge C_E\sum_{k=1}^\infty k^{-1}\|(I-T)T^{k-1}f\|_E. \tag{19}$$

Proof. From the definition of $F(z)$ one sees that $\|F(z)^{-1}\|\le C|1-z|^{-1}$ when $|z|<1$. Consequently, we have $\|H(z)f\|_E\ge C^{-1}\|(1-z)G(z)f\|_E$. We set $z=re^{i\theta}$ and integrate with respect to θ to get

$$\int_{\mathbb{T}}\|H(re^{i\theta})f\|_E\,\frac{d\theta}{2\pi} \ge c\int_{\mathbb{T}}\Big\|A_0Y_{-1}f+\sum_{k=1}^\infty r^ke^{ik\theta}(A_kY_{k-1}f-A_{k-1}Y_{k-2}f)\Big\|_E\frac{d\theta}{2\pi}. \tag{20}$$

It follows from Bourgain's extension of the Hausdorff–Young inequality [3, Theorem 2.5] that if E is B-convex then this latest integral may be bounded below by

$$C_E\sum_{k=1}^\infty k^{-1}r^k\|A_kY_{k-1}f-A_{k-1}Y_{k-2}f\|_E.$$

It is observed in [3, Cor. 2.2] that a similar inequality holds for analytic functions taking values in L^1 (which is the predual of a C^*-algebra). Taking expectations we obtain

$$\mathbb{E}\lim_{r\to 1-}\|H(re^{i\theta})f\|_E\ge C_E\sum_{k=1}^\infty k^{-1}\mathbb{E}\|A_kY_{k-1}f-A_{k-1}Y_{k-2}f\|_E. \tag{21}$$

The existence of the limit as $r\to 1-$ follows from standard facts about Hardy spaces since the norm on any Banach space defines a subharmonic function. We can bound each summand in (21) by taking conditional expectations to get

$$\begin{aligned}\mathbb{E}\|A_kY_{k-1}f-A_{k-1}Y_{k-2}f\|_E &=\mathbb{E}\mathbb{E}(\|A_kY_{k-1}f-A_{k-1}Y_{k-2}f\|\,|\mathcal{F}_{k-1})\\ &\ge\mathbb{E}\|\mathbb{E}(A_kY_{k-1}f|\mathcal{F}_{k-1})-A_{k-1}Y_{k-2}f\|_E=\mathbb{E}\|A_{k-1}Y_{k-2}f\|_E\end{aligned} \tag{22}$$

since $\mathbb{E}(A_k|\mathcal{F}_{k-1})=0$. The inequality (18) follows from this.

A similar, but easier, argument shows that

$$\mathbb{E}\lim_{r\to 1-}\|H(re^{i\theta})f\|_{L^1_\theta E}\ge c\sum_{k=1}^\infty k^{-1}\mathbb{E}\|(Y_k-Y_{k-1})f\|_E. \tag{23}$$

Taking expectations through the norm and using independence gives that

$$\mathbb{E}\|(Y_k-Y_{k-1})f\|\ge\|(\mathbb{E}Y_k-\mathbb{E}Y_{k-1})f\|_E=\|(T^k-T^{k-1})f\|_E, \tag{24}$$

from which (19) follows.

Remark. The inequality (19) shows that if T is power bounded so that $\|T^k\|_{B(E)}\le C$ and if the Abel means $H(re^{i\theta})f$ have $\mathbb{E}\|H(re^{i\theta})f\|_{L^\infty_rL^1_\theta E}<\infty$ then $\|T^nf-T^{n-1}f\|_E$ converges to zero as $n\to\infty$. This may be compared with the conclusion of [12, Theorem 1.1]. It is shown there that if T is a positive linear contraction operator on L^1 for which T^nf and $T^{n+1}f$ intersect slightly but uniformly in f in the unit sphere of L^1, then $T^nf-T^{n+1}f$ converges to zero in norm. See also [1, Theorem 5] and [10, Theorems 1, 6].

3. Upper estimates on $L^p_\theta E$. In this section E will be a *UMD* Banach space.

THEOREM 3. *Let $f\in E$ and suppose that*

(a) $\mathbb{E}\|A_kY_{k-1}f\|_E^2\le C$ *for all* k, *and*
(b) $\|F(z)\|_{B(E)}\le C$ *for* $z\in D$.

Then if $1<p<\infty$ there is a $q>1$ for which

$$\mathbb{E}\|H(re^{i\theta})f\|_{L^p_\theta E}=O((1-r)^{-1/q})\quad \text{as } r\to 1-. \tag{25}$$

Proof. Our basic estimation is carried out in the vector-valued Lebesgue space $L^p_\theta E$ where $1<p<\infty$. Since E is a *UMD* space, $L^p_\theta E$ is also *UMD* by [5] and [6, Theorem 1]. One can easily prove that $L^p_\theta E$ has type q, the minimum of p and the type of E itself. By the factorization lemma and hypothesis (b) we have

$$\begin{aligned}\|H(re^{i\theta})f\|_{L^p_\theta E}&\le\sup_\phi\|F(re^{i\phi})\|_{B(E)}\|G(re^{i\theta})f\|_{L^p_\theta E}\\ &\le C\|G(re^{i\theta})f\|_{L^p_\theta E}.\end{aligned} \tag{26}$$

Using the *UMD* property we obtain

$$\begin{aligned}\mathbb{E}\|G(re^{i\theta})f\|^q_{L^p_\theta E}&=\mathbb{E}\Big\|\sum_{k=0}^\infty r^ke^{ik\theta}A_kY_{k-1}f\Big\|^q_{L^p_\theta E}\\ &\le C\mathbb{E}\Big\|\sum_{k=0}^\infty\mp r^ke^{ik\theta}A_kY_{k-1}f\Big\|^q_{L^p_\theta E}\end{aligned} \tag{27}$$

by [6, Theorem 1] for any choice of signs $\mp$. Taking the average over all choices of signs, we get

$$\mathrm{Ave}_\mp\Big\|\sum_{k=0}^\infty\mp r^ke^{ik\theta}A_kY_{k-1}f\Big\|^q_{L^p_\theta E}\le C\Big(\sum_{k=0}^\infty r^{kq}\|A_kY_{k-1}f\|^q_E\Big) \tag{28}$$

since $L^p_\theta E$ has type q. We now take expectations and use Minkowski's inequality to get

$$\mathbb{E}\|G(re^{i\theta})f\|_{L^p_\theta E}\le C\Big(\sum_{k=0}^\infty r^{kq}\mathbb{E}\|A_kY_{k-1}f\|^q_E\Big)^{1/q}\le C(1-r)^{-1/q}. \tag{29}$$

This estimate, when combined with (26), leads to the stated result.

Remark. The hypothesis (b) of Theorem 3 holds if there is an $s < 1$ for which the (spatial) numerical range of T is contained in the convex hull of $\{1\}$ and the closed disc $\{|z| \le s\}$. It is immediate from the definitions that the numerical range of $T = \mathbb{E}X$ is contained in the closed convex hull of $\bigcup_{\omega\in\Omega} \operatorname{ran}(X(\omega))$. By a theorem of Crabb [4, p. 22], the convex hull of the spectrum $\sigma(T)$ is contained in the closure of $\operatorname{ran}(T)$. It is easy to construct examples of random operators $\Omega \ni \omega \mapsto X(\omega)$ for which $\sigma(T)$ is not contained in $\bigcup_{\omega\in\Omega} \sigma(X(\omega))$.

4. The coefficients of $G(z)$. Following [2] we introduce the Bergman space J_β $(-1<\beta<2)$ as the space of analytic functions $k(z)$ on the unit disc D for which the norm

$$J_\beta(k) = |k(0)| + \int_0^1 (1-r)^\beta \|k'(re^{i\theta})\|_{L^1_\theta}\,dr \qquad (-1<\beta<2), \tag{30}$$

or respectively

$$J_\beta(k) = \int_0^1 (1-r)^{\beta-1} \|k(re^{i\theta})\|_{L^1_\theta}\,dr \qquad (0<\beta<2), \tag{31}$$

is finite. The norms are equivalent where they are both defined. The Lipschitz space $\mathcal{A}_\beta$ is defined to be the space of analytic functions $g(z)$ on the disc for which the norm

$$\mathcal{A}_\beta(g) = |g(0)| + |g'(0)| + \sup_{r,\theta}(1-r)^{2-\beta}|g''(re^{i\theta})| \tag{32}$$

is finite. The dual space J'_β is equivalent to $\mathcal{A}_\beta$ under the natural pairing

$$\langle g,k\rangle = \sum_{n=1}^\infty \widehat{g}(n)\widehat{k}(n) = \int_0^1 (1-r^2)^2 \int_{-\pi}^{\pi} g_1''(re^{i\theta})k'(re^{-i\theta})e^{-i\theta}\frac{d\theta}{2\pi}\,dr \tag{33}$$

where $g_1(z) = z^2(g(z)-g(0))$. For this range of β the analytic polynomials form a dense linear subspace of J_β, so the pairing may be defined initially for polynomials and extended to the complete function spaces using the integral formula (33). The infinite sum is convergent in the sense of Abel summation. If $g(z)$ is an analytic function taking values in a Banach space E we can define $\mathcal{A}_\alpha(g) = \|g(0)\|_E + \|g'(0)\|_E + \sup_{|z|<1}(1-|z|)^{-\alpha-2}\|g''(z)\|_E$. One can then form the Banach space $\mathcal{A}_\alpha(E)$ of functions $g(z)$ for which $\mathcal{A}_\alpha(g) < \infty$.

A fact which will be used repeatedly in the following proofs is contained in [7, Theorem 39]. If $\beta < 0$, a function $g(z)$ has $\mathcal{A}_\beta(g) < \infty$ if and only if $\sup_{|z|<1}(1-|z|)^{-\beta}|g(z)| < \infty$. This fails to hold in general for $\beta \ge 0$.

In this section we obtain a sufficient condition for a Taylor series with martingale coefficients to belong to $\mathcal{A}_\alpha$.

LEMMA 4. *Let d_n be a bounded martingale difference sequence and let*

$$W_N(r,\theta) = (1-r^2)^{1/2}\sum_{m=0}^N d_m r^m e^{im\theta}. \tag{34}$$

Then $\mathbb{E}\|W_N(r,\theta)\|^2_{L^\infty_r L^\infty_\theta} = O((\log N)^2)$.

Proof. By estimating real and imaginary parts separately it suffices to consider

$$U(r,\theta) = (1-r^2)^{1/2}\sum_{m=1}^N d_m r^m \cos m\theta \tag{35}$$

and the corresponding sum involving sines where d_n is a real martingale difference sequence. By the contraction principle (4) stated above, we can replace each d_m by the Rademacher function r_m. We introduce the Chebyshev polynomial T_m of degree m by the relation $T_m(\cos\phi) = \cos m\phi$. We set $r = \cos\phi$ and use the standard identity $r^m = \sum_{j=0}^m c_{jm}T_j(r)$ where $c_{jm} \ge 0$ and $\sum_{j=0}^m c_{jm} = 1$ to substitute for r in the formula for $U(r,\theta)$. We obtain an expression $U(r,\theta) = V(\phi,\theta)$ where $V(\phi,\theta)$ is a trigonometric polynomial of degree $2N+1$ in θ,ϕ. We observe that

$$\sup_{0\le r<1}\sup_{-\pi\le\theta\le\pi}|U(r,\theta)| \le \sup_{-\pi\le\phi\le\pi}\sup_{-\pi\le\theta\le\pi}|V(\phi,\theta)| = M. \tag{36}$$

Using Bernstein's inequality as in the proof of the classical Littlewood–Salem Theorem [9, 6.2] one can show that

$$\mathbb{E}\exp\lambda M/2 \le CN^2\exp C\lambda^2 \qquad (\lambda>0).$$

From this, Chebyshev's inequality gives the distributional estimate

$$\mathbb{P}[M\lambda/2 - C\lambda^2 - \log N^2 \ge u] \le Ce^{-u} \qquad (\lambda, u > 0). \tag{37}$$

The required estimate on $\mathbb{E}M^2$ follows by a straightforward calculation.

PROPOSITION 4. *Let d_n be a bounded martingale difference sequence and let $\beta < -1/2$. Then the function*

$$g(z) = \sum_{n=1}^\infty d_n z^n \tag{38}$$

belongs to $\mathcal{A}_\beta$ almost surely.

Proof. To reduce to the situation of the lemma we set

$$K(r,\theta) = (1-r^2)^{-\beta}\sum_{n=1}^\infty d_n r^n e^{in\theta}. \tag{39}$$

We split this series into dyadic blocks $\Delta_j = \{n : 2^j \le n < 2^{j+1}\}$ by introducing

$$V_j(r,\theta) = (1-r^2)^{1/2} \sum_{n\in\Delta_j} d_n r^{n-2^j} e^{in\theta} \tag{40}$$

and writing

$$K(r,\theta) = \sum_{j=1}^{\infty} (1-r^2)^{-\beta-1/2} r^{2^j} j^2 \Big(j^{-2}(1-r^2)^{1/2} \sum_{n\in\Delta_j} d_n r^{n-2^j} e^{in\theta}\Big). \tag{41}$$

By the Cauchy–Schwarz inequality

$$|K(r,\theta)| \le \Big(\sum_{j=1}^{\infty} (1-r^2)^{-2\beta-1} r^{2^{j+1}} j^4\Big)^{1/2} \Big(\sum_{j=1}^{\infty} j^{-4} |V_j(r,\theta)|^2\Big)^{1/2} \tag{42}$$

Taking expectations and using the Cauchy–Schwarz inequality for integrals gives

$$\begin{aligned} &\mathbb{E}\|K(r,\theta)\|_{L_r^\infty L_\theta^\infty} \\ &\le \Big\| \sum_{j=1}^{\infty} (1-r^2)^{-2\beta-1} r^{2^{j+1}} j^4 \Big\|_{L_r^\infty}^{1/2} \Big(\sum_{j=1}^{\infty} j^{-4} \mathbb{E}\|V_j(r,\theta)\|^2_{L_r^\infty L_\theta^\infty}\Big)^{1/2} \end{aligned} \tag{43}$$

By the lemma, $\mathbb{E}\|V_j(r,\theta)\|^2_{L_r^\infty L_\theta^\infty} = O(j^2)$, so the series in the second factor in (43) converges. Using some elementary estimation one shows that the first factor is finite.

THEOREM 4. *Let E be any Banach space and let $f \in E$, $f' \in E'$. Suppose that*

(a) *T is a nuclear operator from E to itself,*
(b) *$\|((T^t)^n - (T^t)^{n+1})f'\|_{E'} \le Cn^{-\alpha-1}$ for some $\alpha \le 0$ and all $n \ge 1$, and*
(c) *$\|A_k Y_{k-1} f\|_E \le C$ for all k.*

Then $\mathbb{E}\|\langle H(re^{i\theta})f, f'\rangle\|_{\mathcal{A}_\gamma} < \infty$ for each $\gamma < \alpha - 1/2$.

Proof. It is readily verified that $F(z) = (1-z)I + zTF(z)$, so we can use the factorization lemma to write

$$\langle H(z)f, f'\rangle = (1-z)\langle G(z)f, f'\rangle + z\langle TG(z)f, F(z)^t f'\rangle. \tag{44}$$

Since T is a nuclear operator there are $f_j \in E$ and $f'_j \in E'$ with $T = \sum_j f'_j \otimes f_j$ and $\sum_j \|f_j\|_E \|f'_j\|_{E'} \le C$. We write the second summand in (44) as

$$\langle H_1(z)f, f'\rangle = \sum_{j=1}^{\infty} z\langle G(z)f, f'_j\rangle \langle f_j, F(z)^t f'\rangle. \tag{45}$$

This is a decoupling formula, for each summand has the form of the product of a random coefficient involving f and a non-random coefficient involving f'. We proceed to estimate each summand. If $\gamma < \alpha - 1/2$ where α is as in the theorem we can write $\gamma = \delta + \beta$ with $\beta < -1/2$ and $\delta < \alpha$. It follows from [7, Theorem 39] and an elementary calculation that

$$\|z\langle G(z)f, f'_j\rangle \langle f_j, F(z)^t f'\rangle\|_{\mathcal{A}_\gamma} \le C\|\langle G(z)f, f_j\rangle\|_{\mathcal{A}_\beta} \|\langle f_j, F(z)^t f'\rangle\|_{\mathcal{A}_\delta}. \tag{46}$$

Using this estimate and the triangle inequality in the decoupling formula (45) we get

$$\begin{aligned} \mathbb{E}\|\langle H_1(z)f, f'\rangle\|_{\mathcal{A}_\gamma} &\le C\sum_j \mathbb{E}\|\langle G(z)f, f'_j\rangle\langle f_j, F(z)^t f\rangle\|_{\mathcal{A}_\gamma} \\ &\le C\sum_j \mathbb{E}\|\langle G(z)f, f'_j\rangle\|_{\mathcal{A}_\beta} \|\langle f_j, F(z)^t f'\rangle\|_{\mathcal{A}_\delta}. \end{aligned} \tag{47}$$

By (43) of the proof of Proposition 4 we have

$$\mathbb{E}\|\langle G(z)f, f'_j\rangle\|_{\mathcal{A}_\beta} \le C\|f'_j\|_{E'}. \tag{48}$$

By the assumption (b) we have

$$\begin{aligned} \Big\|\sum_{n=1}^{\infty} z^n((T^t)^n - (T^t)^{n+1})f'\Big\|_{E'} &\le \sum_{n=1}^{\infty} |z|^n \|((T^t)^n - (T^t)^{n+1})f'\|_{E'} \\ &\le C\sum_{n=1}^{\infty} |z|^n n^{-\alpha-1} = O((1-|z|)^\delta) \end{aligned}$$

since $\delta < \alpha$ and $\delta < 0$. On considering the power series expansion of $F(z)$ it is evident that the estimate

$$\|\langle f_j, F(z)^t f'\rangle\|_{\mathcal{A}_\delta} \le C\|f_j\|_E \|F(z)^t f'\|_{\mathcal{A}_\delta(E')} \le C\|f_j\|_E \tag{49}$$

follows from this. Substituting (48) and (49) in (47) we get

$$\mathbb{E}\|\langle H_1(z)f, f'\rangle\|_{\mathcal{A}_\gamma} \le C\sum_j \|f_j\|_E \|f'_j\|_{E'} < \infty. \tag{50}$$

Proposition 4 also deals with the first summand in (44).

5. Hilbert space estimates. The result of this section appears to be special to Hilbert space as it uses a spectral theorem and Parseval's identity.

THEOREM 5. *Let f, f' be elements of a Hilbert space E and suppose that*

(a) *T is a self-adjoint operator on E with $0 \le T \le I$,*
(b) *T is a Hilbert–Schmidt operator, and*
(c) *$\|A_k Y_{k-1} f\|_E \le C$.*

Then

$$\mathbb{E}\|(1-r)^{-\beta}\langle H(re^{i\theta})f, f'\rangle\|_{L_r^\infty L_\theta^2} < \infty \quad (\beta < -1/2). \tag{51}$$

Proof. The proof follows that of the previous Theorem 4. One checks that

$$F(z) = I - z(I-T) - z^2T(I-T)(I-zT)^{-1}$$

so the factorization lemma gives

$$H(z)f = (I + z(T-I))G(z)f - z^2(I-T)(I-zT)^{-1}TG(z)f. \tag{52}$$

We denote the final summand in (52) by $H_2(z)f$. It suffices to estimate this term since the proof will include an estimate on the first summand.

By hypotheses (a), (b) we can write $T = \sum_k f'_k \otimes f_k$ where (f_k) is an orthonormal basis and $\sum_k \|f'_k\|_E^2 \le C$. The decoupling formula for $\langle H_2(re^{i\theta})f, f'\rangle$ is obtained by taking adjoints and using the expansion of T to give

$$\begin{aligned}(53)\quad &\langle (I-T)(I-re^{i\theta}T)^{-1}TG(re^{i\theta})f, f'\rangle \\ &= \sum_{k=1}^{\infty} \langle G(re^{i\theta})f, f'_k\rangle\langle f_k, (I-re^{-i\theta}T)^{-1}(I-T)f'\rangle.\end{aligned}$$

We apply the Cauchy–Schwarz inequality to get

$$\begin{aligned}&|\langle H_2(re^{i\theta})f, f'\rangle| \\ &\le \Big(\sum_k |\langle G(re^{i\theta})f, f'_k\rangle|^2\Big)^{1/2}\Big(\sum_k |\langle f_k, (I-re^{-i\theta}T)^{-1}(I-T)f'\rangle|^2\Big)^{1/2}\end{aligned}$$

Using Bessel's inequality for the orthonormal basis (f_k) one deduces that

$$\begin{aligned}(54)\quad &\|\langle H_2(re^{i\theta})f, f'\rangle\|_{L^2_\theta} \\ &\le \Big(\sum_k \|\langle G(re^{i\theta})f, f'_k\rangle\|^2_{L^\infty_\theta}\Big)^{1/2} \|(I-re^{-i\theta}T)^{-1}(I-T)f'\|_{L^2_\theta E}.\end{aligned}$$

We multiply (54) through by $(1-r)^{-\beta}$ and take the expectation of its square. This gives

$$\begin{aligned}(55)\quad &\mathbb{E}\|(1-r)^{-\beta}\langle H_2(re^{i\theta})f, f'\rangle\|^2_{L^\infty_r L^2_\theta} \\ &\le C\mathbb{E}\sum_k \|(1-r)^{-\beta}\langle G(re^{i\theta})f, f'_k\rangle\|^2_{L^\infty_r L^\infty_\theta} \|(I-re^{i\theta}T)^{-1}(I-T)f'\|^2_{L^\infty_r L^2_\theta E}.\end{aligned}$$

One evaluates the last factor in (55) by using Parseval's identity in $L^2_\theta E$ to obtain

$$\|(I-re^{-i\theta}T)(I-T)f'\|^2_{L^2_\theta E} = \sum_{n=0}^{\infty} r^{2n}\langle T^{2n}(I-T)^2f', f'\rangle.$$

By the spectral theorem and hypothesis (a) the right-hand side is at most

$$\langle (I-r^2T^2)^{-1}(I-T)^2f', f'\rangle \le \langle (I+T)^{-1}(I-T)f', f'\rangle \le \|f'\|_E^2. \tag{56}$$

By Proposition 4 and (c) the first factor in each summand in (55) satisfies

$$\mathbb{E}\|(1-r)^{-\beta}\langle G(re^{i\theta})f, f'_k\rangle\|^2_{L^\infty_r L^\infty_\theta} \le C\|f'_k\|_E^2. \tag{57}$$

Using the Cauchy–Schwarz inequality for integrals, the stated estimate (51) follows on substituting (56) and (57) in (55).

6. A converse to Proposition 4

Lemma 6. *If u_n is a bounded martingale difference sequence, then the function*

$$k(z) = \sum_{n=1}^{\infty} \frac{u_n z^n}{n} \tag{58}$$

belongs to J_β almost surely for $\beta > -1/2$.

Proof. Once again this is a consequence of the contraction principle (4). After replacing the u_n by Rademacher functions r_n we can use Fubini's Theorem to derive the estimate

$$\mathbb{E}\|k'(re^{i\theta})\|_{L^1_\theta} \le C(1-r)^{-1/2}, \tag{59}$$

from which the stated result follows.

The method of proof of the following result was suggested by [7, p. 418]. See also [7, p. 421]. Henceforth we let

$$g(z) = \langle G(z)f, f'\rangle = \sum_{n=0}^{\infty} z^n\langle A_nY_{n-1}f, f'\rangle. \tag{60}$$

Theorem 6. *Suppose that there is a constant C for which*

$$\mathbb{E}(|\widehat{g}(n)|^2|\mathcal{F}_{n-1}) \le C(\mathbb{E}(|\widehat{g}(n)||\mathcal{F}_{n-1}))^2 \tag{61}$$

and that there is a $\beta > -1/2$ for which

$$\mathbb{E}\|g''(re^{i\theta})\|_{L^\infty_\theta} = O((1-r)^{-2+\beta}). \tag{62}$$

Then

$$\mathbb{E}\Big(\sum_{n=1}^{\infty} \frac{|\widehat{g}(n)|}{n}\Big)^{1/2} < \infty. \tag{63}$$

Proof. We introduce a function $k(z) = \sum_{n=1}^{\infty} u_nz^n/n$ where u_n is a bounded martingale difference sequence so chosen that $\Re(\widehat{g}(n)u_n) \ge |\widehat{g}(n)|$. To obtain u_n we introduce the polar decomposition $\widehat{g}(n) = v_n|\widehat{g}(n)|$ and set $u_n = \overline{v}_n - \mathbb{E}(\overline{v}_n|\mathcal{F}_{n-1})$. This forms a bounded martingale difference sequence. Since $\widehat{g}(n)$ is a martingale difference sequence, it follows on using the triangle inequality that

$$\lambda|\mathbb{E}(v_n|\mathcal{F}_{n-1})| \le \mathbb{E}(|\lambda v_n - \widehat{g}(n)|\,|\mathcal{F}_{n-1}) \le (\mathbb{E}((\lambda - |\widehat{g}(n)|)^2|\mathcal{F}_{n-1}))^{1/2} \quad (\lambda > 0).$$

Optimizing this over λ we obtain

$$(\mathbb{E}(|\widehat{g}(n)|\,|\mathcal{F}_{n-1}))^2 \le (1 - |\mathbb{E}(v_n|\mathcal{F}_{n-1})|^2)\mathbb{E}(|\widehat{g}(n)|^2|\mathcal{F}_{n-1}).$$

By the assumption (61) on the expectation and variance of $|\widehat{g}(n)|$ it follows that

$$\Re(u_n\widehat{g}(n)) = |\widehat{g}(n)| - \Re(\widehat{g}(n)\mathbb{E}(\bar{v}_n|\mathcal{F}_{n-1})) \ge |\widehat{g}(n)|(1 - (1 - C^{-2})^{1/2}) \ge c|\widehat{g}(n)| \tag{64}$$

where $c > 0$.

By the duality formula (33) the pairing of $g(z)$ and $k(z)$ may be represented as

$$J = \sum_{n=1}^{\infty} \widehat{g}(n)\widehat{k}(n) = \langle g(z), k(z)\rangle = \int_0^1 (1 - r^2)^2 U_r\, dr, \tag{65}$$

where

$$U_r = \int_{-\pi}^{\pi} g_1''(re^{i\theta})k'(re^{-i\theta})e^{-i\theta}\,\frac{d\theta}{2\pi}. \tag{66}$$

We estimate this by introducing a dyadic decomposition. Let $r_j = 1 - 2^{-j}$ for $j \ge 0$. We observe that $1 - r_j$, $1 - r_{j+1}$ and $r_{j+1} - r_j$ lie between 2^{-j} and 2^{-j-1}. Hence the ratios of these quantities are bounded above and below by absolute constants. Splitting up the integral in (65) gives

$$|J| \le C\sum_{j=0}^{\infty} \int_{r_j}^{r_{j+1}} (1 - r)^2|U_r|\, dr, \tag{67}$$

so

$$|J| \le C\sum_j (1 - r_j)^2(r_{j+1} - r_j) \sup_{r_j \le r < r_{j+1}} |U_r|. \tag{68}$$

By Minkowski's inequality with exponent $1/2 < 1$ we have

$$|J|^{1/2} \le C\sum_j (1 - r_j)(r_{j+1} - r_j)^{1/2} \sup_{r_j \le r < r_{j+1}} |U_r|^{1/2} \tag{69}$$

and on taking expectations and substituting for r_j we have

$$\mathbb{E}|J|^{1/2} \le C\sum_j 2^{-3j/2}\mathbb{E} \sup_{r_j \le r < r_{j+1}} |U_r|^{1/2}. \tag{70}$$

We use the formula (66) to estimate the integrands in this latest series by

$$\sup_{r_j \le r < r_{j+1}} |U_r| \le \sup_{r_j \le r < r_{j+1}} \|g''(re^{i\theta})\|_{L^\infty_\theta} \sup_{r_j \le r < r_{j+1}} \|k'(re^{i\theta})\|_{L^1_\theta} \le \|g''(r_{j+1}e^{i\theta})\|_{L^\infty_\theta} \|k'(r_{j+1}e^{i\theta})\|_{L^1_\theta} \tag{71}$$

using standard properties of the Poisson integral formula. We take expectations and use the Cauchy–Schwarz inequality for integrals to show that

$$\mathbb{E} \sup_{r_j \le r < r_{j+1}} |U(r)|^{1/2} \le (\mathbb{E}\|g''(r_{j+1}e^{i\theta})\|_{L^\infty_\theta}\mathbb{E}\|k'(r_{j+1}e^{i\theta})\|_{L^1_\theta})^{1/2}. \tag{72}$$

It follows from estimate (59) in the proof of Lemma 6 that there is a constant C with

$$\mathbb{E}\|k'(r_{j+1}e^{i\theta})\|_{L^1_\theta} \le C(1 - r_{j+1})^{-1/2} = O(2^{j/2}). \tag{73}$$

Combining this with the assumption (62) on $g(z)$ we conclude from (72) that

$$\mathbb{E} \sup_{r_j \le r < r_{j+1}} |U_r|^{1/2} = O(2^{j - \beta j/2 + j/4}). \tag{74}$$

Substituting this into (70) we get

$$\mathbb{E}|J|^{1/2} = \sum_j O(2^{-3j/2 + j - \beta j/2 + j/4}) = O(1), \tag{75}$$

where the exponents in the series (75) are negative since $-1/4 - \beta/2 < 0$. Returning to (65) and recalling the choice of $k(z)$ we conclude that (63) holds.

PROPOSITION 6. *Suppose that*

(a) $|\widehat{g}(n)| \le C$, *and*
(b) $\mathbb{E}\|g\|_{A_\beta} < \infty$ *for some* $\beta > -1/2$.

Then

$$\mathbb{E}\Big(\sum_{n=1}^{\infty} \frac{|\widehat{g}(n)|^2}{n}\Big)^{1/2} < \infty. \tag{76}$$

Proof. One can prove this in the same way as Theorem 6, now taking u_n to satisfy $\Re(u_n\widehat{g}(n)) = |\widehat{g}(n)|^2$. The hypothesis (b) on the norm of g is stronger than condition (62) of the theorem since

$$(1 - s)^{-\beta+2}\mathbb{E}\|g''(se^{i\theta})\|_{L^\infty_\theta} \le \mathbb{E} \sup_{0<r<1} \sup_{-\pi\le\theta<\pi} (1 - r)^{-\beta+2}|g''(re^{i\theta})|. \tag{77}$$

OPEN PROBLEM. Under what conditions on X does the circle of convergence form the natural boundary of $H(z)f$? See [1, p. 539] and [9].

Acknowledgements. I should like to thank Dr. D. Kershaw, Professor J. A. D. Welsh and Professor S. C. Power for helpful remarks. I am also grateful to the referee who simplified several proofs.

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Received October 13, 1994
Revised version January 10, 1995 (3354)

STUDIA MATHEMATICA 115 (3) (1995)

Nonatomic Lipschitz spaces

by

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Abstract. We abstractly characterize Lipschitz spaces in terms of having a lattice-complete unit ball and a separating family of pure normal states. We then formulate a notion of "measurable metric space" and characterize the corresponding Lipschitz spaces in terms of having a lattice complete unit ball and a separating family of normal states.

Let (X,d) be a metric space. Then the *Lipschitz space* $\mathrm{Lip}(X,d)$ is the Banach space consisting of all bounded scalar-valued Lipschitz functions on X, with norm

$$\|f\|_L = \max(\|f\|_\infty, L(f)).$$

Here $\|f\|_\infty$ denotes the sup norm of f and $L(f)$ denotes the Lipschitz number of f,

$$L(f) = \sup\{|f(x)-f(y)|/d(x,y) : x,y \in X,\ x \neq y\}.$$

Lipschitz spaces have been studied in [1], [3], [5], [8], [9], [10], [11], [12], [13].

The real part of the unit ball of $\mathrm{Lip}(X,d)$ is a completely distributive complete sublattice, and we showed in [11] that this fact characterizes Lipschitz spaces up to isomorphism. Our first aim here is to give another abstract characterization of Lipschitz spaces, this time in terms of order properties which may be more familiar. In the new characterization, complete distributivity of the unit ball is replaced by the existence of a separating family of pure normal states (Theorem 4).

This new result is somewhat unnatural, in that it juxtaposes pureness and normality, two properties not usually seen together. This is actually an advantage, because it suggests a direction for generalization.

To see this, consider the space l^∞ of bounded scalar-valued sequences. It too has a separating family of pure normal states, namely the coordinate evaluations. But l^∞ is merely a special example of the class of spaces $L^\infty(X,\mu)$, which generally have a separating family of normal states (given by integration against functions in $L^1(X,\mu)$). *Pure* normal states exist in

1991 *Mathematics Subject Classification*: 46B42, 46E05, 46E15.