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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
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Local polynomials are polynomials

by

C. K. FONG (Ottawa, Ont.), G. LUMER (Mons),
E. NORDGREN (Durham, N.H.), H. RADJAVI (Halifax, N. Scotia)
and P. ROSENTHAL (Toronto, Ont.)

Abstract. We prove that a function f is a polynomial if $G\circ f$ is a polynomial for every bounded linear functional G. We also show that an operator-valued function is a polynomial if it is locally a polynomial.

We begin with the following general result.

THEOREM. *Let $\mathscr{Y}$ be a Banach space and $\mathscr{W}$ a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$). Let $\mathscr{V}$ be a Banach subspace of the space $\mathscr{B}(\mathscr{Y},\mathscr{W})$ of bounded linear operators from $\mathscr{Y}$ into $\mathscr{W}$. Let $\mathscr{X}$ be any non-empty set and φ any function taking $\mathscr{X}$ into $\mathscr{V}$. Suppose f is a function from a non-empty connected open set of real or complex numbers into $\mathscr{X}$ with the property that $\varphi(f(z))y$ is a polynomial in z (with coefficients in $\mathscr{W}$) for each fixed $y\in\mathscr{Y}$. Then $\varphi(f(z))$ is a polynomial in z (with coefficients in $\mathscr{V}$).*

Proof. For each fixed y,

$$\varphi(f(z))y = T_0(y) + T_1(y)z + \ldots + T_k(y)z^k,$$

where each T_j is a mapping from $\mathscr{Y}$ into $\mathscr{W}$ and the degree k depends upon y. Since

$$\varphi(f(z))(c_1y_1 + c_2y_2) = c_1\varphi(f(z))(y_1) + c_2\varphi(f(z))(y_2)$$

for each fixed z and since two polynomials which agree on an open set are identical, it follows that each T_j is linear. We must show that each T_j is bounded.

We can assume, by replacing z by $z - z_0$ if necessary, that 0 is in the domain of f (a polynomial in $z - z_0$ is also a polynomial in z). Then T_0 is clearly bounded, since $T_0y = \varphi(f(0))y$, and $\varphi(f(0))\in\mathscr{V}$.

Choose $\delta > 0$ such that $\mathscr{S} = \{z : 0 < |z| < \delta\}$ is contained in the domain of f. Then the set of maps $z^{-1}(\varphi(f(z)) - T_0)$, parameterized by $z\in\mathscr{S}$, is

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pointwise bounded on $\mathscr{Y}$. (For each fixed y,

$$\|z^{-1}(\varphi(f(z))-T_0)y\| \le K_1+K_2\delta+\ldots+K_k\delta^{k-1},$$

where k is the degree associated with y and $K_j=\|T_j(y)\|$.) Hence, by the Principle of Uniform Boundedness, there is a K such that

$$\|z^{-1}(\varphi(f(z))-T_0)y\| \le K\|y\|$$

for all $z\in\mathscr{S}$, $y\in\mathscr{Y}$. For any fixed y,

$$\lim_{z\to 0} z^{-1}(\varphi(f(z))-T_0)y=T_1y,$$

so $\|T_1\|\le K$.

Considering $z^{-1}[z^{-1}(\varphi(f(z))-T_0)-T_1]$ shows that T_2 is bounded, and similar considerations yield T_j bounded for all j. This also implies that each $T_j\in\mathscr{V}$.

To finish the proof of the Theorem we must show that there is an N such that $T_j=0$ for $j\ge N$. To see this, for each positive integer n let

$$\mathscr{Y}_n=\{y\in\mathscr{Y}: T_ky=0 \text{ for all } k\ge n\}.$$

Clearly, the $\mathscr{Y}_n$ are all closed subsets of $\mathscr{Y}$. Moreover, the assumption that each $\varphi(f(z))y$ is a polynomial implies that $\bigcup_{n=1}^{\infty}\mathscr{Y}_n=\mathscr{Y}$. Then the Baire Category Theorem implies that there is an N such that $\mathscr{Y}_N$ has non-empty interior. A bounded linear operator which vanishes on a set with non-empty interior is identically 0, so it follows that $T_k=0$ for $k\ge N$ and $\varphi(f(z))$ is a polynomial in z with coefficients in $\mathscr{V}$.

COROLLARY 1. *If f is a function mapping a domain in the complex plane into a normed vector space $\mathscr{X}$ such that $G\circ f$ is a polynomial for every bounded linear functional G on $\mathscr{X}$, then f is a polynomial.*

Proof. In the Theorem, take $\mathscr{X}=\mathscr{X}$, $\mathscr{Y}=\mathscr{X}^*$ and $\mathscr{W}=\mathbb{C}$. Then $\mathscr{B}(\mathscr{Y},\mathscr{W})=\mathscr{X}^{**}$. Let φ be the canonical imbedding of $\mathscr{X}$ into $\mathscr{X}^{**}$, and let $\mathscr{V}$ be the closure of $\varphi(\mathscr{X})$ in $\mathscr{X}^{**}$. Then, for $G\in\mathscr{X}^*$, $\varphi(f(z))G=G(f(z))$, and the Theorem implies that $\varphi\circ f$ is a polynomial with coefficients in $\mathscr{V}$, but since that polynomial is $\varphi(\mathscr{X})$-valued, its coefficients must actually be in $\varphi(\mathscr{X})$.

COROLLARY 2. *If $\mathscr{Y}$ is a Banach space and f is a function mapping a domain in $\mathbb{C}$ into $\mathscr{B}(\mathscr{Y})$ such that the mapping $z\to f(z)y$ is a polynomial for every y in $\mathscr{Y}$, then f is a polynomial.*

Proof. In the Theorem, let $\mathscr{V}=\mathscr{X}=\mathscr{B}(\mathscr{Y})$, φ be the identity map, and $\mathscr{W}=\mathscr{Y}$. Then $\varphi(f(z))y=f(z)y$, and the Theorem applies.

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MATHEMATICS DEPARTMENT
CARLETON UNIVERSITY
OTTAWA, ONTARIO, CANADA

MATHEMATICS DEPARTMENT
DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA, CANADA

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TORONTO, ONTARIO, CANADA

MATHEMATICS DEPARTMENT
UNIVERSITY OF NEW HAMPSHIRE
DURHAM, NEW HAMPSHIRE, U.S.A.

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