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Chaotic behavior of infinitely divisible processes

by

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Abstract. The hierarchy of chaotic properties of symmetric infinitely divisible stationary processes is studied in the language of their stochastic representation. The structure of the Musielak–Orlicz space in this representation is exploited here.

1. Introduction. In this paper we study the chaotic behavior of infinitely divisible (ID) stationary processes with continuous time. A large number of papers on ergodic properties of stochastic processes have been devoted to Gaussian processes starting from Maruyama (1949), Grenander (1950) and Fomin (1950). For stable processes we refer to Cambanis, Hardin and Weron (1987), Weron (1985), Podgórski and Weron (1991), Podgórski (1992), Gross (1994), Hernández and Houdré (1993). For infinitely divisible processes the study of mixing and ergodicity was started by the pioneering work of Maruyama (1970), where he introduced an analytical approach to ID processes, based on the Lévy–Khinchin representation. For harmonizable ID processes he proved that they are never ergodic, gave necessary and sufficient conditions for mixing, and pointed out that mixing and mixing of all orders are equivalent. For stationary Gaussian processes this was already known by the result of Leonov (1960). Recently Gross and Robertson (1993) have examined chaotic properties of ID stationary sequences which can be represented as random measures on stationary sequences of sets.

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In general, the hierarchy of chaotic properties, which exhibits gradually stronger chaotic behavior, is well known (see Cornfeld, Fomin and Sinai (1982), Lasota and Mackey (1994)). Namely, the following properties exhibit gradually stronger chaotic behavior: ergodicity, weak mixing, p-mixing and, finally, Kolmogorov property or exactness for invertible and non-invertible systems, respectively. The open problem in the general theory is the question of proper inclusions between p-mixing systems for different p (usually referred to as strong mixing systems; see Walters (1982)). For Gaussian systems the relations simplify: ergodic systems coincide with systems possessing the weak mixing property and all strong mixing properties are equivalent; between all other classes there are proper inclusions (cf. Newton (1968)).

The results obtained here for symmetric ID processes reveal exactly the same hierarchy as in the Gaussian case. In contrast to Maruyama (1970) here we employ, as a simple tool, the concept of the dynamical functional and we combine it with the stochastic representation of ID processes developed by Rajput and Rosiński (1989). As a result we are able to present, in a fairly simple way, a systematic study of the chaotic behavior of non-Gaussian ID stationary processes. In Section 5 we give a characterization of ergodic ID processes (Th. 1) and prove that ergodicity and weak mixing are equivalent (Th. 2). Section 6 contains a new characterization of mixing for ID processes and a new proof of the Maruyama result that mixing and mixing of all orders are equivalent (Th. 3). We also discuss some examples in Section 7. It turns out that each ID moving average process is mixing (Ex. 1) and there exists a non-Gaussian moving average process which has the Kolmogorov property or is exact (Ex. 3). Examples of Gross and Robertson (1993) of weakly mixing but not mixing ID sequences complete the picture of the hierarchy of chaotic properties in this case. A version of Lemma 1 as well as characterizations of ergodicity and mixing similar to (iii) of Theorem 1 and (ii) of Theorem 3 were also obtained in Cambanis and Ławniczak (1989).

2. Preliminaries. Throughout this paper we use the following notation. $(\Omega,\mathcal{F},P)$ is a probability space and $\mathbf{X}:\Omega\to\mathbb{R}^{\mathbb{R}}$ is a stochastic process, which can be identified with a family of real random variables $\{X_t\}_{t\in\mathbb{R}}$. Here we consider only real, stationary and measurable stochastic processes. We denote by $\mathcal{L}_0(\mathbf{X})$ the space of real random variables which are measurable with respect to $\mathcal{F}_{\mathbf{X}}=\sigma(X_t:t\in\mathbb{R})$. It is a complete metric space with the topology of convergence in probability. The closure of the linear span $\text{lin}\{X_t:t\in\mathbb{R}\}$ with respect to this topology is denoted by $\mathbf{L}_0(\mathbf{X})$. The group of transformations $(T_t)_{t\in\mathbb{R}}$ of $\mathcal{L}_0(\mathbf{X})$ is generated by the group of shifts $(S_t)_{t\in\mathbb{R}}$ on $\mathbb{R}^{\mathbb{R}}$ via $T_tY=f(S_t\mathbf{X})$, where f is a measurable function from $\mathbb{R}^{\mathbb{R}}$ to $\mathbb{R}$ and $Y=f(\mathbf{X})$. For $p\in\mathbb{N}$ we let $\mathcal{L}_p(\mathbf{X})=\{Y\in\mathcal{L}_0(\mathbf{X}):E|Y|^p<\infty\}$.

For a stationary and measurable process $\mathbf{X}$ we consider the following chaotic properties. The process $\mathbf{X}$ is *ergodic* if for each $Y\in\mathcal{L}_0(\mathbf{X})$ with $T_tY=Y$ a.s. for all $t\in\mathbb{R}$ we have $Y=\text{const}$ a.s. The process $\mathbf{X}$ is *weakly mixing* if for all $Y,Z\in\mathcal{L}_2(\mathbf{X})$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|E\{(T_tY)Z)\}-EY\,EZ|\,dt=0,$$

and *mixing of order* $p\in\mathbb{N}$ if for all $Y_0,Y_1,\ldots,Y_p$ in $\mathcal{L}_{p+1}(\mathbf{X})$ we have

$$E\{(T_{t_0}Y_0)\ (T_{t_1}Y_1)\ldots(T_{t_p}Y_p)\}\to EY_0\,EY_1\ldots EY_p$$

as $\min_{1\le j\le p}(t_j-t_{j-1})\to\infty$. Mixing of order 1 is called plain *mixing*.

Further, the process $\mathbf{X}$ has the *Kolmogorov property* (*K-property*) if there exists a σ-field $\mathcal{F}_0\subseteq\mathcal{F}_{\mathbf{X}}$ such that $\mathcal{F}_0\subseteq T_t\mathcal{F}_0$ for all $t\in\mathbb{R}$, the σ-field generated by $\bigcup_{t\in\mathbb{R}}T_t\mathcal{F}_0$ is equal to $\mathcal{F}_{\mathbf{X}}$, and $\bigcap_{t\in\mathbb{R}}T_t\mathcal{F}_0$ is the trivial σ-field; here the action of T_t on a measurable set is defined by the action of T_t on its indicator function. The exactness property is defined for a positive time process $\mathbf{X}:\Omega\to\mathbb{R}^{[0,\infty)}$. $\mathbf{X}$ is *exact* if the σ-field $\bigcap_{t\in(0,\infty)}T_t\sigma(X_u:u\ge 0)$ is trivial (see Rokhlin (1964)). It is clear that if a positive time stationary process $\{X_t\}_{t\ge0}$ has a stationary extension to $\{X_t\}_{t\in\mathbb{R}}$, then its exactness implies the K-property of $\{X_t\}_{t\in\mathbb{R}}$ (taking $\mathcal{F}_0=\sigma\{X_t:t\ge0\}$).

Let us recall that a random vector $\mathbf{V}$ in $\mathbb{R}^d$ with characteristic function $\phi_{\mathbf{V}}$ is *infinitely divisible* (ID) if for each $n\in\mathbb{N}$ there exists a characteristic function ϕ_n such that $\phi_{\mathbf{V}}=(\phi_n)^n$. A stochastic process $\{X_t\}_{t\in\mathbb{R}}$ is ID if for each $n\in\mathbb{N}$ and $(t_1,\ldots,t_n)\in\mathbb{R}^n$ the random vector $(X_{t_1},\ldots,X_{t_n})$ is ID. In this paper we will only deal with symmetric and stochastically continuous ID processes, i.e. additionally the random vector $(X_{t_1},\ldots,X_{t_n})$ is symmetric and X_{u_n} converges in probability to X_{u_0} whenever $u_n\to u_0$. Since stochastically continuous processes have measurable modifications, we assume, without further mention, that *all processes under consideration are measurable and stochastically continuous.*

The Lévy representation of the characteristic function of a symmetric ID random vector $\mathbf{Z}\in\mathbb{R}^d$ has the form

$$\phi_{\mathbf{Z}}(\mathbf{t})=Ee^{i(\mathbf{Z},\mathbf{t})}=\exp\Big(-\frac{1}{2}(\mathbf{R}\mathbf{t},\mathbf{t})+\int_{\mathbb{R}_0^d}(1-\cos(\mathbf{x},\mathbf{t}))\,Q(d\mathbf{x})\Big),$$

where $\mathbf{R}$ is a positive definite $d\times d$ matrix and Q is a symmetric σ-finite measure on $\mathbb{R}_0^d=\mathbb{R}^d\setminus\{0\}$ such that $\int|\mathbf{x}|^2(1+|\mathbf{x}|^2)^{-1}\,Q(d\mathbf{x})<\infty$. We refer to $(Q,\mathbf{R})$ as the *characteristics* of the ID vector $\mathbf{Z}$. Sometimes, especially when weak convergence is considered, it is convenient to use the equivalent characteristics $(S,\mathbf{R})$, where S is a finite measure on the Borel subsets of

$(0,\infty)\times\mathbb{S}_{d-1}$ defined by

$$S(A)=\int_{(|\mathbf{x}|,\mathbf{x}/|\mathbf{x}|)\in A}|\mathbf{x}|^2(1+|\mathbf{x}|^2)^{-1}\,Q(d\mathbf{x}).$$

Here $\mathbb{S}_{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.

The full description of weak convergence of ID random vectors in terms of their characteristics is given in Maruyama (1970), Proposition 5.1. It states that a sequence of ID random vectors $\mathbf{Z}_n$ with characteristics $(S_n,\mathbf{R}_n)$ converges weakly if and only if $\mathbf{R}_n$ is convergent as an element of a finite-dimensional vector space and S_n converges weakly, i.e. there exists a measure S_0 on $(0,\infty)\times\mathbb{S}_{d-1}$ such that for any continuous bounded function g we have $\lim_{n\to\infty}\int g\,dS_n=\int g\,dS_0$. The limit distribution is also ID with characteristics $(S_0,\mathbf{R}_0)$ and

$$\mathbf{R}_0=\lim_{n\to\infty}\mathbf{R}_n+\widetilde{\mathbf{R}},$$

where $\widetilde{\mathbf{R}}_{i,j}=\int_{\mathbb{S}_{d-1}}x_ix_j\,S_0(d\mathbf{x})$.

The stochastic representation of symmetric ID processes described below is the basic tool used in this paper. For further details see Rajput and Rosiński (1989).

Let $(S,\mathcal{S})$ be a measurable space and let Λ be a symmetric ID independently scattered stochastic measure on a δ-ring which generates $\mathcal{S}$. There is a one-to-one correspondence (written explicitly at the end of the section) between Λ and a triple $(\lambda,\sigma^2,\varrho)$, where λ is a σ-finite measure on $\mathcal{S}$, called the *control measure* of Λ, σ is a non-negative function in $\mathbf{L}_2(S,\mathcal{S},\lambda)$ and $\varrho:S\times\mathcal{B}_{\mathbb{R}}\to[0,\infty]$ is such that for each fixed $s\in S$, $\varrho(s,\cdot)$ is a symmetric Lévy measure and for each fixed B in the σ-field $\mathcal{B}_{\mathbb{R}}$ of Borel sets of the real line the function $\varrho(\cdot,B)$ is measurable and finite, whenever 0 does not belong to the closure of B. The function

$$\Psi(t,s)=t^2\sigma^2(s)+\int(1\wedge(tx)^2)\,\varrho(s,dx)$$

generates the Musielak–Orlicz space $\mathbf{L}_\Psi(S,\lambda)$ consisting of all measurable functions $f:S\to\mathbb{R}$ such that $\int_S\Psi(|f(s)|,s)\,\lambda(ds)<\infty$ with a Fréchet norm defined by

$$\|f\|_\Psi=\inf\Big\{c>0:\int_S\Psi(|f(s)|/c,s)\,\lambda(ds)\le c\Big\}.$$

Detailed information on Musielak–Orlicz spaces, also called generalized Orlicz spaces, can be found in Musielak (1983). A measurable function f on S is integrable with respect to Λ if and only if $f\in\mathbf{L}_\Psi(S,\lambda)$ and then $Y=\int_S f(s)\,\Lambda(ds)$ has characteristic function

$$\phi_Y(t)=\exp\{-N_\Psi(tf)\},$$

where

$$N_\Psi(g)=\int_S\Big[\frac{1}{2}g^2(s)\sigma^2(s)+\int_{\mathbb{R}}(1-\cos(g(s)x))\,\varrho(s,dx)\Big]\lambda(ds),\qquad g\in\mathbf{L}_\Psi(S,\lambda).$$

When $\mathbf{X}$ is a symmetric stochastically continuous ID process, there exist a measurable space $(S,\mathcal{S})$, a symmetric ID independently scattered random measure Λ on $(S,\mathcal{S})$ with corresponding triple (λ,σ,ϱ), a closed subspace $\mathbf{L}_\Psi(\mathbf{X})$ of $\mathbf{L}_\Psi(S,\lambda)$ and a linear topological isomorphism of $\mathbf{L}_0(\mathbf{X})$ onto $\mathbf{L}_\Psi(\mathbf{X})$, such that the processes $\{X_t\}_{t\in\mathbb{R}}$ and $\{\int_S f_t\,d\Lambda\}_{t\in\mathbb{R}}$ have the same finite-dimensional distributions, where for each $t\in\mathbb{R}$, f_t corresponds to X_t by the above isomorphism. We refer to $\{\int_S f_t\,d\Lambda\}_{t\in\mathbb{R}}$ as the *stochastic representation* of $\{X_t\}_{t\in\mathbb{R}}$. We denote by $(\mathbf{T}_t)_{t\in\mathbb{R}}$ the group of transformations of $\mathbf{L}_\Psi(\mathbf{X})$ which corresponds to $(T_t)_{t\in\mathbb{R}}$ by the stochastic representation so that $f_t=\mathbf{T}_tf_0$. If the process $\mathbf{X}$ is stationary, then for each $Y\in\mathbf{L}_0(\mathbf{X})$ and $u\in\mathbb{R}$ we have $\phi_Y=\phi_{T_uY}$ and consequently, for each $f\in\mathbf{L}_\Psi(\mathbf{X})$ and $u,t\in\mathbb{R}$ we have $N_\Psi(tf)=N_\Psi(t\mathbf{T}_uf)$.

We end this section with the relationship between the characteristics of the finite-dimensional distributions of $\mathbf{X}$ and their description in terms of $\mathbf{L}_\Psi(\mathbf{X})$. If $(Q,\mathbf{R})$ are the characteristics of $(X_{s_1},\dots,X_{s_m})$, then

$$\mathbf{R}_{ij}=\int f_{s_i}f_{s_j}\sigma^2\,d\lambda=\int f_{s_i-s_j}f_0\sigma^2\,d\lambda$$

and the measure Q satisfies

$$\int_{\mathbb{R}^m}g(y)\,Q(dy)=\int_S\Big[\int_{\mathbb{R}}g(xf_{s_1}(s),\dots,xf_{s_n}(s))\,\varrho(s,dx)\Big]\lambda(ds);$$

this will be used in the proof of equivalence of mixing and p-mixing in Section 6.

3. Dynamical functional. It is very convenient to express the ergodic and mixing properties of a stationary process $\mathbf{X}$ through its *dynamical functional* $\Phi:\mathcal{L}_0(\mathbf{X})\times\mathbb{R}\to\mathbb{C}$ defined by

$$\Phi(Y,t)=E\exp\{i(T_tY-Y)\}$$

(see Podgórski and Weron (1991)). For each $Y\in\mathcal{L}_0(\mathbf{X})$ the function $\Phi(Y,\cdot)$ is symmetric and positive definite. If the process $\mathbf{X}$ is stochastically continuous, then the group $(T_t)_{t\in\mathbb{R}}$ is continuous on $\mathbf{L}_0(\mathbf{X})$ with respect to the topology of convergence in probability, and consequently Φ is continuous in the product topology on $\mathbf{L}_0(\mathbf{X})\times\mathbb{R}$.

We have the following simple characterizations of ergodic and mixing properties of a stationary process.

PROPOSITION 1. *Let* $\mathbf{X}$ *be a stationary measurable stochastic process.*

(i) $\mathbf{X}$ *is ergodic if and only if for each* $Y \in \mathcal{L}_0(\mathbf{X})$ *or, equivalently, for each* $Y \in \operatorname{lin}\{X_t : t \in \mathbb{R}\}$, *we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \Phi(Y,t)\,dt = |Ee^{iY}|^2.$$

(ii) $\mathbf{X}$ *is weakly mixing if and only if for each* $Y \in \mathcal{L}_0(\mathbf{X})$, *or, equivalently, for each* $Y \in \operatorname{lin}\{X_t : t \in \mathbb{R}\}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\Phi(Y,t) - |Ee^{iY}|^2|\,dt = 0.$$

(iii) $\mathbf{X}$ *is mixing if and only if for each* $Y \in \mathcal{L}_0(\mathbf{X})$ *or, equivalently, for each* $Y \in \operatorname{lin}\{X_t : t \in \mathbb{R}\}$,

$$\lim_{T\to\infty} \Phi(Y,T) = |Ee^{iY}|^2.$$

(iv) $\mathbf{X}$ *is mixing of order* p *if and only if for all* $Y_0, \ldots, Y_p$ *in* $\mathcal{L}_0(\mathbf{X})$ *or, equivalently, in* $\operatorname{lin}\{X_t : t \in \mathbb{R}\}$, *we have*

$$E\exp\{i(T_{t_0}Y_0 + \ldots + T_{t_p}Y_p)\} \to Ee^{iY_0}\ldots Ee^{iY_p}$$

as $\min_{0\le i\le p}(t_i - t_{i-1}) \to \infty$.

Proof. Part (i) was proven in Podgórski and Weron (1991). The proofs of the first three parts are very similar and the proof of part (iv) follows at once from the definition of mixing of order p by approximation arguments. Thus we present here only the proof of part (ii).

It is obvious that weak mixing implies the condition given in (ii). Since any element of $\mathcal{L}_2(\mathbf{X})$ can be approximated by linear combinations of random variables of the form $\exp(iY)$, where $Y \in \operatorname{lin}\{X_t : t \in \mathbb{R}\}$, it suffices to prove that if for all V from a linearly dense subset $\mathcal{E}$ of $\mathcal{L}_2(\mathbf{X})$ we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |E\{(T_tV)V\} - (EV)^2|\,dt = 0,$$

then $\mathbf{X}$ is weakly mixing. Note that although the functions $\exp(iY)$ are complex it is enough to consider real V in the above condition as it stands in the definition of $\mathcal{L}_2(\mathbf{X})$.

For $V \in \mathcal{E}$ we define

$$\mathcal{E}_V = \left\{ Z \in \mathcal{L}_2(\mathbf{X}) : \lim_{T\to\infty} \frac{1}{T} \int_0^T |E\{(T_tV)Z\} - EV\cdot EZ|\,dt = 0\right\}.$$

Now it is enough to prove that

$$\{V \in \mathcal{L}_2(\mathbf{X}) : \mathcal{E}_V = \mathcal{L}_2(\mathbf{X})\} = \mathcal{L}_2(\mathbf{X}).$$

$\mathcal{E}_V$ is a closed subspace of $\mathcal{L}_2(\mathbf{X})$ and by assumption both $\mathbf{1} \in \mathcal{E}_V$ and $T_tV \in \mathcal{E}_V$ for each $t \in \mathbb{R}$. Thus if Z is orthogonal to $\mathcal{E}_V$, then $E\{(T_tV)Z\} = 0$ and $EZ = 0$ implying that $Z \in \mathcal{E}_V$ and, as a consequence, that $Z = 0$. It follows that $\mathcal{E}_V = \mathcal{L}_2(\mathbf{X})$ for each $V \in \mathcal{E}$. Consequently,

$$\mathcal{E} \subseteq \{V \in \mathcal{L}_2(\mathbf{X}) : \mathcal{E}_V = \mathcal{L}_2(\mathbf{X})\}.$$

Since $\mathcal{E}$ is a linearly dense subset of $\mathcal{L}_2(\mathbf{X})$ and the set on the right hand side of the above inclusion is a closed subspace of $\mathcal{L}_2(\mathbf{X})$, it follows that $\mathcal{E}$ has to be equal to the whole space $\mathcal{L}_2(\mathbf{X})$. ∎

Note that the dynamical functional of an ID stochastic process $\mathbf{X}$ with stochastic representation of the form $(\int_S f_t\,d\Lambda)_{t\in\mathbb{R}}$ is given by

$$\Phi(Y,t) = \exp\{-N_\Psi(\mathbf{T}_tf - f)\},$$

where $f \in \mathbf{L}_\Psi(\mathbf{X})$ corresponds to $Y \in \mathbf{L}_0(\mathbf{X})$. When $\mathbf{X}$ is a symmetric α-stable process ($0 < \alpha \le 2$), then its dynamical functional for $Y \in \mathbf{L}_0(\mathbf{X})$ takes the form

$$\Phi(Y,t) = \exp\{-\|\mathbf{T}_tf - f\|_\alpha^\alpha\},$$

where in this case $\mathbf{L}_\Psi(S,\lambda) = \mathbf{L}_\alpha(S,\lambda)$. In the Gaussian case $\alpha = 2$ we also have for $Y \in \mathbf{L}_0(\mathbf{X})$,

$$\Phi(Y,t) = \exp\{\operatorname{Cov}(T_tY,Y) - \operatorname{Var}(Y)\}.$$

4. Lemmas. To prove our main results we need the following technical lemmas.

With the notation of Section 2 for each $f \in \mathbf{L}_\Psi(\mathbf{X})$ we define the functions $R_f^G, R_f^P : \mathbb{R} \to \mathbb{R}$ by

$$R_f^G(t) = \int_S (\mathbf{T}_tf)f\sigma^2\,d\lambda,$$
$$R_f^P(t) = 2N_\Psi(f) - N_\Psi(\mathbf{T}_tf - f) - R_f^G(t).$$

LEMMA 1. *For each* $f \in \mathbf{L}_\Psi(\mathbf{X})$ *the functions* R_f^G *and* R_f^P *are continuous and positive definite.*

Proof. Since $\mathbf{X}$ is stochastically continuous, i.e. $\lim_{t\to t_0} X_t = X_{t_0}$ in probability, it follows that $\lim_{t\to t_0} f_t = f_{t_0}$ in the Fréchet norm $\|\cdot\|_\Psi$ (see Section 2). Thus

$$\lim_{t\to t_0} \int_S \Psi(|f_t(s) - f_{t_0}(s)|, s)\,\lambda(ds) = 0.$$

Consequently, $\lim_{t\to t_0} \int_S (f_t - f_{t_0})\sigma^2\,d\lambda = 0$. This implies the continuity of R_f^G. Moreover, $\Phi(Y,t) = \exp\{-N_\Psi(\mathbf{T}_tf - f)\}$ is continuous and thus so is R_f^P. Now since R_f^G is clearly positive definite, it is enough to show that R_f^P

is positive definite. From the stationarity of $\mathbf{X}$ and the symmetry of $\varrho(s,\cdot)$ we have for $u,t\in\mathbb{R}$,

$$R_f^p(t-u)=\int_S\int_{\mathbb{R}}(1-e^{ix(\mathbf{T}_tf)(s)})(1-e^{-ix(\mathbf{T}_uf)(s)})\,\varrho(s,dx)\,\lambda(ds).$$

Thus for $n\in\mathbb{N}$ and $a_1,\ldots,a_n\in\mathbb{R}$,

$$\sum_{i,j=1}^n a_ia_jR_f^p(t_i-t_j)=\int_S\int_{\mathbb{R}}\Big|\sum_{i=1}^n a_i(1-e^{-ix(\mathbf{T}_{t_i}f)(s)})\Big|^2\varrho(s,dx)\,\lambda(ds)\geq 0.\ ■$$

We use the following standard notation. If ν is a finite measure, then $\widehat{\nu}$ denotes its Fourier transform and $e^\nu=\sum_{k=0}^\infty \nu^{*k}/k!$, where ν^{*k} denotes the k-fold convolution of the measure ν. With this notation $\widehat{e^\nu}=e^{\widehat{\nu}}$.

LEMMA 2. *If ν is a symmetric finite measure on $\mathcal{B}_{\mathbb{R}}$, then the following conditions are equivalent*:

(i) $e^\nu(\{0\})=1$,
(ii) $\nu^{*2}(\{0\})=0$,
(iii) ν *has no atom.*

Proof. Since $e^\nu(\{0\})=\sum_{k=0}^\infty \nu^{*k}(\{0\})/k!$ and ν^{*k} is a positive measure, clearly (i)⇒(ii).

Let $\mathcal{A}$ be the set of atoms of ν. By the symmetry of ν we have $\nu^{*2}(0)=\sum_{a\in\mathcal{A}}\nu(-a)\nu(a)=\sum_{a\in\mathcal{A}}[\nu(a)]^2$ and condition (ii) implies that the set $\mathcal{A}$ is empty.

If ν has no atom, then for each $k\in\mathbb{N}$, the measure ν^{*k} has no atom, so (iii)⇒(i). ■

The next lemma expresses some analytical properties of real functions; nevertheless it is proven with the use of probabilistic methods.

LEMMA 3. *For each bounded measurable function $\zeta:\mathbb{R}\to\mathbb{R}$ the following conditions are equivalent*:

(i) $$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{c\zeta(t)}\,dt=1\quad \textit{for each } c>0,$$

(ii) $$\lim_{T\to\infty}\frac{1}{T}\int_0^T |e^{c\zeta(t)}-1|\,dt=0\quad \textit{for each } c>0,$$

(iii) $$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\zeta(t)|\,dt=0,$$

(iv) $$\lim_{T\to\infty}\frac{1}{T}|\{t\in[0,T]:|\zeta(t)|>\varepsilon\}|=0\quad \textit{for each } \varepsilon>0,$$

(v) *there exist natural numbers $k\neq j$, at least one of them odd, and $b\in\mathbb{R}$ such that*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\{\zeta(t)+b\}^n\,dt=b^n\quad \textit{for } n=k,j,$$

(vi) *there exists $b\in\mathbb{R}$ such that*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\{\zeta(t)+b\}^n\,dt=b^n\quad \textit{for all } n\in\mathbb{N}.$$

Proof. Since $|\zeta(t)|\leq M$ for all $t\in\mathbb{R}$, the normalized occupation measure μ_T of ζ over $[0,T]$, defined on a Borel subset B of $[-M,M]$ by $\mu_T(B)=|\zeta^{-1}B\cap[0,T]|/T$, is a probability measure corresponding to a random variable Z_T with $|Z_T|\leq M$. By the transformation theorem

$$\frac{1}{T}\int_0^T G(\zeta(t))\,dt=\int_{-M}^M G(x)\,d\mu_T(x)=EG(Z_T)$$

for any measurable function G for which either integral exists. In this framework the proof of Lemma 3 follows from standard properties of convergence of random variables.

First, (i) says that the Laplace transform $E\exp\{-c(M-Z_T)\}$ converges to e^{-cM}. The continuity theorem for Laplace transforms implies weak convergence of the distributions of Z_T to δ_0, which implies convergence in probability of Z_T to zero, i.e. (iv).

Now, the boundedness of the family $(Z_T)_{T\geq 0}$ and its convergence in probability to zero imply that $\lim_{T\to\infty}EG(Z_T)=G(0)$ for any continuous function G on $\mathbb{R}$. Thus (iv) implies all the other conditions.

Since the implications (iii)⇒(iv), (ii)⇒(i) and (vi)⇒(v) are obvious, it is enough to prove (v)⇒(iv). By the tightness of the family of distributions of Z_T, it is sufficient to show that if Z_{T_n} converges weakly to, say, Z_0 then $Z_0=0$ a.s. Indeed, if Z_0 is not degenerate, then for $k>j$ we have the sharp Jensen inequality

$$E(Z_0+b)^k>\{E(Z_0+b)^j\}^{k/j}$$

and thus (v) cannot be satisfied since it implies that both sides must be equal to b^k. Consequently, we must have $Z_0=a$ a.s. for some real a and $(a+b)^k=b^k$, $(a+b)^j=b^j$. Since either j or k is odd, this implies $a=0$. ■

The final lemma is used in Section 6 in the proof of equivalence of mixing and p-mixing.

LEMMA 4. *Let $\mathbf{Y}_n$ be a sequence of ID random vectors in $\mathbb{R}^k$ with characteristics $(Q_n,\mathbf{R}_n)$. If $\mathbf{Y}_n$ converges weakly to an ID random vector with* *characteristics* $(Q_0, \mathbf{R}_0)$, *then for each bounded function* $F : \mathbb{R}^k \to \mathbb{R}$ *such that* $F(\mathbf{x}) = o(|\mathbf{x}|^2)$ *as* $\mathbf{x} \to 0$, *we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}_0^k} F(\mathbf{x})\, Q_n(d\mathbf{x}) = \int_{\mathbb{R}_0^k} F(\mathbf{x})\, Q_0(d\mathbf{x}).$$

Proof. For $r \in (0,\infty)$ and $\omega \in \mathbb{S}_{k-1}$ define a continuous function $h(r,\omega) = r\omega$. If a function g on $\mathbb{R}_0^k$ is continuous and bounded, then so is $g \circ h$. As a consequence, weak convergence of S_n to a measure S_0 implies weak convergence of measures defined on Borel sets of $\mathbb{R}_0^k$ by $\widetilde{S}_n = S_n \circ h^{-1}$ to $\widetilde{S}_0 = S_0 \circ h^{-1}$. Note that

$$\widetilde{S}_n(A) = \int_A \frac{|\mathbf{x}|^2}{1+|\mathbf{x}|^2}\, Q_n(d\mathbf{x}).$$

Thus in view of remarks on weak convergence of ID vectors in Section 2, it is enough to note that $F(\mathbf{x})(1+|\mathbf{x}|^2)/|\mathbf{x}|^2$ is bounded and continuous on $\mathbb{R}_0^k$. ■

5. Ergodicity and weak mixing. For a symmetric ID process $\mathbf{X}$ we have the following characterization of ergodicity.

THEOREM 1. *Let* $\mathbf{X}$ *be a stationary symmetric ID stochastic process with stochastic representation* $\{\int_S \mathbf{T}_t f_0\, d\Lambda\}_{t\in\mathbb{R}}$. *Then the following are equivalent*:

(i) $\mathbf{X}$ *is ergodic*,

(ii) *for each* $f \in L_\Psi(\mathbf{X})$ *or, equivalently, for each* $f \in \mathrm{lin}\{f_t : t \in \mathbb{R}\}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \exp\{2N_\Psi(f) - N_\Psi(\mathbf{T}_t f - f)\}\, dt = 1,$$

(iii) *for each natural number* n *or, equivalently, for* $n = 1,2$, *and for each* $f \in L_\Psi(\mathbf{X})$ *or, equivalently, for each* $f \in \mathrm{lin}\{f_t : t \in \mathbb{R}\}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T N_\Psi^n(\mathbf{T}_t f - f)\, dt = 2^n N_\Psi^n(f),$$

(iv) *for each* $f \in L_\Psi(\mathbf{X})$ *or, equivalently, for each* $f \in \mathrm{lin}\{f_t : t \in \mathbb{R}\}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |N_\Psi(\mathbf{T}_t f - f) - 2N_\Psi(f)|\, dt = 0.$$

Proof. By Proposition 1 and the form of the dynamical functional for ID processes, $\mathbf{X}$ is ergodic if and only if for each $f \in L_\Psi(\mathbf{X})$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \exp\{-N_\Psi(\mathbf{T}_t f - f)\}\, dt = \exp\{-2N_\Psi(f)\},$$

which proves the equivalence (i)⇔(ii).

By Lemma 1 and Bochner's theorem, for each $f \in \mathbf{L}_\Psi(\mathbf{X})$ there exists a finite symmetric measure ν_f such that $\widehat{\nu}_f(t) = 2N_\Psi(f) - N_\Psi(\mathbf{T}_t f - f)$. Condition (ii) and the fact that for any finite measure ν defined on $\mathcal{B}_{\mathbb{R}}$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \widehat{\nu}(t)\, dt = \nu(\{0\}),$$

imply that $e^{\nu_f}(\{0\}) = 1$. By Lemma 2 this is equivalent to $\nu_f^{*2}(\{0\}) = 0$, which implies $\nu_f(\{0\}) = 0$. Applying the above relation to ν_f and to ν_f^{*2} we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \{N_\Psi(\mathbf{T}_t f - f) - 2N_\Psi(f)\}\, dt = 0,$$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \{N_\Psi(\mathbf{T}_t f - f) - 2N_\Psi(f)\}^2\, dt = 0.$$

Thus

$$0 = \lim_{T\to\infty} \frac{1}{T} \int_0^T \{N_\Psi^2(\mathbf{T}_t f - f) - 4N_\Psi(\mathbf{T}_t f - f)N_\Psi(f) + 4N_\Psi^2(f)\}\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T N_\Psi^2(\mathbf{T}_t f - f)\, dt - 4N_\Psi^2(f)$$

and we have obtained condition (iii) for $n = 1,2$. Consequently, by Lemma 3, condition (iii) holds for each natural number n.

The remaining implications (iii)⇒(iv) and (iv)⇒(ii) follow immediately from Lemma 3.

It is easy to notice from the way we used Proposition 1, that in the above argument we can replace the space $\mathbf{L}_\Psi(\mathbf{X})$ by $\mathrm{lin}\{f_t : t \in \mathbb{R}\}$. ■

While, in general, weak mixing is stronger than ergodicity, we now prove that for ID processes they coincide.

THEOREM 2. *Ergodicity of a stationary symmetric ID process implies weak mixing.*

Proof. According to Proposition 1, a symmetric ID process is weakly mixing if and only if for each $f \in \mathbf{L}_\Psi(\mathbf{X})$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\exp\{2N_\Psi(f) - N_\Psi(\mathbf{T}_t f - f)\} - 1|\, dt = 0.$$

If $\mathbf{X}$ is ergodic, condition (iv) of Theorem 1 shows that condition (iii) of Lemma 3 is satisfied with $\zeta(t) = 2N_\Psi(f) - N_\Psi(\mathbf{T}_t f - f)$, hence so is condition (ii) of Lemma 3 and putting $c = 1$ implies that $\mathbf{X}$ is weakly mixing. ∎

6. Mixing. In this section we consider strong mixing properties of a symmetric ID process. Mixing and p-mixing are equivalent for such a process as shown by Maruyama (1970), where the result is proven in the case of general, not necessarily symmetric, ID processes. One of the goals of this paper is to present relations between chaotic behavior of ID processes and properties of the related Musielak–Orlicz space. For this reason we decided to sketch the proof of the mentioned equivalence even though its main idea is very similar to the one used by Maruyama. As a by-product, we provide a characterization of mixing alternative to the one given in Theorem 6 of Maruyama (1970).

THEOREM 3. *Let $\mathbf{X}$ be a stationary symmetric ID process with stochastic representation $\{\int_S \mathbf{T}_t f_0\, d\Lambda\}_{t\in\mathbb{R}}$. Then the following are equivalent:*

(i) $\mathbf{X}$ *is mixing,*

(ii) *for each $f \in \mathbf{L}_\Psi(\mathbf{X})$ or, equivalently, for each $f \in \operatorname{lin}\{f_t : t \in \mathbb{R}\}$,*

$$\lim_{T\to\infty} N_\Psi(\mathbf{T}_T f - f) = 2N_\Psi(f),$$

(iii) $\mathbf{X}$ *is p-mixing.*

Sketch of proof. (i)⇔(ii) follows from part (iii) of Proposition 1 and the form of the dynamical functional. Since (iii)⇒(i), it is enough to prove that (i) and (ii) imply (iii).

Let $Y_0, \ldots, Y_p \in \operatorname{lin}\{X_t : t \in \mathbb{R}\}$ and let $g_0, \ldots, g_p \in \operatorname{lin}\{f_t : t \in \mathbb{R}\}$ correspond to them through the stochastic representation. For $\mathbf{t} = (t_0, \ldots, t_p) \in \mathbb{R}^{p+1}$ let $\delta(\mathbf{t}) = \min_{1\le i\le p}(t_i - t_{i-1})$. Then the condition for p-mixing described in (iv) of Proposition 1 can be written as

$$\lim_{\delta(\mathbf{t})\to\infty} \{G(\mathbf{t}) + P(\mathbf{t})\} = 0,$$

where $G(\mathbf{t})$ and $P(\mathbf{t})$ are the differences between the Gaussian parts and Poisson parts respectively in the exponents of $E\exp(\sum_{k=0}^p T_{t_k} Y_k)$ and $Ee^{iY_0} \ldots Ee^{iY_p}$. We will show that (i) and (ii) imply that both $G(\mathbf{t}) \to 0$ and $P(\mathbf{t}) \to 0$ as $\delta(\mathbf{t}) \to \infty$.

The mixing condition (ii) implies that (X_t, X_0) converges in distribution to (X_0', X_0), where X_0' is an independent copy of X_0. Thus the function $R_{f_0}^G$ defined in Section 4 is convergent at infinity (see remarks on weak convergence of ID random vectors in Section 2) and since it is positive definite, the limit is non-negative. So by (ii) both $R_{f_0}^G$ and $R_{f_0}^P$ must converge to zero. Now for $g_i = \sum_{k=1}^n a_{ik} f_{s_{ik}}$, $i = 0, \ldots, p$, by stationarity of $\mathbf{X}$, we have

$$G(\mathbf{t}) = \int_S \sum_{i>j}^p \mathbf{T}_{t_i} g_i \mathbf{T}_{t_j} g_j \sigma^2\, d\lambda = \sum_{i>j}^p \sum_{k,l}^n a_{ik} a_{jl} R_{f_0}^G(t_i - t_j + s_{ik} - s_{jl}).$$

Thus $G(\mathbf{t}) \to 0$ since for $i > j$, $t_i - t_j + s_{ik} - s_{jl}$ tends to infinity as $\delta(\mathbf{t}) \to \infty$.

To prove $\lim_{\delta(\mathbf{t})\to\infty} P(\mathbf{t}) = 0$ we use induction and assume that $\mathbf{X}$ is mixing of order $p-1$. For $\mathbf{u} \in \mathbb{R}^{p+1}$ let $C_p(\mathbf{u}) = p - \sum_{i=0}^p \cos(u_i) + \cos(\sum_{i=0}^p u_i)$, $S_p(\mathbf{u}) = \sum_{i=0}^p \sin(u_i) - \sin(\sum_{i=0}^p u_i)$. Note the relation

$$C_p(\mathbf{u}) = C_{p-1}(u_0, \ldots, u_{p-2}, u_p) + C_1(u_0 + \ldots + u_{p-2} + u_p, u_{p-1}).$$

By this relation, symmetry of the measures $\varrho(s, dx)$ and the equality $(1 - e^a)(1 - e^{b+c}) = (1 - e^a)(1 - e^c) + e^c(1 - e^a)(1 - e^b)$ one can obtain

$$\begin{aligned}
P(\mathbf{t}) &= \int_S\int_{\mathbb{R}} C_p(\mathbf{T}_{t_0} g_0(s)x, \ldots, \mathbf{T}_{t_p} g_p(s)x)\, \varrho(s, dx)\, \lambda(ds) \\
&= \int_S\int_{\mathbb{R}} \Big[C_{p-1}((\mathbf{T}_{t_i} g_i(s)x)_{i\ne p-1}^p) \\
&\quad + C_1\Big(\sum_{i\ne p-1} \mathbf{T}_{t_i} g_i(s)x, \mathbf{T}_{t_{p-1}} g_{p-1}(s)x\Big)\Big]\, \varrho(s, dx)\, \lambda(ds) \\
&= \int_S\int_{\mathbb{R}} [C_{p-1}((\mathbf{T}_{t_i} g_i(s)x)_{i\ne p-1}^p) \\
&\quad + C_1(\mathbf{T}_{t_{p-1}} g_{p-1}(s)x, \mathbf{T}_{t_p} g_p(s)x)]\, \varrho(s, dx)\, \lambda(ds) \\
&\quad + \int_S\int_{\mathbb{R}} \cos(\mathbf{T}_{t_p} g_p(s)x) C_1\Big(\sum_{i=0}^{p-2} \mathbf{T}_{t_i} g_i(s)x, \mathbf{T}_{t_{p-1}} g_{p-1}(s)x\Big)\, \varrho(s, dx)\, \lambda(ds) \\
&\quad + \int_S\int_{\mathbb{R}} \sin(\mathbf{T}_{t_p} g_p(s)x) S_1\Big(\sum_{i=0}^{p-2} \mathbf{T}_{t_i} g_i(s)x, \mathbf{T}_{t_{p-1}} g_{p-1}(s)x\Big)\, \varrho(s, dx)\, \lambda(ds).
\end{aligned}$$

The first summand converges to zero by the inductive assumption. The second summand is upper bounded in absolute value by

$$\int_S \int_{\mathbb{R}} \Big| C_1\Big(\sum_{i=0}^{p-2} \mathbf{T}_{t_i} g_i(s)x, \mathbf{T}_{t_{p-1}} g_{p-1}(s)x\Big)\Big| \varrho(s,dx)\,\lambda(ds),$$

which converges, as $\delta(\mathbf{t}) \to \infty$, by the relation at the end of Section 2 and by Lemma 4, to

$$\int_{\mathbb{R}_0^p} \Big| C_1\Big(\sum_{i=0}^{p-2} x_i, x_{p-1}\Big)\Big| Q_0(dx),$$

where Q_0 is the characteristic of a vector of independent ID random variables in $\mathbb{R}^p$. Now $Q_0(\{x \in \mathbb{R}^p : x_j x_i \neq 0\}) = 0$ for any $i \neq j$ (if the characteristic Q_0 is the sum of the characteristics of the coordinates restricted to the coordinate axes in $\mathbb{R}^p$). This implies $C_1(\sum_{i=0}^{p-2} x_i, x_{p-1}) = 0$ a.e. $[Q_0]$. The same arguments apply to the third summand.

Remark 1. The basic idea of this and Maruyama's proofs was to establish the following fact. If the limiting distribution of some multidimensional ID sequence has a multidimensional Lévy measure whose restriction to any two coordinates is concentrated along the axes then also the whole measure is concentrated along the axes (which corresponds to mutual independence of the coordinate variables). This, in particular, implies that in this case pairwise independence of a family of random variables is equivalent to their mutual independence.

Remark 2. For the underlying Musielak–Orlicz space $L_\Psi(\mathbf{X})$ mixing means that for $g_0, g_1 \in L_\Psi(\mathbf{X})$ we have

$$\lim_{t\to\infty} N_\Psi(\mathbf{T}_{t_1} g_1 + g_0) = N_\Psi(g_1) + N_\Psi(g_0).$$

But this also implies (for example by application of Lemma 4) that

$$\lim_{t\to\infty} \int_S \Psi(\mathbf{T}_{t_1} g_1(s) + g_0(s), s)\,\lambda(ds) = \int_S \Psi(g_1(s), s)\,\lambda(ds) + \int_S \Psi(g_0(s), s)\,\lambda(ds).$$

In general, p-mixing translates to

$$\lim_{\delta(\mathbf{t})\to\infty} N_\Psi(g_0 + \ldots + \mathbf{T}_{t_p} g_p) = N_\Psi(g_0) + \ldots + N_\Psi(g_p),$$

which implies

$$\lim_{\delta(\mathbf{t})\to\infty} \int_S \Psi(g_0(s) + \ldots + \mathbf{T}_{t_p} g_p(s), s)\lambda(ds) = \int_S \Psi(g_0(s), s)\,\lambda(ds) + \ldots + \int_S \Psi(g_p(s), s)\,\lambda(ds).$$

The opposite implications, in general, do not hold.

Remark 3. The following additional characterization of mixing can be easily obtained from the proof of Theorem 6 of Maruyama (1970):

(iv) *for each* $a_1, a_2 \in \mathbb{R}$,

$$\lim_{t\to\infty} N_\Psi(a_1 f_0 + a_2 \mathbf{T}_t f_0) = N_\Psi(a_1 f_0) + N_\Psi(a_2 f_0).$$

7. Examples. Here we give the explicit form of the conditions in Theorems 1 and 3 for some examples of stationary ID processes. In all these examples the symmetric ID random measure Λ on $(S, \mathcal{S})$ with control measure λ has ϱ and σ of the form described below. Let r be a symmetric Lévy measure on $\mathbb{R}$ and for each $s \in \mathbb{R}$, let $\varrho(s, A) = r(A)$ for all $A \in \mathcal{B}_{\mathbb{R}}$ and $\sigma^2(s) = \sigma_0^2 \geq 0$. In this case $f \in \mathbf{L}_\Psi(\mathbf{X})$ if and only if

$$\sigma_0^2 \int_S |f(s)|^2\,\lambda(ds) < \infty, \qquad \int_S \int_{\mathbb{R}} \{1 \wedge |xf(s)|^2\}\, r(dx)\,\lambda(ds) < \infty.$$

Notice that if $\sigma_0^2 > 0$ then $\mathbf{L}_\Psi \subseteq \mathbf{L}_2(S, \lambda)$. For $A \in \mathcal{S}$ we have

$$Ee^{it\Lambda(A)} = \exp\Big\{-\lambda(A)\Big[\frac{1}{2}t^2\sigma_0^2 + \int (1 - \cos tx)\, r(dx)\Big]\Big\}.$$

Recall that $E|\Lambda(A)|^q < \infty$ if and only if

$$\int_A \int_{|x|>1} |x|^q\, \varrho(s, dx)\,\lambda(ds) = 2\lambda(A) \int_1^\infty x^q\, r(dx) < \infty$$

(cf. Rajput and Rosiński (1989)).

Example 1. *Moving averages are mixing.* Assume that $S = \mathbb{R}$ and λ is Lebesgue measure on $\mathcal{B}_{\mathbb{R}}$. Then Λ is stationarily scattered, and corresponds to a process L_s with stationary independent increments and $L_0 = 0$ via $\Lambda\{(s,t]\} = L_t - L_s$ or $\Lambda(ds) = dL_s$. $\{L_s\}_{s\in\mathbb{R}}$ is called a *Lévy motion.* By the invariance of λ under the action of the shift transformation, if $f_0(\cdot)$ is Λ-integrable then so is $f_0(\cdot - t)$ for all $t \in \mathbb{R}$. A symmetric ID process is called a *moving average* if it has the representation

$$\int_{-\infty}^{\infty} f_0(t - s)\, dL_s, \qquad t \in \mathbb{R}.$$

In this case we have

$$N_\Psi(f) = \frac{1}{2}\sigma_0^2 \int_{-\infty}^{\infty} f^2(s)\,ds + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [1 - \cos(f(s)x)]\, r(dx)\,ds.$$

It is clear that for every $f \in \mathbf{L}_\Psi(\mathbb{R}, \lambda)$, $N_\Psi\{f(\cdot + t)\} = N_\Psi\{f(\cdot)\}$ for all $t \in \mathbb{R}$, so that every moving average process is stationary.

We now prove that every symmetric ID moving average process is mixing, or, according to Theorem 3, that

$$N_\Psi\{f(\cdot - T) - f(\cdot)\} \to 2N_\Psi\{f(\cdot)\} \quad \text{as } T \to \infty.$$

However, this follows immediately from the expression for N_Ψ when f has compact support, and for general f in $\mathbf{L}_\Psi(\mathbb{R},\lambda)$ from the fact that $f(\cdot)1_{(|\cdot|\le c)} \to f(\cdot)$ in $\mathbf{L}_\Psi(\mathbb{R},\lambda)$ as $c \to \infty$. An alternative proof establishes likewise the condition in (iii) of Proposition 1:

$$\lim_{T\to\infty} E\exp\Big\{i\int_{-\infty}^{\infty}[f(s-T)-f(s)]\,\Lambda(ds)\Big\} = \Big|E\exp\Big\{i\int_{-\infty}^{\infty} f(s)\,\Lambda(ds)\Big\}\Big|^2.$$

EXAMPLE 2. *Processes with Poisson, gamma and compound Poisson stochastic representations.* We say that a symmetric ID process $\mathbf{X}$ has a *Poisson* (or *gamma*) *stochastic representation* $\int_S f_t\,d\Lambda$ if the random measure Λ has a Poisson (or gamma) distribution, respectively. In these cases $\sigma^2(s) = 0$ and $\varrho(s,A) = r(A)$ for all $s \in \mathbb{R}$ and $A \in \mathcal{S}$ with $\lambda(A) < \infty$.

The Poisson case corresponds to $r = \delta_{\{-1\}} + \delta_{\{1\}}$. Then $f \in \mathbf{L}_\Psi(S,\lambda)$ if and only if $\lambda\{|f| > 1\} < \infty$ and $\int_{\{|f|\le 1\}} f^2\,d\lambda < \infty$. Moreover, we have

$$N_\Psi(f) = 2\int_S \{1 - \cos[f(s)]\}\,\lambda(ds).$$

Now let $\{\mathbf{T}_t\}_{t\in\mathbb{R}}$ be any group of transformations of $\mathbf{L}_\Psi(S,\lambda)$ such that for $t,u \in \mathbb{R}$,

$$\int_S \{1 - \cos[tf(s)]\}\,\lambda(ds) = \int_S \{1 - \cos[tT_u f(s)]\}\,\lambda(ds)$$

(one can take, for example, $\{\mathbf{T}_t\}_{t\in\mathbb{R}}$ induced by pointwise and measure preserving transformations of $(S,\mathcal{S},\lambda)$). The conditions in Theorems 1 and 3 take a more concrete form upon using the expression for $N_\Psi(f)$. Thus the condition in (iv) of Theorem 1 becomes

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big|\int_S \{1 - 2\cos[f(s)] + \cos[(\mathbf{T}_t f - f)(s)]\}\,\lambda(ds)\Big|\,dt = 0$$

and similarly the condition in Theorem 3 becomes

$$\lim_{t\to\infty}\int_S \{1 - \cos[(\mathbf{T}_t f - f)(s)]\}\,\lambda(ds) = 2\int_S \{1 - \cos[f(s)]\}\,\lambda(ds).$$

The gamma process corresponds to $r(dx) = |x|^{-1}e^{-\theta|x|}\,dx$ for some $\theta > 0$. In this case $f \in \mathbf{L}_\Psi(S,\lambda)$ if and only if

$$\int_S e^{-\theta/|f|}(e^{\theta/|f|} - 1 - \theta/|f|)f^2\,d\lambda < \infty, \qquad \int\int_{1/|f(s)|}^{\infty} x^{-1}e^{-\theta x}\,dx\,\lambda(ds) < \infty.$$

Since $2\int_0^\infty[1-\cos(tx)]x^{-1}e^{-\theta x}\,dx = \ln[1 + (t/\theta)^2]$, it follows that

$$N_\Psi(f) = \int_S \ln[1 + \theta^{-2}f^2]\,d\lambda.$$

The corresponding conditions for ergodicity and mixing can be expressed respectively as follows:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big|\int_S \ln[\{1 + \theta^{-2}(\mathbf{T}_t f - f)^2\}(1 + \theta^{-2}f^2)^{-2}]\,d\lambda\Big|\,dt = 0$$

and

$$\lim_{t\to\infty}\int_S \ln[1 + \theta^{-2}(\mathbf{T}_t f - f)^2]\,d\lambda = 2\int_S \ln[1 + \theta^{-2}f^2]\,d\lambda.$$

Both examples considered so far have finite second moment: $E\Lambda^2(A) < \infty$ for all A with $\lambda(A) < \infty$ (as $\int_1^\infty x^2\,r(dx) < \infty$). An example with infinite second moment is the symmetric α-stable case $(0 < \alpha < 2)$ which corresponds to $r(dx) = |x|^{-2-\alpha}dx$. Defining $\int_0^\infty[1-\cos(y)]y^{-1-\alpha}\,dy = c_\alpha$, we have

$$N_\Psi(f) = c_\alpha\int_S |f|^\alpha\,d\lambda$$

and $\mathbf{L}_\Psi(S,\lambda) = \mathbf{L}_\alpha(S,\lambda)$. For more details in this case see Cambanis, Hardin and Weron (1987) and Podgórski (1992).

The next process provides a non-stable example with infinite second moment for Λ. Let $r(dx) = p_\alpha(x)dx$, where p_α is the density of a standard symmetric α-stable distribution. We have

$$Ee^{it\Lambda(A)} = \exp\Big\{-\lambda(A)\int_{-\infty}^{\infty}(1 - e^{itx})p_\alpha(x)\,dx\Big\} = \exp\{-\lambda(A)(1 - e^{-|t|^\alpha})\}$$

and $\Lambda(A)$ has a compound Poisson distribution, i.e. the same distribution as $Y_0 + \ldots + Y_N$, where $Y_0 = 0$, $\{Y_i\}_{i=1}^\infty$ are i.i.d. random variables with standard symmetric α-stable distribution and N is a Poisson random variable with mean $\lambda(A)$ and independent of $\{Y_i\}_{i=1}^\infty$. Since for $0 < \alpha < 2$, the second moment of an α-stable random variable does not exist, $\int_1^\infty x^2 p_\alpha(x)\,dx = \infty$, the second moment of $\Lambda(A)$ does not exist either; in fact, $E|\Lambda(A)|^q < \infty$ if and only if $\lambda(A) < \infty$ and $0 < q < \alpha$. We have $\int_0^\infty[1 - \cos(tx)]p_\alpha(x)\,dx = 1 - e^{-|t|^\alpha}$, and thus

$$N_\Psi(f) = \int_S \{1 - e^{-|f|^\alpha}\}\,d\lambda,$$

and one can likewise simplify the formulas in Theorems 1 and 3.

EXAMPLE 3. *The symmetric α-stable Ornstein–Uhlenbeck process is exact on positive time (or has the K-property on real time).* Let us recall that the

Ornstein–Uhlenbeck process is the moving average process

$$X_t = \int_{-\infty}^{\infty} e^{-(t-s)} 1_{[0,\infty)}(t-s)\, dL_s = e^{-t} \int_{-\infty}^{t} e^s\, dL_s, \qquad -\infty < t < \infty,$$

where $\{L_t\}_{-\infty<t<\infty}$ is a symmetric α-stable Lévy motion. It can also be obtained from a Lévy motion on the positive line by time change, namely

$$X_t = e^t L_{e^{-\alpha t}}, \qquad -\infty < t < \infty$$

(cf. Adler, Cambanis and Samorodnitsky (1990)).

First we prove that the process $\{X_t\}_{t\ge 0}$ is exact. Since

$$T_t \mathcal{F}_{\mathbf{X}} = \sigma(X_s : s \ge t) = \sigma(L_u : 0 \le u \le e^{-\alpha t})$$

and the Lévy motion is stochastically continuous with independent increments, it follows from the zero-one law of Blumenthal (1957) for such a process that the σ-field $\bigcap_{v\ge 0} \sigma(L_u : u \le v)$ is trivial and consequently $\bigcap_{t\ge 0} T_t \mathcal{F}_{\mathbf{X}}$ is also trivial. By the definition and remarks from Section 2 the process $\{X_t\}_{t\ge 0}$ is exact. For the proof of this property in the Gaussian case see Lasota and Mackey (1994). It is clear that the process $\{X_t\}_{-\infty<t<\infty}$ has the K-property.

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