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# Pointwise ergodic theorems in Lorentz spaces $L(p,q)$ for null preserving transformations

by

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**Abstract.** Let $(X,\mathcal{F},\mu)$ be a finite measure space and $\tau$ a null preserving transformation on $(X,\mathcal{F},\mu)$. Functions in Lorentz spaces $L(p,q)$ associated with the measure $\mu$ are considered for pointwise ergodic theorems. Necessary and sufficient conditions are given in order that for any $f$ in $L(p,q)$ the ergodic average $n^{-1}\sum_{i=0}^{n-1} f\circ\tau^i(x)$ converges almost everywhere to a function $f^*$ in $L(p_1,q_1)$, where $(p,q)$ and $(p_1,q_1)$ are assumed to be in the set $\{(r,s): r=s=1,\ \text{or}\ 1<r<\infty \text{ and } 1\le s\le\infty,\ \text{or}\ r=s=\infty\}$. Results due to C. Ryll-Nardzewski, S. Gładysz, and I. Assani and J. Woś are generalized and unified.

**1. Introduction and results.** Let $(X,\mathcal{F},\mu)$ be a finite measure space and $\tau$ a *null preserving transformation* on $(X,\mathcal{F},\mu)$ (i.e., $\tau^{-1}\mathcal{F}\subset\mathcal{F}$ and $\mu(\tau^{-1}A)=0$ whenever $\mu(A)=0$). We define an operator $T$ by putting

$$Tf = f\circ\tau.$$

$T$ is said to satisfy *the pointwise ergodic theorem from* $L(p,q)$ *to* $L(p_1,q_1)$ if for any $f$ in $L(p,q)$ the ergodic average

$$M_n(T)f=\frac{1}{n}\sum_{i=0}^{n-1}T^if$$

converges a.e. to a function $f^*$ in $L(p_1,q_1)$, where $L(p,q)$ and $L(p_1,q_1)$ are the Lorentz spaces associated with the measure $\mu$. Throughout this paper we shall assume that $(p,q)$ and $(p_1,q_1)$ are in the set

$$\{(r,s): r=s=1,\ \text{or}\ 1<r<\infty \text{ and } 1\le s\le\infty,\ \text{or}\ r=s=\infty\}.$$

The basic properties of Lorentz spaces $L(p,q)$ are explained in Hunt [5]. In particular, the following are used in the argument below.

(I) $f\in L(p,q)$ if and only if $\|f\|_{pq}<\infty$, where

$$\|f\|_{pq}=\begin{cases}(q\int_0^\infty(\mu(\{|f|>t\}))^{q/p}t^{q-1}\,dt)^{1/q} & (q\ne\infty),\\ \sup_{t>0} t(\mu(\{|f|>t\}))^{1/p} & (q=\infty).\end{cases}$$

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(II) $\|f\|_{pp} = (\int_X |f|^p\, d\mu)^{1/p} = \|f\|_p$ and hence $L(p,p) = L_p$.

(III) $L(p,q)$ is a Banach space with norm equivalent to $\|f\|_{pq}$; and further if $q \neq \infty$, then the conjugate space of $L(p,q)$ is $L(p',q')$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

(IV) There exists a constant $C = C(p,q)$, depending only on $(p,q)$, such that

$$C^{-1}\|f\|_{pq} \le \sup\Big\{\Big|\int_X fg\, d\mu\Big| : \|g\|_{p'q'} \le 1\Big\} \le C\|f\|_{pq}.$$

(V) $L_\infty \subset L(p,q) \subset L_1$ (because $\mu$ is a finite measure).

Since $T$ maps $L_\infty$ into $L_\infty$ and satisfies $Tf_n \downarrow 0$ a.e. on $X$ whenever $f_n \downarrow 0$ a.e. on $X$, the adjoint operator $T^*$ acting on $L_1$ can be defined by the relation

$$\int (Tf)u\, d\mu = \int f(T^*u)\, d\mu \quad (f \in L_\infty \text{ and } u \in L_1).$$

$T^*$ is often referred to as *the Frobenius–Perron operator associated with* $\tau$. The basic properties of $T^*$ and $\tau$ are given in Krengel's book [7]. Using this $T^*$, we characterize $T$ (and hence $\tau$) which satisfies the pointwise ergodic theorem from $L(p,q)$ to $L(p_1,q_1)$. The main result is as follows.

THEOREM 1. *Let $(X,\mathcal{F},\mu)$ be a finite measure space and $\tau$ a null preserving transformation on $(X,\mathcal{F},\mu)$. Let $(p,q)$ and $(p_1,q_1)$ be in the set $\{(r,s) : r = s = 1,\ \text{or } 1 < r < \infty \text{ and } 1 \le s \le \infty,\ \text{or } r = s = \infty\}$. Then the following conditions are equivalent.*

(i) *The operator $Tf = f\circ\tau$ satisfies the pointwise ergodic theorem from $L(p,q)$ to $L(p_1,q_1)$.*

(ii) *$T^*$ satisfies the mean ergodic theorem in $L_1$, and further for any $0 \le u \in L(p_1',q_1')$ the limit function $u_0^* = \lim_n M_n(T^*)u$ is in $L(p',q')$, where $M_n(T^*)u = n^{-1}\sum_{i=0}^{n-1} T^{*i}u$ and the limit can be taken in the sense of almost everywhere convergence.*

(iii) *$\mu(A) > 0$ and $\tau^{-1}A = A$ imply*

$$\int_A (\liminf_n M_n(T^*)1)\, d\mu > 0,$$

*and further for any $0 \le u \in L(p_1',q_1')$, $\liminf_n M_n(T^*)u$ is a function in $L(p',q')$.*

(iv) *To each $0 \le u \in L(p_1',q_1')$ there corresponds a functional $F_u$ defined on $X^+(\mathcal{F})$, where $X^+(\mathcal{F})$ denotes the space of all nonnegative simple measurable functions on $X$, such that*

$$(1)\quad \begin{cases} F_u(f) \le K_u\|f\|_{pq}, & K_u \text{ being a constant depending only on } u \\ & \qquad (\text{if } (p,q) \neq (\infty,\infty)), \\ F_u(f_n) \to 0 & \text{whenever } f_n \downarrow 0 \text{ a.e. on } X\ (\text{if } (p,q) = (\infty,\infty)), \end{cases}$$



$$(2)\quad \begin{cases} F_u(tf) = tF_u(f) & \text{for constants } t \ge 0, \\ F_u(f+g) \le F_u(f) + F_u(g), \end{cases}$$

$$(3)\quad 0 \le F_u(f) \le F_u(f+g),$$

$$(4)\quad F_u(Tf) \le F_u(f),$$

$$(5)\quad F_u(f) = \int_X fu\, d\mu \quad \text{whenever } f = Tf.$$

Remarks. (a) In condition (ii), the validity of the $L_1$-mean ergodic theorem for $T^*$ is essential. To see this, the following example may be interesting.

EXAMPLE 1. Let $X$ be the set of all integers and $\mu$ any finite measure on the subsets of $X$ such that $\mu(\{x\}) > 0$ for each $x \in X$. Define $\tau : X \to X$ by $\tau(x) = x + 1$. Then the operator $T^*$ on $L_1$ is *dissipative*, i.e., for any $0 \le u \in L_1$ we have

$$\sum_{i=0}^{\infty} T^{*i}u(x) < \infty \quad (x \in X).$$

It follows that $\lim_n M_n(T^*)u(x) = 0$ for all $x \in X$, and thus $T^*$ satisfies the pointwise ergodic theorem from $L(p_1',q_1')$ to $L(p_1,q_1)$. On the other hand, as is easily seen, $T^*$ does not satisfy the mean ergodic theorem in $L_1$.

This example shows that the validity of the $L_1$-mean ergodic theorem for $T^*$ cannot be removed in condition (ii). (Cf. the Lemma below.) We also note that the validity of the $L_1$-mean ergodic theorem for $T^*$ implies the validity of the pointwise ergodic theorem for $T^*$ from $L_1$ to $L_1$ (cf. Ito [6]).

(b) The special cases of Theorem 1 where $p = q$ and either $p_1 = q_1 = 1$ or $p_1 = q_1 = p$ were studied by C. Ryll-Nardzewski [10], S. Gładysz [4], and I. Assani and J. Woś [2]. The present theorem generalizes and unifies their results.

As a corollary to the proof of Theorem 1 we have

THEOREM 2. *Let $(X,\mathcal{F},\mu)$ be a finite measure space and $\tau$ a null preserving transformation on $(X,\mathcal{F},\mu)$. Let $(p,q) \in \{(r,s) : r = s = 1,\ \text{or } 1 < r < \infty \text{ and } 1 \le s \le \infty\}$. If the operator $Tf = f\circ\tau$ satisfies $\liminf_n \|M_n(T)\|_{pq} < \infty$ then $T$ satisfies the pointwise ergodic theorem from $L(p,q)$ to itself.*

Theorem 2 generalizes and improves results due to I. Assani [1] and P. Ortega Salvador [8]. A simple example given at the end of this paper shows that we may have $\sup_n \|M_n(T)\|_{pq} = \infty$, even though $T$ satisfies $\liminf_n \|M_n(T)\|_{pq} < \infty$.

## 2. Proofs of Theorems 1 and 2

LEMMA. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $\tau$ a null preserving transformation on $(X, \mathcal{F}, \mu)$. Then the operator $Tf = f \circ \tau$ satisfies the pointwise ergodic theorem from $L_\infty$ to $L_\infty$ if and only if one of the following conditions holds.*

(a) *For any $f \in L_\infty$, $M_n(T)f$ converges in measure.*
(b) *For any $A \in \mathcal{F}$, $\lim_n n^{-1} \sum_{i=0}^{n-1} \mu(\tau^{-i}A)$ exists.*
(c) *$T^*$ satisfies the mean ergodic theorem in $L_1$.*

Sketch of proof. Clearly, if $T$ satisfies the pointwise ergodic theorem from $L_\infty$ to $L_\infty$, then (a) holds. Since (a) implies the $L_1$-norm convergence of $M_n(T)1_A$, where $1_A$ is the indicator function of $A$, (b) follows from (a).

(b)$\Rightarrow$(c). The Vitali–Hahn–Saks theorem implies that

$$\lambda(A) = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \mu(\tau^{-i}A) \quad (A \in \mathcal{F})$$

defines a countably additive measure absolutely continuous with respect to $\mu$. Let $v_0^* = d\lambda/d\mu \in L_1$. Since $M_n(T^*)1$ converges weakly to $v_0^*$, we can apply a mean ergodic theorem (cf., for example, Theorem VIII.5.1 of [3], p. 661) to infer that $M_n(T^*)1$ converges in $L_1$-norm. By an approximation argument, (c) follows.

Lastly, suppose (c) holds. Put $v_0^* = \text{strong-}\lim_n M_n(T^*)1$. Since $T^*v_0^* = v_0^*$, the finite measure $\lambda = v_0^* \, d\mu$ is invariant under $\mu$. Therefore, putting $C = \{v_0^* > 0\}$, we have $\mu(C \setminus \tau^{-1}C) = 0$. Hence by a standard argument we may suppose without loss of generality that $C \subset \tau^{-1}C$. Then the set

$$A = X \setminus \lim_n \tau^{-n}C$$

satisfies $\tau^{-1}A = A$, and thus we see that $\mu(A) = \lim_n \int M_n(T)1_A \, d\mu = \lim_n \int_A M_n(T^*)1 \, d\mu = \int_A v_0^* \, d\mu = 0$. This together with the Birkhoff ergodic theorem proves that $T$ satisfies the pointwise ergodic theorem from $L_\infty$ to $L_\infty$, completing the proof.

The following proposition is an immediate corollary to the proof above, and may be regarded as a refinement of the Lemma.

PROPOSITION. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Assume that $\tau$ is an ergodic null preserving transformation on $(X, \mathcal{F}, \mu)$. Let $(p, q) \in \{(r, s) : r = s = 1, \text{ or } 1 < r < \infty \text{ and } 1 \le s \le \infty\}$. Then the operator $Tf = f \circ \tau$ satisfies the pointwise ergodic theorem in $L(p, q)$ (i.e., $\lim_n M_n(T)f$ exists and is finite a.e. on $X$ for any $f$ in $L(p, q)$) if and only if $T^*$ satisfies the mean ergodic theorem in $L_1$ and the pointwise and $L_1$-norm limit $v_0^* = \lim_n M_n(T^*)1$ is a function in $L(p', q')$, where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.*



Sketch of proof. Since $L_\infty \subset L(p, q)$, if $T$ satisfies the pointwise ergodic theorem in $L(p, q)$, then $T^*$ satisfies the mean ergodic theorem in $L_1$ by the Lemma. In order to show that $v_0^* = \lim_n M_n(T^*)1$ is a function in $L(p', q')$, suppose the contrary: $v_0^* \notin L(p', q')$. Then we can choose a function $0 \le f \in L(p, q)$ so that

$$\int f v_0^* \, d\mu = \infty.$$

Since $\tau$ is ergodic, the Birkhoff ergodic theorem implies $\lim_n M_n(T)f(x) = \infty$ a.e. on $X$, a contradiction.

The converse implication is immediate from the last part of the proof of the Lemma.

Remarks. (a) Without assuming the ergodicity of $\tau$, the Proposition does not hold. To see this, we construct the following example.

EXAMPLE 2. Let $X = [0, 1)$ and $\mathcal{F} =$ the Lebesgue measurable subsets of $[0, 1)$. For $k \ge 1$, let $X_k = [2^{-k}, 2^{-k+1})$ and $\tau_k : X_k \to X_k$ be any invertible ergodic measure preserving transformation with respect to the Lebesgue measure $m$ on $X_k$. Take a probability measure $\mu$ defined on $\mathcal{F}$, equivalent to the Lebesgue measure $m$, so that (here, $L(p', q')$ is defined for $\mu$)

(i) $\mu(X_k) = m(X_k) = 2^{-k}$ for each $k \ge 1$,
(ii) the function $w = dm/d\mu$ is not in $L(p', q')$,
(iii) the function $w_k = w1_{X_k}$ is in $L(p', q')$ for each $k \ge 1$.

Finally, define $\tau : X \to X$ by $\tau(x) = \tau_k(x)$ for $x \in X_k$. Clearly, $\tau$ is a null preserving transformation on $(X, \mathcal{F}, \mu)$ for which the operator $Tf = f \circ \tau$ satisfies the pointwise ergodic theorem in $L(p, q)$. Nevertheless, the function

$$v_0^* = \lim_n M_n(T^*)1 \in L_1(X, \mathcal{F}, \mu)$$

is not in $L(p', q')$, because the ergodicity of $\tau_k$ on $X_k$, together with (i), implies that

$$v_0^* = w_k \quad \text{a.e. on } X_k.$$

(b) When $(p, q) \ne (\infty, \infty)$, another characterization of the operator $T$ which satisfies the pointwise ergodic theorem in $L(p, q)$ is that $\sup_n |M_n(T)f| < \infty$ a.e. on $X$ for all $f \in L(p, q)$. This is true even if $\tau$ is not ergodic. For this and related results we refer the reader to [11].

Proof of Theorem 1. (i)$\Rightarrow$(ii). Since $L_\infty \subset L(p, q)$, $T^*$ satisfies the mean ergodic theorem in $L_1$ by the Lemma. For $0 \le u \in L(p_1', q_1')$ we denote by $u_0^*$ the pointwise and $L_1$-norm limit of $M_n(T^*)u$. We then have, for any $0 \le f \in L_\infty$,

$$\int f u_0^* \, d\mu = \lim_n \int f M_n(T^*) u \, d\mu = \lim_n \int (M_n(T) f) u \, d\mu$$
$$= \int (\lim_n M_n(T) f) u \, d\mu = \int f^* u \, d\mu$$
$$\le C(p_1, q_1) \|f^*\|_{p_1 q_1} \|u\|_{p_1' q_1'},$$

where the third equality is due to the fact that $\|M_n(T) f\|_\infty \le \|f\|_\infty < \infty$. Therefore, if $0 \le f \in L(p,q)$ then, by putting $f_N = f \wedge N$ (hence $f_N \uparrow f$ a.e. on $X$), we obtain

$$\int f u_0^* \, d\mu = \lim_N \int f_N u_0^* \, d\mu \le \lim_N C(p_1, q_1) \|f_N^*\|_{p_1 q_1} \|u\|_{p_1' q_1'}$$
$$\le C(p_1, q_1) \|f^*\|_{p_1 q_1} \|u\|_{p_1' q_1'} < \infty,$$

which proves that $u_0^* \in L(p', q')$.

(ii)$\Rightarrow$(iii). Immediate.

(iii)$\Rightarrow$(ii). Let us define $\widetilde{v}_0 = \liminf_n M_n(T^*)1$ a.e. on $X$. By Fatou's lemma, we get $0 \le \widetilde{v}_0 \in L_1$; and since $T^* \widetilde{v}_0 \le \widetilde{v}_0$ and $\|T^* \widetilde{v}_0\|_1 = \|\widetilde{v}_0\|_1$ it follows that $T^* \widetilde{v}_0 = \widetilde{v}_0$. Thus the measure $\lambda = \widetilde{v}_0 \, d\mu$ is invariant with respect to $\tau$. And as in the proof of the Lemma we may suppose without loss of generality that the set $C = \{\widetilde{v}_0 > 0\}$ satisfies $C \subset \tau^{-1} C$. Then the set $A = X \setminus \lim_n \tau^{-n} C$ is invariant under $\tau$, and clearly $\int_A \widetilde{v}_0 \, d\mu = 0$. By (iii) we get $\mu(A) = 0$. This and the Birkhoff ergodic theorem imply that $T$ satisfies the pointwise ergodic theorem from $L_\infty$ to $L_\infty$. Hence by the Lemma, $T^*$ satisfies the mean ergodic theorem in $L_1$.

(ii)$\Rightarrow$(iv). For $0 \le u \in L(p_1', q_1')$, denote the pointwise and $L_1$-norm limit of $M_n(T^*) u$ by $u_0^*$. Since $T^* u_0^* = u_0^*$ and $u_0^* \in L(p', q')$ by (ii), it is easy to see that the functional $F_u(f) = \int f u_0^* \, d\mu$ defined for $f \in X^+(\mathcal{F})$ satisfies all the requirements (1)–(5).

(iv)$\Rightarrow$(i). Let $u \in L(p_1', q_1')$ and $u > 0$ a.e. on $X$. Using the functional $F_u$ defined on $X^+(\mathcal{F})$, we define for $f \in X(\mathcal{F})$, where $X(\mathcal{F})$ denotes the space of all simple measurable functions on $X$,

$$P_u(f) = F_u(f^+), \quad \text{with } f^+(x) = \max\{f(x), 0\}.$$

From (2) and (3) it follows that

$$P_u(f+g) \le P_u(f) + P_u(g) \quad \text{and} \quad P_u(tf) = t P_u(f)$$

for $f, g \in X(\mathcal{F})$ and a constant $t \ge 0$. Since $(Tf)^+ = T(f^+)$, (4) gives

$$P_u(Tf) = F_u((Tf)^+) = F_u(T(f^+)) \le F_u(f^+) = P_u(f)$$

for any $f \in X(\mathcal{F})$. We now consider the linear functional $F_u^*(f) = \int f u \, d\mu$ defined on the linear subspace $\{f \in X(\mathcal{F}) : Tf = f\}$ of $X(\mathcal{F})$. Since $Tf = f$ implies $T(f^+) = f^+$, it follows from (5) that for any $f \in X(\mathcal{F})$ with $Tf = f$,

$$F_u^*(f) = \int f u \, d\mu \le \int f^+ u \, d\mu = F_u(f^+) = P_u(f).$$



Thus, as in Gładysz [4], we may apply a variant of the Hahn–Banach theorem (cf., for example, 10.5 in [9]) to infer that the linear functional $F_u^*$ can be linearly extended from the subspace $\{f \in X(\mathcal{F}) : Tf = f\}$ to the whole space $X(\mathcal{F})$ so that for all $f \in X(\mathcal{F})$,

$$F_u^*(f) \le P_u(f) = F_u(f^+) \quad \text{and} \quad F_u^*(Tf) = F_u^*(f).$$

In particular, if $f \in X^+(\mathcal{F})$, then, since $P_u(-f) = F_u((-f)^+) = 0$, we get $F_u^*(f) = -F_u^*(-f) \ge -P_u(-f) = 0$. Consequently,

$$(6) \qquad 0 \le F_u^*(f) = F_u^*(Tf) \le F_u(f) \quad \text{for } f \in X^+(\mathcal{F});$$

and further by (5),

$$(7) \qquad F_u^*(f) = \int f u \, d\mu = F_u(f) \quad \text{for } f \in X^+(\mathcal{F}) \text{ with } Tf = f.$$

We next set $\lambda(A) = F_u^*(1_A)$ for $A \in \mathcal{F}$. By (1) and (6), $\lambda$ is a finite measure absolutely continuous with respect to $\mu$ and invariant under $\tau$. Thus the function $0 \le u_0^* = d\lambda/d\mu \in L_1$ is invariant under $T^*$, and the set $C = \{u_0^* > 0\}$ satisfies $C \subset \tau^{-1} C$. Since the set $A = X \setminus \lim_n \tau^{-n} C$ is invariant under $\tau$, we have, by (7),

$$\int_A u \, d\mu = F_u(1_A) = F_u^*(1_A) = \int_A u_0^* \, d\mu = 0.$$

Since $u > 0$ a.e. on $X$, we must have $\mu(A) = 0$. This together with the Birkhoff ergodic theorem proves that for any measurable $f \ge 0$ on $X$ the pointwise limit

$$f^*(x) = \lim_n M_n(T) f(x)$$

exists a.e. on $X$ (but may be equal to $\infty$). Here we note that if $0 \le f_n \uparrow f$ a.e. on $X$ then we have $f_n^* \uparrow f^*$ a.e. on $X$.

Now, let $0 \le f \in L(p,q)$ be fixed arbitrarily, and put $f_N = f \wedge N$. Since $0 \le f_N^* \uparrow f^*$ a.e. on $X$, it follows that

$$\int f^* u \, d\mu = \lim_N \int f_N^* u \, d\mu = \lim_N \int f_N^* u_0^* \, d\mu,$$

where the last equality is due to (7), by a standard approximation argument. Since $\|M_n(T) f_N\|_\infty \le N$ for all $n \ge 1$, it follows that

$$\int f_N^* u_0^* \, d\mu = \lim_n \int (M_n(T) f_N) u_0^* \, d\mu = \lim_n \int f_N M_n(T^*) u_0^* \, d\mu$$
$$= \int f_N u_0^* \, d\mu \quad (\text{because } u_0^* = T^* u_0^*).$$

Hence we obtain, for $0 \le f \in L(p,q)$,

$$(8) \qquad \int f^* u \, d\mu = \lim_N \int f_N u_0^* \, d\mu = \int f u_0^* \, d\mu.$$

Next, choose $g_n \in X^+(\mathcal{F})$ so that $g_n \uparrow f$ a.e. on $X$. By (6) and (1), we have (when $(p,q)=(\infty,\infty)$, we put $K_u = F_u(1)$ here)

$$\int f u_0^*\,d\mu = \lim_n \int g_n u_0^*\,d\mu = \lim_n F_u^*(g_n) \le \lim_n F_u(g_n)$$
$$\le K_u \lim_n \|g_n\|_{pq} \le K_u \|f\|_{pq} < \infty.$$

Hence we get for $0 \le f \in L(p,q)$ and $0 \le u \in L(p_1', q_1')$,

$$\int f^* u\,d\mu = \int f u_0^*\,d\mu < \infty.$$

This proves that $f^* \in L(p_1,q_1)$ and also that $u_0^* \in L(p',q')$.

The proof is complete.

Remark. It follows from the proof of (i)$\Rightarrow$(ii) of Theorem 1 that if $T$ satisfies the pointwise ergodic theorem from $L_\infty$ to $L_\infty$ and if there exists a constant $K$ such that $\|f^*\|_{p_1q_1} \le K\|f\|_{pq}$ for all $0 \le f \in L_\infty$, then $T$ already satisfies the pointwise ergodic theorem from $L(p,q)$ to $L(p_1,q_1)$.

Proof of Theorem 2. Since $\liminf_n \|M_n(T)\|_{pq} < \infty$ by hypothesis, we can choose a subsequence $\{n'\}$ of $\{n\}$ so that $\sup_{n'} \|M_{n'}(T)\|_{pq} \le K < \infty$. Then define a functional $F$ on $X^+(\mathcal{F})$ by putting

$$F(f) = \limsup_{n'} \int M_{n'}(T) f\,d\mu \quad \text{for } f \in X^+(\mathcal{F}).$$

Clearly we have

$$F(Tf) = F(f), \quad 0 \le F(f) \le C(p,q) K (\mu(X))^{1/p'} \|f\|_{pq}$$

and

$$F(f) = \int f\,d\mu \quad \text{whenever } f = Tf.$$

It is also immediate that

$$F(f) \le F(f+g) \le F(f) + F(g) \quad \text{and} \quad F(tf) = tF(f)$$

for $f, g \in X^+(\mathcal{F})$ and a constant $t \ge 0$. Thus, as in the proof of (iv)$\Rightarrow$(i) of Theorem 1, we see that for any $0 \le f \in L(p,q)$ the pointwise limit

$$f^*(x) = \lim_n M_n(T) f(x)$$

exists a.e. on $X$. Then, by Fatou's lemma, we obtain

$$\|f^*\|_{pq} \le \liminf_n \|M_n(T) f\|_{pq} \le K\|f\|_{pq} < \infty.$$

The proof is complete.

**3. A counterexample.** We shall prove by a counterexample that *if* $(p,q) \ne (\infty,\infty)$, *then the condition* $\liminf_n \|M_n(T)\|_{pq} < \infty$ *does not imply* $\sup_n \|M_n(T)\|_{pq} < \infty$.



For this purpose we shall define, as in Ryll-Nardzewski [10], a finite measure $\mu$ on the subsets $\mathcal{F}$ of a countable set $X$ and a one-to-one transformation $\tau$ from $X$ onto itself as follows.

Let $X$ be the union of a sequence $X_k$ of finite disjoint sets such that for each $k \ge 1$, $X_k$ possesses $a_k$ points, where the sequence $\{a_k\}$ is chosen according to the later conditions. Let $X_k = \{(k,i) : 1 \le i \le a_k\}$ and $a_0 = 1$. We shall assume that $a_k$ and $a_k/a_{k-1}$ are integers so large that there exists a positive function $w_k$ on $X_k$ which satisfies the following

PROPERTY. *For each* $k \ge 1$ *there exists* $b_k$, $1 < b_k < a_k$, *such that*

$$w_k(k,i) = C_k^{i-1} \quad \text{if } 1 \le i \le b_k, \tag{9}$$

*where* $C_k > 1$ *is a constant depending only on* $k$, *and* $w_k$ *is decreasing on the set* $\{(k,i) : b_k \le i \le a_k\}$; *and further we have*

$$w_k(k,1) = w_k(k,a_k) = 1, \quad \frac{1}{a_k}\sum_{i=1}^{a_k} w_k(k,i) \le 2, \tag{10}$$

$$\frac{1}{b_k}\Big(\sum_{i=1}^{b_k} w_k(k,i)\Big)^{1/p} = \frac{1}{b_k}\Big(\sum_{i=0}^{b_k-1} C_k^i\Big)^{1/p} > k, \tag{11}$$

$$C_k^{a_{k-1}} < 2. \tag{12}$$

Define a transformation $\tau : X \to X$ and a measure $\mu$ in $X$ by putting

$$\tau(k,i) = \begin{cases} (k,a_k) & \text{if } i = 1, \\ (k,i-1) & \text{if } 2 \le i \le a_k, \end{cases}$$

and

$$\mu(A) = \sum_{k=1}^\infty \left[\frac{\sum_{(k,i)\in A} w_k(k,i)}{2^k\big(\sum_{i=1}^{a_k} w_k(k,i)\big)}\right].$$

Taking (9) and (12) into consideration we see that if $f \in L(p,q)$ and if $\{f \ne 0\} \subset \bigcup\{X_k : k > n\}$, then

$$\Big\|\frac{1}{a_n}\sum_{i=0}^{a_n-1} T^i f\Big\|_{pq} \le K(p,q)\|f\|_{pq} \tag{13}$$

with

$$K(p,q) = \begin{cases} 2 & \text{if } p = q = 1, \\ 2^{1/p} p/(p-1) & \text{if } 1 < p < \infty \text{ and } 1 \le q \le \infty, \end{cases}$$

where the inequality for the case $1 < p < \infty$ and $1 \le q \le \infty$ is due to the inequalities (2.2) in Hunt [5], with $r = 1$.

On the other hand, since $a_n/a_k$, $1 \le k \le n$, are integers, it follows that if $f \in L(p,q)$ and $\{f \ne 0\} \subset \bigcup\{X_k : 1 \le k \le n\}$, then $a_n^{-1}\sum_{i=0}^{a_n-1} T^i f$ is a constant function on each $X_k$ with $1 \le k \le n$, and thus by (10) we have the same inequality (13) for this $f$. It follows that $\liminf_n \|M_n\|_{pq} < \infty$.

But (11) gives, for $f = 1_E$ with $E = \{(k,1)\}$,

$$\frac{1}{\|f\|_{pq}} \left\| \frac{1}{b_k} \sum_{i=0}^{b_k-1} T^i f \right\|_{pq} = \frac{1}{b_k} (\mu(\{(k,i) : 1 \le i \le b_k\}))^{1/p} > k,$$

and hence we have $\sup_n \|M_n(T)\|_{pq} = \infty$.

Remark. A slight modification of the above example shows that if $(p,q) \ne (\infty,\infty)$, then we may have $\lim_n \|M_n(T)\|_{pq} = \infty$ for $T$ which satisfies the pointwise ergodic theorem from $L(p,q)$ to itself. It follows that the converse of Theorem 2 does not hold.

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# Automatic extensions of functional calculi

by

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**Abstract.** Given a Banach algebra $\mathcal{F}$ of complex-valued functions and a closed, linear (possibly unbounded) densely defined operator $A$, on a Banach space, with an $\mathcal{F}$ functional calculus we present two ways of extending this functional calculus to a much larger class of functions with little or no growth conditions. We apply this to spectral operators of scalar type, generators of bounded strongly continuous groups and operators whose resolvent set contains a half-line. For $f$ in this larger class, one construction measures how far $f(A)$ is from generating a strongly continuous semigroup, while the other construction measures how far $f(A)$ is from being bounded. We apply our constructions to evolution equations.

**I. Introduction and preliminaries.** Suppose $\mathcal{F}$ is a Banach algebra of complex-valued functions on a subset of the complex plane. If $A$ is in $B(X)$, the space of bounded linear operators from the Banach space $X$ into itself, and $\mathcal{F}$ contains both $f_0(z) \equiv 1$ and $f_1(z) \equiv z$, then an $\mathcal{F}$ functional calculus for $A$ is a continuous algebra homomorphism, $f \mapsto f(A)$, from $\mathcal{F}$ into $B(X)$, such that $f_0(A) = I$, the identity operator, and $f_1(A) = A$.

When $A$ is unbounded, then we cannot have $f_1 \in \mathcal{F}$. Something more indirect is required to involve $A$ in its functional calculus. We will essentially use the definition of a functional calculus given in [8], except that we will also consider Banach algebras $\mathcal{F}$ that may not contain $f_0$; thus in (3) of Definition 1.2 we stipulate that functions $z \mapsto (\lambda - z)^{-m}$ are mapped where one would expect.

It is convenient to introduce terminology and important concepts before proceeding further.

TERMINOLOGY AND HYPOTHESES 1.1. All operators considered are linear. Throughout, we will assume that $A$ is a closed, densely defined operator on a Banach space $X$. We will write $\mathcal{D}(A)$ for the domain of $A$, $\varrho(A)$ for the resolvent set of $A$, $B(X)$ for the Banach space of bounded operators from $X$ to itself. We will write $\mathrm{Im}(B)$ for the image of an operator $B$. The space $\mathcal{F}$ will always be a Banach algebra of complex-valued functions on a subset

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