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Trivial bundles of spaces of probability measures and countable-dimensionality

by

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Abstract. The probability measure functor P carries open continuous mappings $f : X \xrightarrow{\text{onto}} Y$ of compact metric spaces into Q-bundles provided Y is countable-dimensional and all fibers $f^{-1}(y)$ are infinite. This answers a question raised by V. Fedorchuk.

1. Introduction. The probability measure functor P is a covariant functor acting from the category of compact Hausdorff spaces and continuous mappings into itself. For a compact Hausdorff space X, the space $P(X)$ can be defined as follows: Let $C(X)$ denote the space of continuous functions on X and let $M(X)$ denote the linear space dual to $C(X)$. The space $P(X)$ is the subspace of $M(X)$ consisting of all nonnegative functionals μ ($\mu(\varphi) \geq 0$ for every nonnegative function $\varphi \in C(X)$) with norm 1. As a topological space, we consider $P(X)$ in the weak*-topology. Then $P(X)$ is compact with $w(P(X)) = w(X) \cdot \aleph_0$. It should be noted that $P(X)$ is, in fact, the space of probability measures on X because, by the Riesz theorem (F. Riesz for a closed interval, Banach and Saks for a compact metric space, and, finally, Kakutani for a compact Hausdorff space), the linear space $M(X)$ is isomorphic to the space of countably additive finite regular Borel measures on X. In view of that, we sometimes use $\int_X \varphi \, d\mu$ to denote the value $\mu(\varphi)$ of a functional $\mu \in M(X)$ on a function $\varphi \in C(X)$.

Recall finally that, for a continuous $f : X \to Y$, the corresponding mapping $P(f) : P(X) \to P(Y)$ is defined by

$$(P(f)\mu)\varphi = \int_X \varphi \circ f \, d\mu, \quad \mu \in P(X) \text{ and } \varphi \in C(Y).$$

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Let us now especially mention the well-known fact that, for a metrizable X, there are only two possibilities for the space $P(X)$. If X consists of n points, then $P(X)$ is naturally homeomorphic to an $(n-1)$-dimensional simplex. In case X is infinite, $P(X)$ is affinely homeomorphic to an infinite-dimensional compact convex set in the separable Hilbert space ℓ_2. So, by a result of Keller [10], $P(X)$ is homeomorphic to the Hilbert cube Q.

Suppose $f : X \to Y$ is a continuous mapping between compact metric spaces. Concerning the mapping $P(f)$, there is a series of papers ([3]–[8]) where the following question is treated: *When does the corresponding mapping $P(f)$ become a trivial bundle with fibers homeomorphic to the Hilbert cube Q (i.e., a Q-bundle)?*

Below we quickly quote all we know concerning this question. First of all, *if $P(f)$ is a Q-bundle, then f is certainly an open surjection with all fibers $f^{-1}(y)$ infinite* [5, Theorem 2.1 and Remark 2.3]. Unfortunately, these conditions are only necessary. As shown in [3] (see also [4]), the mapping $P(\eta)$ is not a Q-bundle, where

$$\eta = \prod\{\nu_k : k = 2^n \text{ and } n = 0, 1, \ldots\}$$

and the mapping $\nu_k : \mathbb{S}^k \to \mathbb{RP}^k$ is a 2-fold covering mapping of the k-dimensional sphere onto the real projective k-space. But, under some dimension-type restrictions on the spaces X and Y, these conditions are already sufficient. For finite-dimensional X and Y this result is due to V. Fedorchuk [5, Theorem 2.1]. In [7], Fedorchuk has also shown that the finite-dimensionality of X can be omitted.

The purpose of the present paper is to give a further generalization of the above results by proving the following theorem.

THEOREM 1.1. *Let f be an open continuous mapping from a compact metric space X onto a countable-dimensional metric space Y with all fibers $f^{-1}(y)$ infinite. Then $P(f) : P(X) \to P(Y)$ is a Q-bundle.*

Here, a space Y is said to be of *countable dimension* provided that it is a countable union of finite-dimensional subsets.

Remark. Theorem 1.1 answers in the affirmative a question of V. Fedorchuk [8, Question 6.18′]. On the other hand, the question of the necessity of the countable-dimensionality of Y in Theorem 1.1 remains open.

The paper is arranged as follows. A general scheme of the proof of Theorem 1.1 is presented in the next section. Two auxiliary lemmas (Lemmas 2.1 and 2.2) needed for that proof are established in Sections 3 and 4, respectively.

2. General scheme of the proof of Theorem 1.1. The purpose of this section is to show how the general scheme of the proof of [5, Theorem 2.1] can be slightly modified to work in our present situation.

Suppose X, Y and f are as in Theorem 1.1. We check that $P(f)$ is a Q-bundle using the West–Toruńczyk criterion [16]. Let $\varepsilon > 0$. Following closely the proof of [5, Theorem 2.1] (see [5, Remark 0.13 and Proposition 0.14]), to do that it suffices to construct two continuous mappings

$$g_i : P(X) \to P(X), \quad i = 1, 2,$$

satisfying the conditions:

(1) $P(f) \circ g_i = P(f)$,
(2) $\varrho(g_i, \mathrm{id}_{P(X)}) < \varepsilon$,
(3) $g_1(P(X)) \cap g_2(P(X)) = \emptyset$,

where ϱ is a fixed metric on $P(X)$.

Using a lemma of Milyutin [14], [15], we first fix a zero-dimensional compact metric space Z and a Milyutin epimorphism $g : Z \to Y$. Recall that a continuous mapping $g : Z \to Y$ is a *Milyutin epimorphism* [15] if it admits a regular averaging operator $u : C(Z) \to C(Y)$, i.e., a linear mapping u such that

(i) $\|u\| = 1$,
(ii) $u(\varphi)$ is nonnegative provided $\varphi \in C(Z)$ is nonnegative,
(iii) $u(\varphi \circ g) = \varphi$ for every $\varphi \in C(Y)$.

Next, let $T = \{(z, x) \in Z \times X : g(z) = f(x)\}$ be the fiber product of the spaces Z and X with respect to the mappings g and f, and let $g_0 : T \to X$ and $f_0 : T \to X$ be the projections.

Finally, let S be a zero-dimensional compact metric space and $h : S \to T$ be a Milyutin epimorphism.

Throughout most of this paper, we will find ourselves in the situation of the following commutative diagram.

$$\begin{tikzcd} S \arrow[r, "h"] & T \arrow[r, "g_0"] \arrow[d, "f_0"'] & X \arrow[d, "f"] \\ & Z \arrow[r, "g"'] & Y \end{tikzcd}$$

Keeping in mind this diagram, we now state two lemmas which give us a basic component of the construction of the mappings g_i. Suppose q is a metric on T agreeing with the topology of T. For $\xi > 0$ and $F \subset T$ we use $B_\xi(F)$ to denote the ξ-neighbourhood of F in (T, q).

LEMMA 2.1. *For every $\xi > 0$ there exist two continuous selections $\sigma_1, \sigma_2 : Z \to T$ for f_0^{-1} such that*

(a) $q(\sigma_1(z), \sigma_2(z)) < \xi$ *for every* $z \in Z$, *and*
(b) $g_0\sigma_1(Z) \cap g_0\sigma_2(Z) = \emptyset$.

LEMMA 2.2. *Let $\xi > 0$, and let $\sigma_1, \sigma_2 : Z \to T$ be as in Lemma 2.1. Then there exists a retraction $r : S \to S$ such that*

(a) $hr(S) \cap g_0^{-1}g_0\sigma_1(Z) = \emptyset$,
(b) $f_0 \circ h \circ r = f_0 \circ h$, *and*
(c) $hrh^{-1}(t) \subset B_\xi(t)$ *for every* $t \in T$.

It should be remarked that Lemma 2.1 is a countable-dimensional version of [5, Lemma 2.2]. The proof of Lemma 2.1 is contained in the next Section 3. This proof illustrates the power of the technique of set-valued semicontinuous selections and is quite different from that of [5, Lemma 2.2]. As for Lemma 2.2, it refines a part of the proof of [5, Theorem 2.1]. The principal improvement is condition (c) (compare with [5, (2.17)]), which will be essentially used at the end of this section. The proof of Lemma 2.2 is postponed until Section 4.

We now turn to the definition of the mappings g_i. Let $u : C(Z) \to C(Y)$ be a regular averaging operator for the Milyutin epimorphism g. This u gives rise to an embedding $u^* : P(Y) \to P(Z)$ that is a selection for $(P(g))^{-1}$ and is defined by

$$u^*(\mu)(\varphi) = \mu(u(\varphi)) \quad \text{for every } \mu \in P(Y) \text{ and } \varphi \in C(Z).$$

Repeating [5, (2.3)], we define $g_1 : P(X) \to P(X)$ by setting

$$g_1(\mu) = (1-\xi)\mu + \xi P(g_0 \circ \sigma_1)u^*P(f)(\mu).$$

This definition is correct and it guarantees that g_1 is continuous and has the following properties:

$(1; g_1)$ $P(f) \circ g_1 = P(f)$,
$(2; g_1)$ $\varrho(g_1, \mathrm{id}_{P(X)}) < \varepsilon$ for small ξ,
$(3; g_1)$ $\int_{g_0\sigma_1(Z)} dg_1(\mu) > 0$ for any $\mu \in P(X)$.

For details concerning the verification of these properties see [5].

We now proceed to the definition of g_2. Let $v : C(S) \to C(T)$ be a regular averaging operator corresponding to the Milyutin epimorphism h.

By [5, Lemma 0.11], the mapping g_0 parallel to the Milyutin epimorphism g is also a Milyutin epimorphism. Let then $w : C(T) \to C(X)$ be the corresponding regular averaging operator.

Next, let $w^* : P(X) \to P(T)$ and $v^* : P(T) \to P(S)$ be the mappings adjoint to w and v (see the definition of u^*). Note that w^* and v^* are selections for $(P(g_0))^{-1}$ and $(P(h))^{-1}$, respectively.

Repeating [5, (2.9)], we define $g_2 : P(X) \to P(X)$ by

$$g_2 = P(g_0) \circ P(h) \circ P(r) \circ v^* \circ w^*.$$

The continuous mapping g_2 so defined has the following properties:

$(1; g_2)$ $P(f) \circ g_2 = P(f)$,
$(2; g_2)$ $\varrho(g_2, \mathrm{id}_{P(X)}) < \varepsilon$ for small ξ,
$(3; g_2)$ $\int_{g_0\sigma_1(Z)} dg_2(\mu) = 0$ for any $\mu \in P(X)$.

Indeed, $(1; g_2)$ follows from Lemma 2.2(b) (see [5, (2.10)]). To verify $(3; g_2)$, we note that, by Lemma 2.2(a), $g_0hr(S) \cap g_0\sigma_1(Z) = \emptyset$. Therefore, for every measure $\nu \in P(S)$,

$$\int\limits_{g_0\sigma_1(Z)} dP(g_0 \circ h \circ r)(\nu) = 0,$$

which is $(3; g_2)$.

Following [5], to check finally $(2; g_2)$, it suffices to show that $P(h \circ r) \circ v^*$ is close to the identity because $P(g_0) \circ w^* = \mathrm{id}_{P(X)}$. Towards this end, we first note that the space $P(T)$ is embedded (in a natural way) in $\mathbb{R}^{C(T)}$. So, $P(T)$ admits a base consisting of all sets of the form

$$O(\mu, \varphi_1, \ldots, \varphi_k, \delta) = \{\nu \in P(T) : |\mu(\varphi_i) - \nu(\varphi_i)| < \delta,\ i = 1, \ldots, k\},$$

where $\delta > 0$, $\varphi_1, \ldots, \varphi_k \in C(T)$ and $\mu \in P(T)$. Next, by virtue of [5, Proposition 0.2], every finite open cover of $P(T)$ has a refinement

$$\{O(\mu, \varphi_1, \ldots, \varphi_k, \delta) : \mu \in P(T)\}$$

for some $\delta > 0$ and $\varphi_1, \ldots, \varphi_k \in C(T)$. Therefore, for a given $\delta > 0$ and functions $\varphi_1, \ldots, \varphi_k \in C(T)$, it suffices to find a $\xi > 0$ such that the mapping $\chi = P(h \circ r) \circ v^*$, constructed for this particular ξ, satisfies the condition

$(2; g_2)'$ $\chi(\mu) \in O(\mu, \varphi_1, \ldots, \varphi_k, \delta)$, $\mu \in P(T)$.

Pick a $\xi > 0$ for which $d(t, t') < \xi$ implies $|\varphi_i(t) - \varphi_i(t')| < \delta$, $i = 1, \ldots, k$, and let us check that this ξ works. Notice that $\chi(\mu)(\varphi) = P(h \circ r)v^*(\mu)(\varphi) = \mu(v(\varphi \circ h \circ r))$. Hence, to show $(2; g_2)'$, it is now sufficient to show that

$(2; g_2)''$ $|\varphi_i(t) - v(\varphi_i \circ h \circ r)(t)| < \delta$, $i = 1, \ldots, k$.

Remembering that v is an averaging operator for h and using [5, Proposition 0.6], we get

$$v(\varphi_i \circ h \circ r)(t) \in \operatorname{conv} \varphi_i(hrh^{-1}(t)), \qquad i = 1, \ldots, k.$$

This, together with Lemma 2.2(c), implies $(2; g_2)''$.

The g_1 and g_2 constructed for a small ξ satisfy all our requirements.

3. Proof of Lemma 2.1. Throughout this section X, Y and f are as in Theorem 1.1. Set

$$\mathcal{F}(X) = \{S \subset X : S \neq \emptyset \text{ and } S \text{ is closed}\}.$$

A set-valued mapping $\varphi : Y \to \mathcal{F}(X)$ is *lower semicontinuous* (*upper semi-continuous*), or l.s.c. (u.s.c.), if

$$\varphi^{-1}(U) = \{y \in Y : \varphi(y) \cap U \neq \emptyset\}$$

is open (resp., closed) in Y for every open (resp., closed) $U \subset X$. For a family $\mathcal{W}$ of subsets of X and $A \subset X$, we use $\mathrm{St}(A;\mathcal{W})$ to denote the star of A with respect to $\mathcal{W}$.

The most difficult part of the proof of Lemma 2.1 consists in proving the following selection theorem.

THEOREM 3.1. *For every $\delta > 0$ there exist two l.s.c. selections $\varphi_1, \varphi_2 : Y \to \mathcal{F}(X)$ for f^{-1} such that, for every $y \in Y$,*

$$\varphi_1(y) \cap \varphi_2(y) = \emptyset \quad \text{and} \quad H(d)(\varphi_1(y), \varphi_2(y)) < \delta.$$

Here, d is a compatible metric on X and $H(d)$ denotes the Hausdorff metric on $\mathcal{F}(X)$. Recall that, for $F, G \in \mathcal{F}(X)$,

$$H(d)(F,G) = \inf\{\varepsilon > 0 : F \subset B_\varepsilon(G) \text{ and } G \subset B_\varepsilon(F)\}.$$

In preparation for the proof of Theorem 3.1, we need some more terminology. Let $\mathcal{T}(X)$ be the topology of X, $\mathcal{U} = \{U_\alpha : \alpha \in \mathcal{A}\}$ be an open cover of Y, and let $t : \mathcal{A} \to \mathcal{T}(X)$. We shall say that the triple $(t, \mathcal{A}; \mathcal{U})$ is a *$t(\mathcal{A})$-approximate section* for f [9] if $\mathcal{A}$ is finite and $\mathrm{cl}(U_\alpha) \subset f(t(\alpha))$ for every $\alpha \in \mathcal{A}$.

Consider the set

$$\Omega(f) = \{(t, \mathcal{A}; \mathcal{U}) : (t, \mathcal{A}; \mathcal{U}) \text{ is a } t(\mathcal{A})\text{-approximate section for } f\}.$$

For every $(t, \mathcal{A}; \mathcal{U}) \in \Omega(f)$, we let

$$\mathrm{mesh}(t, \mathcal{A}; \mathcal{U}) = \max\{\mathrm{diam}(t(\alpha)) : \alpha \in \mathcal{A}\}.$$

Suppose $\theta : Y \to \mathcal{F}(X)$ is a set-valued selection for f^{-1} and $(t, \mathcal{A}; \mathcal{U}) \in \Omega(f)$.

(3.2) We shall say that θ is *subject* to $(t, \mathcal{A}; \mathcal{U})$ ($\theta \approx (t, \mathcal{A}; \mathcal{U})$ for short) if $\theta(y) \cap t(\alpha) \neq \emptyset$ for every $y \in U_\alpha$ and $\theta(y) \subset \bigcup\{t(\alpha) : y \in U_\alpha\}$.

Let $\varphi : Y \to \mathcal{F}(X)$ and $\psi : Y \to \mathcal{F}(X)$ be two set-valued mappings. We shall say that (φ, ψ) is a *Michael pair* for f^{-1} provided φ is l.s.c., ψ is u.s.c. and $\varphi(y) \subset \psi(y) \subset f^{-1}(y)$ for every $y \in Y$. Set

$$\mathrm{Mp}(f) = \{(\varphi, \psi) : (\varphi, \psi) \text{ is a Michael pair for } f^{-1}\}.$$

Finally, for $(\varphi, \psi) \in \mathrm{Mp}(f)$ and $(t, \mathcal{A}; \mathcal{U}) \in \Omega(f)$ we write $(\varphi, \psi) \approx (t, \mathcal{A}; \mathcal{U})$ provided φ and ψ are subject to $(t, \mathcal{A}; \mathcal{U})$ simultaneously.

PROPOSITION 3.3. *Let $(t, \mathcal{A}; \mathcal{U}) \in \Omega(f)$, and let $\varphi_1 : Y \to \mathcal{F}(X)$ and $\varphi_2 : Y \to \mathcal{F}(X)$ be selections for f^{-1} such that $\varphi_i \approx (t, \mathcal{A}; \mathcal{U})$, $i = 1, 2$. Then*

$$H(d)(\varphi_1(y), \varphi_2(y)) < \mathrm{mesh}(t, \mathcal{A}; \mathcal{U}) \quad \text{for every } y \in Y.$$

Proof. This follows immediately from the fact that, for every $y \in Y$ and every $i, j \in \{1, 2\}$, (3.2) implies

$$\varphi_i(y) \subset \bigcup\{t(\alpha) : y \in U_\alpha\} \subset \mathrm{St}(\varphi_j(y); t(\mathcal{A})) \subset B_{\mathrm{mesh}(t,\mathcal{A};\mathcal{U})}(\varphi_j(y)). \ ■$$

PROPOSITION 3.4. *Let $(t', \mathcal{A}; \mathcal{U}), (t, \mathcal{A}; \mathcal{U}) \in \Omega(f)$ be such that $t'(\alpha) \subset t(\alpha)$ for every $\alpha \in \mathcal{A}$, and let $\varphi : Y \to \mathcal{F}(X)$ be a selection for f^{-1} with $\varphi \approx (t', \mathcal{A}; \mathcal{U})$. Then $\varphi \approx (t, \mathcal{A}; \mathcal{U})$.*

Proof. Simply note that $\varphi(y) \cap t'(\alpha) \neq \emptyset$ implies $\varphi(y) \cap t(\alpha) \neq \emptyset$. ■

LEMMA 3.5. *For every $(t, \mathcal{A}; \mathcal{U}) \in \Omega(f)$ there exists a $(\varphi, \psi) \in \mathrm{Mp}(f)$ such that $(\varphi, \psi) \approx (t, \mathcal{A}; \mathcal{U})$.*

Proof. Let $\alpha \in \mathcal{A}$ be such that U_α is non-empty. Define a set-valued mapping $\Phi_\alpha : \mathrm{cl}(U_\alpha) \to \mathcal{F}(t(\alpha))$ by

$$\Phi_\alpha(y) = f^{-1}(y) \cap t(\alpha) \quad \text{for every } y \in \mathrm{cl}(U_\alpha).$$

Note that Φ_α is l.s.c. because f is open. Since now $t(\alpha)$ is completely metrizable, by a result of E. Michael [13, Theorem 1.1], there exist two compact-valued selections $\varphi_\alpha, \psi_\alpha : \mathrm{cl}(U_\alpha) \to \mathcal{F}(t(\alpha))$ for Φ_α such that φ_α is l.s.c., ψ_α is u.s.c. and $\varphi_\alpha(y) \subset \psi_\alpha(y)$ for every $y \in \mathrm{cl}(U_\alpha)$. Define $\varphi, \psi : Y \to \mathcal{F}(X)$ by $\varphi(y) = \bigcup\{\varphi_\alpha(y) : y \in U_\alpha\}$ and $\psi(y) = \bigcup\{\psi_\alpha(y) : y \in \mathrm{cl}(U_\alpha)\}$. These φ and ψ satisfy all our requirements. ■

LEMMA 3.6. *For every $(t, \mathcal{A}; \mathcal{U}) \in \Omega(f)$ there exists a u.s.c. finite-valued selection $\theta : Y \to \mathcal{F}(X)$ for f^{-1} such that $\theta \approx (t, \mathcal{A}; \mathcal{U})$.*

Proof. As in the previous proof, whenever $\alpha \in \mathcal{A}$ with $U_\alpha \neq \emptyset$, we define an l.s.c. mapping $\Phi_\alpha : \mathrm{cl}(U_\alpha) \to \mathcal{F}(t(\alpha))$ by $\Phi_\alpha(y) = f^{-1}(y) \cap t(\alpha)$ for every $y \in \mathrm{cl}(U_\alpha)$. Since $\mathrm{cl}(U_\alpha)$ is countable-dimensional, by [9, Theorem 2.1], each Φ_α admits a finite-valued u.s.c. selection θ_α. Define, finally, the required θ by $\theta(y) = \bigcup\{\theta_\alpha(y) : y \in \mathrm{cl}(U_\alpha)\}$. ■

Proof of Theorem 3.1. Let $\mathcal{V}$ be an open cover of X such that $\mathrm{diam}(V) < \delta$ for every $V \in \mathcal{V}$. Since X is compact, there exists a finite open cover $\mathcal{A}$ of X which is a closure-refinement of $\mathcal{V}$. Define, in a natural fashion, a mapping $t : \mathcal{A} \to \mathcal{V} \subset \mathcal{T}(X)$ such that $\mathrm{cl}(\alpha) \subset t(\alpha)$ for every $\alpha \in \mathcal{A}$. Set $W_\alpha = f(\alpha)$, and let $\mathcal{W} = \{W_\alpha : \alpha \in \mathcal{A}\}$. Since f is open, $\mathcal{W}$ is an open cover of Y. The following holds.

(i) $(t, \mathcal{A}; \mathcal{W}) \in \Omega(f)$. Indeed, $\alpha \in \mathcal{A}$ implies $\mathrm{cl}(W_\alpha) = \mathrm{cl}(f(\alpha)) \subset f(\mathrm{cl}(\alpha)) \subset f(t(\alpha))$.

(ii) $\mathrm{mesh}(t,\mathcal{A};\mathcal{W})<\delta$ because $\mathrm{diam}(t(\alpha))<\delta$ for every $\alpha\in\mathcal{A}$.

(iii) Whenever $y\in Y$, there is an $\alpha\in\mathcal{A}$ such that $f^{-1}(y)\cap\alpha$ is infinite. By assumption, $f^{-1}(y)$ is infinite. Then (iii) follows immediately from the fact that $\mathcal{A}$ is finite and $f^{-1}(y)\subset\bigcup\mathcal{A}$.

Let now $\theta:Y\to\mathcal{F}(X)$ be a u.s.c. finite-valued selection for f^{-1} such that $\theta\approx(t,\mathcal{A};\mathcal{W})$. By virtue of Lemma 3.6, such a θ certainly exists. Since θ is u.s.c., $F_0=\bigcup\{\theta(y):y\in Y\}\subset X$ is closed. Let us check that $\{f(\alpha\backslash F_0):\alpha\in\mathcal{A}\}$ covers Y. Take a point $y\in Y$. By (iii), there exists an $\alpha\in\mathcal{A}$ such that $f^{-1}(y)\cap\alpha$ is infinite. Then $f^{-1}(y)\cap(\alpha\backslash F_0)\neq\emptyset$ because $f^{-1}(y)\cap F_0=\theta(y)$ is finite. Hence, $y\in f(\alpha\backslash F_0)$.

Next, let $\mathcal{U}=\{U_\alpha:\alpha\in\mathcal{A}\}$ be an open cover of Y such that

(iv) $\mathrm{cl}(U_\alpha)\subset f(\alpha\backslash F_0)$ $(\subset f(t(\alpha)\backslash F_0))$ for every $\alpha\in\mathcal{A}$.

This immediately implies that (see also (ii))

(v) $(t,\mathcal{A};\mathcal{U})\in\Omega(f)$ and $\mathrm{mesh}(t,\mathcal{A};\mathcal{U})=\mathrm{mesh}(t,\mathcal{A};\mathcal{W})<\delta$.

Define first $t_1:\mathcal{A}\to\mathcal{T}(X)$ by $t_1(\alpha)=t(\alpha)\backslash F_0$ for every $\alpha\in\mathcal{A}$. By (iv), $(t_1,\mathcal{A};\mathcal{U})\in\Omega(f)$. Then, by Lemma 3.5, there exists a pair $(\varphi_1,\psi_1)\in\mathrm{Mp}(f)$ with $(\varphi_1,\psi_1)\approx(t_1,\mathcal{A};\mathcal{U})$. Since ψ_1 is u.s.c., $F_1=\bigcup\{\psi_1(y):y\in Y\}\subset X$ is closed. Define then $t_2:\mathcal{A}\to\mathcal{T}(X)$ by $t_2(\alpha)=t(\alpha)\backslash F_1$ for every $\alpha\in\mathcal{A}$. We claim that $(t_2,\mathcal{A};\mathcal{U})\in\Omega(f)$. Indeed, let $\alpha\in\mathcal{A}$ and let $y\in\mathrm{cl}(U_a)$. By (iv), $y\in\mathrm{cl}(U_\alpha)\subset f(\alpha)=W_\alpha$. Therefore,

$$f^{-1}(y)\cap t_2(\alpha)=f^{-1}(y)\cap(t(\alpha)\backslash F_1)\supset f^{-1}(y)\cap F_0\cap t(\alpha)=\theta(y)\cap t(\alpha)\neq\emptyset$$

because $\theta\approx(t,\mathcal{A};\mathcal{W})$ (see (3.2)) and because $F_1\cap F_0=\emptyset$. So, $\mathrm{cl}(U_\alpha)\subset f(t_2(\alpha))$. Let finally $(\varphi_2,\psi_2)\in\mathrm{Mp}(f)$ be such that $(\varphi_2,\psi_2)\approx(t_2,\mathcal{A};\mathcal{U})$, which exists by virtue of Lemma 3.5. It only remains to verify that φ_1 and φ_2 satisfy all our requirements. Indeed, by Proposition 3.4, $\varphi_i\approx(t,\mathcal{A};\mathcal{U})$ $(i=1,2)$ because $t_i(\alpha)\subset t(\alpha)$ for every $\alpha\in\mathcal{A}$. Then, by (v) and Proposition 3.3,

$$H(d)(\varphi_1(y),\varphi_2(y))<\mathrm{mesh}(t,\mathcal{A};\mathcal{U})<\delta.$$

That $\varphi_1(y)\cap\varphi_2(y)=\emptyset$ is now obvious because $\varphi_1(y)\subset\psi_1(y)\subset F_1$ and $\varphi_2(y)\subset\bigcup\{t_2(\alpha):\alpha\in\mathcal{A}\}=\bigcup\{t(\alpha)\backslash F_1:\alpha\in\mathcal{A}\}=\bigcup\{t(\alpha):\alpha\in\mathcal{A}\}\backslash F_1$. ■

Having established Theorem 3.1, we now proceed to the proof of Lemma 2.1. By Theorem 3.1 (with $\delta=\xi/2>0$), there exist two l.s.c. selections $\varphi_1,\varphi_2:Y\to\mathcal{F}(X)$ for f^{-1} such that, for every $y\in Y$, $\varphi_1(y)\cap\varphi_2(y)=\emptyset$ and $\varphi_2(y)\subset B_{\xi/2}(\varphi_1(y))$. Since $\dim(Z)=0$, by a result of Michael [11, Theorem 1], the l.s.c. mapping $\varphi_2\circ g:Z\to\mathcal{F}(X)$ admits a continuous selection $\kappa_2:Z\to X$. Note that $\kappa_2(z)\in\varphi_2(g(z))\subset B_{\xi/2}(\varphi_1(g(z)))$. Hence, $B_{\xi/2}(\kappa_2(z))\cap\varphi_1(g(z))\neq\emptyset$. Define then a set-valued mapping $\Phi:Z\to\mathcal{F}(X)$ by letting

$$\Phi(z)=\mathrm{cl}(\varphi_1(g(z))\cap B_{\xi/2}(\kappa_2(z)))\quad\text{for every } z\in Z.$$

By [12, Propositions 2.3 and 2.5], Φ is l.s.c. Then, by the same arguments as before, Φ admits a continuous selection $\kappa_1:Z\to X$. We have

(A) $d(\kappa_1(z),\kappa_2(z))<\xi$ for every $z\in Z$

because $\kappa_1(z)\in\mathrm{cl}(B_{\xi/2}(\kappa_2(z)))\subset B_\xi(\kappa_2(z))$. Moreover,

(B) $\kappa_1(Z)\cap\kappa_2(Z)=\emptyset$.

Indeed, let $x\in\kappa_1(Z)\cap\kappa_2(Z)$. Then there are points $z_i\in Z$ for which $\kappa_i(z_i)=x$. Since $x=\kappa_i(z_i)\in\varphi_i(g(z_i))\subset f^{-1}(g(z_i))$, it follows that $g(z_1)=g(z_2)=y$. But then $x=\kappa_2(z_2)\in\varphi_2(y)$ and $x=\kappa_1(z_1)\in\varphi_1(y)$, which is impossible because $\varphi_1(y)\cap\varphi_2(y)=\emptyset$.

Define finally $\sigma_i:Z\to T$ by $\sigma_i(z)=(z,\kappa_i(z))$. This definition is correct because $\kappa_i(z)\in f^{-1}(g(z))$ and therefore

$$f(\kappa_i(z))=ff^{-1}(g(z))=g(z).$$

Now 2.1(a) and 2.1(b) follow immediately from (A) and (B), respectively (notice that we may assume $q((z,x_1),(z,x_2))=d(x_1,x_2)$ for every $z\in Z$).

4. Proof of Lemma 2.2.

Suppose $\xi>0$ and σ_1 and σ_2 are as in Lemma 2.1. Define

$$\Sigma_1=g_0^{-1}g_0\sigma_1(Z)\quad\text{and}\quad S_1=h^{-1}(\Sigma_1).$$

In preparation for the proof of Lemma 2.2, we show

PROPOSITION 4.1. *There is a continuous mapping $l:S_1\to T\backslash\Sigma_1$ such that, for every $s\in S_1$,*

(a) *$l(s)\in f_0^{-1}f_0h(s)$, and*

(b) *$q(h(s),l(s))<\xi$.*

Proof. Define $X_i=g_0\sigma_i(Z)$, $i=1,2$. Observe that $X_i\subset X$ is closed and $X_1\cap X_2=\emptyset$. Then there exists an $\eta>0$ such that $B_\eta(X_2)\cap X_1=\emptyset$. Since Z is compact, there is a $\delta\in(0,\xi)$ such that $q(\sigma_1(z),\sigma_2(z))<\delta$ for every $z\in Z$. First, define a set-valued mapping $\varphi:S_1\to\mathcal{F}(X)$ by

$$\varphi(s)=\mathrm{cl}(B_\delta(g_0h(s))\cap f^{-1}fg_0h(s))\quad\text{for every } s\in S_1.$$

Next, define another set-valued mapping $\Phi:S_1\to\mathcal{F}(B_\eta(X_2))$ by $\Phi(s)=\varphi(s)\cap B_\eta(X_2)$ for every $s\in S_1$. Note that Φ is correctly defined. Take a point $s\in S_1$. Since $g_0h(s)\in X_1=g_0\sigma_1(Z)$, there is a $z\in Z$ for which $g_0h(s)=g_0\sigma_1(z)$. We claim that $g_0\sigma_2(z)\in\Phi(s)$. Indeed, on the one hand,

$$g_0\sigma_2(z)\in f^{-1}fg_0\sigma_1(z)\cap X_2=f^{-1}fg_0h(s)\cap X_2.$$

On the other hand, $q(\sigma_2(z),\sigma_1(z))<\delta$ implies $d(g_0\sigma_2(z),g_0\sigma_1(z))<\delta$. Therefore, $g_0\sigma_2(z)\in B_\delta(g_0\sigma_1(z))=B_\delta(g_0h(s))$.

Note now that, by [12, Propositions 2.3 and 2.5], φ is l.s.c. Hence, by [12, Proposition 2.4], so is Φ because $B_\eta(X_2)$ is open. Then, by [11, Theorem 1], Φ admits a continuous selection $p: S_1 \to B_\eta(X_2)$ because $B_\eta(X_2)$ is completely metrizable and $\dim(S_1) = 0$. We are now ready to define our l. Namely, for every $s \in S_1$, put $l(s) = (f_0h(s), p(s))$. To show that $l(s) \in T\backslash\Sigma_1$, note first that $p(s) \in \varphi(s) \subset f^{-1}fg_0h(s)$. Hence,

$$fp(s) = fg_0h(s) = gf_0h(s).$$

So, $l(s) \in T$. Next, note that $g_0l(s) = p(s) \in B_\eta(X_2)$, and therefore $l(s) \notin \Sigma_1 = g_0^{-1}(X_1)$ because $X_1 \cap B_\eta(X_2) = \emptyset$. It follows from the definition of l that (a) holds. Finally, (b) follows immediately from the fact that $d(g_0h(s), p(s)) \le \delta < \xi$. ■

PROPOSITION 4.2. *There exist a clopen subset $W \subset S$, containing S_1, and a continuous mapping $w: W \to S\backslash W$ such that*

(a) $f_0hw(s) = f_0h(s)$ *for every* $s \in W$, *and*
(b) $hwh^{-1}(t) \subset B_\xi(t)$ *for every* $t \in h(W)$.

Proof. Let $l: S_1 \to T\backslash\Sigma_1$ be as in Proposition 4.1. Define a set-valued mapping $\varphi: S \to \mathcal{F}(T)$ by $\varphi(s) = \{l(s)\}$ if $s \in S_1$ and $\varphi(s) = f_0^{-1}f_0h(s)$ otherwise. Note that the mapping f_0 is open as a "parallel" to the open mapping f (second lemma on parallels of [1]). This implies that the mapping φ is l.s.c. because, by 4.1(a), l is a selection for $f_0^{-1}f_0h|S_1$ (see [12, Example* 1.3]). Then, by [11, Theorem 1], φ admits a continuous selection $k: S \to T$. Note that $k(s) = l(s)$ for every $s \in S_1$. Therefore, $kh^{-1}(\Sigma_1) \cap \Sigma_1 = \emptyset$. Take then a neighbourhood V of Σ_1 in T such that $kh^{-1}(V) \cap V = \emptyset$. This is possible because h is perfect. Since $q(k(s), h(s)) < \xi$ for every $s \in S_1$ and $\dim(S) = 0$, there now exists a clopen subset W of S such that $S_1 \subset W \subset h^{-1}(V)$ and $q(k(s), h(s)) < \xi$ for every $s \in W$. Remembering that h is a Milyutin epimorphism and using Ditor's theorem [2], we can find an l.s.c. selection $\Phi: T \to \mathcal{F}(S)$ for h^{-1}. Finally, again by [11, Theorem 1], let $w: W \to S\backslash W$ be a continuous selection for the mapping $\Phi \circ k|W: W \to \mathcal{F}(S\backslash W)$. It only remains to check that this w works. Take a point $s \in W$. Note that $w(s) \in \Phi(k(s)) \subset h^{-1}(k(s))$. Hence,

$$f_0hw(s) = f_0hh^{-1}k(s) = f_0k(s) = f_0h(s).$$

So, (a) holds. Let now $t \in h(W)$ and $s \in h^{-1}(t) \cap W$. Since $hw(s) = k(s)$ and since $q(k(s), h(s)) < \xi$, it follows that $hw(s) \in B_\xi(h(s)) = B_\xi(t)$. Therefore, $hwh^{-1}(t) \subset B_\xi(t)$. ■

We are now ready for the proof of Lemma 2.2. Let W and w be as in Proposition 4.2. Define $r: S \to S$ by $r|W = w$ and $r|S\backslash W = \mathrm{id}_{S\backslash W}$. Since W is clopen, r is continuous. Moreover, $r \circ r = \mathrm{id}_{S\backslash W}$, so r is a retraction. By 4.2(a), $f_0 \circ h \circ r = f_0 \circ h$. Finally, $hrh^{-1}(t) \subset B_\xi(t)$ for every $t \in T$ by 4.2(b) and the definition of r.

References

[1] P. S. Aleksandrov and B. A. Pasynkov, *Introduction to Dimension Theory*, Nauka, Moscow, 1973 (in Russian).

[2] S. Z. Ditor, *Averaging operators in $C(S)$ and lower semicontinuous sections of continuous maps*, Trans. Amer. Math. Soc. 175 (1973), 195–208.

[3] A. N. Dranishnikov, *On Q-fibrations without disjoint sections*, Funktsional. Anal. i Prilozhen. 22 (2) (1988), 79–80 (in Russian).

[4] —, *A fibration that does not accept two disjoint many-valued sections*, Topology Appl. 35 (1990), 71–73.

[5] V. Fedorchuk, *Trivial bundles of spaces of probability measures*, Mat. Sb. 129 (171) (1986), 473–493 (in Russian); English transl.: Math. USSR-Sb. 57 (1987), 485–505.

[6] —, *Soft mappings, set-valued retractions and functors*, Uspekhi Mat. Nauk 41 (6) (1986), 121–159 (in Russian).

[7] —, *A factorization lemma for open mappings between compact spaces*, Mat. Zametki 42 (1) (1987), 101–113 (in Russian).

[8] —, *Probability measures in topology*, Uspekhi Mat. Nauk 46 (1) (1991), 41–80 (in Russian).

[9] V. Gutev, *Open mappings looking like projections*, Set-Valued Anal. 1 (1993), 247–260.

[10] O.-H. Keller, *Die Homoiomorphie der kompakten konvexen Mengen im Hilbertschen Raum*, Math. Ann. 105 (1931), 748–758.

[11] E. Michael, *Selected selection theorems*, Amer. Math. Monthly 63 (1956), 233–238.

[12] —, *Continuous selections I*, Ann. of Math. 63 (1956), 361–382.

[13] —, *A theorem on semi-continuous set-valued functions*, Duke Math. J. 26 (1959), 647–656.

[14] A. A. Milyutin, *Isomorphisms of the spaces of continuous functions over compact sets of the cardinality of the continuum*, Teor. Funktsiĭ Funktsional. Anal. i Prilozhen. 2 (1966), 150–156 (in Russian).

[15] A. Pełczyński, *Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions*, Dissertationes Math. 58 (1968).

[16] H. Toruńczyk and J. West, *Fibrations and bundles with Hilbert cube manifold fibers*, Mem. Amer. Math. Soc. 406 (1989).

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