VA^*V^{-1} for every $A \in B(H)$, where V is a bounded invertible conjugate-linear operator on H. On the other hand, $\theta(A)$ is always similar to A. In particular, it would follow that an operator A is one-to-one if and only if A^* is. But this is certainly not true (consider, for instance, the shift operator). Thus, θ is an automorphism. The proof of the theorem is complete.

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The upper bound of the number of eigenvalues for a class of perturbed Dirichlet forms

by

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Abstract. The theory of Markov processes and the analysis on Lie groups are used to study the eigenvalue asymptotics of Dirichlet forms perturbed by scalar potentials.

Introduction. Let $A(x, D) = \sum_{|\alpha| \le m} a_\alpha(x)(i^{-1}\partial/\partial x)^\alpha$ be a selfadjoint differential operator with the symbol $A(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x)\xi^\alpha$. The Bohr–Sommerfeld quantization principle, according to which the volume $\sim h^d$ in the phase space should count for one eigenvalue of $A(x, D)$, leads us to the hypothesis that the number of eigenvalues of $A(x, D)$ which are less than λ should be approximately the volume of the set $A = \{(x, \xi) \mid A(x, \xi) < \lambda\}$. If $A(x, D)$ is elliptic and $\lambda \to \infty$, this hypothesis is asymptotically correct (cf. [10]). For the Schrödinger operator $-\Delta + V$, this "volume-counting" has been fully expressed in the form of the Cwikel–Lieb–Rosenblum inequality (cf. [13]). However, this inequality can also produce grossly inaccurate estimates for systems as simple as two uncoupled harmonic oscillators. Following Fefferman (cf. [5]), it is better to count the number of distorted unit cubes which can be packed disjointly inside the subset A instead of measuring the importance of A. This idea, called the SAK-principle, led to sharp estimates of eigenvalue asymptotics (cf. [5], [6]). Because counting the number of distorted unit cubes which fit inside A is not easy, this kind of estimate gives us only a qualitative description for the number of eigenvalues. (In [3], it is shown how we can count the number of proper boxes in the case of Schrödinger operators with polynomial potentials.)

The aim of this paper is to redefine the place of "volume-counting type" estimates and to give a quantitative description of the number of eigenvalues for operators defined as $\mathcal{D} + V$, where $\mathcal{D}$ is the infinitesimal generator of a (sub)markovian semigroup and V is a function. For $\mathcal{D}$ being a sum of

squares of vector fields this description is given as a sharper version of the Cwikel–Lieb–Rosenblum inequality.

Preliminaries. Let M be a connected C^∞ manifold. Consider a probability space $(\Omega, \mathcal{F}, P)$ and let $\xi(t) : \Omega \to M$ be a Markov process on M. Assume that the trajectories $t \to \xi(t)$ are right continuous with probability one. Let ξ be homogeneous in time and have the Feller property. If we denote by E^x the expected value with the initial condition $\xi(0) = x \in M$, then $T_t f(x) = E^x f(\xi(t))$ defines a strongly continuous semigroup on the space of bounded continuous functions on M. Let $\mathcal{D}$ denote the infinitesimal generator of the semigroup T_t and V be a bounded continuous function. Consider the strongly continuous semigroup of operators S_t defined by the infinitesimal generator $\mathcal{D} + V$.

LEMMA 0 (The Feynman–Kac formula). *For any bounded continuous function f defined on M and every $x \in M$,*

$$S_t f(x) = E^x f(\xi(t)) \exp\left(- \int_0^t V(\xi(s))\, ds \right).$$

Proof. By the Trotter formula, we have

$$e^{-t(\mathcal{D}+V)} f = \lim_{n\to\infty} (e^{-t\mathcal{D}/n} e^{-tV/n})^n f.$$

By using the Markov property the right-hand side of the above equality is the limit of

$$E^x f(\xi(t)) \exp\left(- \frac{t}{m} \sum_{j=1}^m V\left(\xi\left(j\frac{t}{m} \right) \right) \right).$$

Because the paths of ξ are right continuous, the function $t \to V(\xi(t))$ is integrable in the Riemann sense. So this expected value tends to

$$E^x f(\xi(t)) \exp\left(- \int_0^t V(\xi(s))\, ds \right)$$

as n tends to infinity.

The Feynman–Kac formula allows us to present the kernel of the semigroup S_t in the form which is useful when exploring the eigenvalue asymptotics of the operator $\mathcal{D} + V$. For this purpose we will construct a "Markov bridge" between any two points $x, y \in M$. The following assumption gives us the possibility of building this Markov bridge: There exists a positive Radon measure m on M such that the Markov transition function of the process ξ is the collection of absolutely continuous (relative to m) probability measures with densities $p(x, y, t)$. If $\xi(t)$ is a Hunt process, i.e. it is right continuous, has the strong Markov property, and is quasi left continuous,

then the transition function of $\xi(t)$ is absolutely continuous relative to any everywhere dense positive Radon measure on M for which T_t is a semigroup of selfadjoint operators on $L_2(M, dm)$ (cf. [8]). Our first assumption is the following:

(A) there exists an everywhere dense positive Radon measure m on M such that $\xi(t)$ is an m-symmetric Hunt process.

Let Ω be the set of all right continuous trajectories on M. We construct the process ξ on Ω by putting

$$P_x\{\xi : \xi(t_1) \in A_1, \ldots, \xi(t_n) \in A_n\}$$
$$= \int dm(x_1) \ldots dm(x_n) \prod_{i=1}^n p(x_{i-1}, x_i, t_i - t_{i-1}) \chi_{A_i}(x_i),$$

where $x_0 = x$, $0 < t_1 < \ldots < t_n$. For any $t > 0$ the probability $P_{x,t}$ can be decomposed into a family of conditional measures $\{P_{x,y,t} : y \in M\}$ given by

$$P_{x,y,t}\{\xi : \xi(t_1) \in A_1, \ldots, \xi(t_n) \in A_n\}$$
$$= \int dm(x_1) \ldots dm(x_n) \left(\prod_{i=1}^n p(x_{i-1}, x_i, t_{i-1} - t_i) \chi_{A_i}(x_i) \right) p(x_n, y, t - t_n),$$

where $x_0 = x$, $y \in M$ and $0 < t_1 < \ldots < t_n < t$. The measure $P_{x,y,t}$ is supported by the set of all right continuous trajectories which start from x and finish in y.

Now we can describe the kernel of the semigroup $S_t = e^{-t(\mathcal{D}+V)}$ in the following form:

$$S_t(x, y) = \int dP_{x,y,t} \exp\left(- \int_0^t V(\xi(s))\, ds \right).$$

The first natural question is whether or not for any fixed $t > 0$ the function $S_t(x, y)$ is continuous in the $M \times M$-topology. The positive answer can be easily proved if we restrict the considered classes of manifolds and Markov processes to the class for which the following conditions hold:

(B) there exists a Lie group G which acts transitively on M,

(C) m is a G-invariant positive Radon measure on M,

(D) T_t commutes with the G-action,

(E) for every $t > 0$, there exists $o \in M$ such that $p(o, o, t) < \infty$.

The conditions (A)–(D) are sufficient to prove that, for fixed $t > 0$ and fixed $o \in M$, the family $\{p(\cdot, og, t) : g \in G\}$ is uniformly integrable. The conditions (A)–(E) and the semigroup property prove that for any $t > 0$ and $o \in M$ the function $p(o, \cdot, t)$ is square integrable, and thus, $p(\cdot, \cdot, t)$ is $M \times M$-continuous. Hence, $S_t(\cdot, \cdot)$ is continuous in the $M \times M$-topology. We concentrate our research on the cases in which the conditions (A)–(E) hold.

DEFINITION 1. Any strongly continuous semigroup of operators satisfying the conditions (A)–(E) is called an *unexplosive G-invariant symmetric semigroup.*

Any strongly continuous semigroup is an approximation of the identity operator. For the considered semigroups we can measure the "velocity of approximation" by the following parameters:

$$\mathrm{rank}_0(\mathcal{D}, x) = \inf\{\delta : \exists_{c=c(\delta)} \forall_{t>0}\; p(x,x,t) \le ct^{-\delta/2}\},$$

$$\mathrm{rank}_\infty(\mathcal{D}, x) = \sup\{\tau : \exists_{c=c(\tau)} \forall_{t>0}\; p(x,x,t) \le ct^{-\tau/2}\}.$$

The condition (D) proves that both ranks are independent of the choice of $x \in M$. So, they are parameters of the global behaviour of the semigroup, and we can write $\mathrm{rank}_0(\mathcal{D})$, $\mathrm{rank}_\infty(\mathcal{D})$.

Let G act transitively on M. Let m be a G-invariant measure on M. We say that the action of G is *unimodular* if the following condition holds:

(F) there exist $o \in M$ and $s : M \to G$ such that for every $x \in M$, $os(x) = x$ (we say that s is a *selector*), and for every $y \in M$ and $\phi \in C_c(M)$,

$$\int dm(x)\,\phi(ys(x)) = \int dm(x)\,\phi(x).$$

For the reasons which will be seen below (in the proof of Lemma 1) we explore only the class of unimodular actions.

The main lemma. We now estimate the number of negative eigenvalues of the operator $\mathcal{D} + V$, $V \le 0$. Denote this number by $N(\mathcal{D} + V, 0)$. In the case of $\mathcal{D} = -\Delta$ and $M = \mathbb{R}^d$, this result is due to Cwikel [4], Lieb [11], and Rosenblum [14]. Our proof is an adaptation of Lieb's method published in [13].

LEMMA 1 (The Cwikel–Lieb–Rosenblum inequality). *Let the action of G on M be unimodular. Let $\mathcal{D}$ be the infinitesimal generator of an unexplosive G-invariant symmetric semigroup. Let $\mathrm{rank}_0(\mathcal{D}) \le \delta \le \mathrm{rank}_\infty(\mathcal{D})$ and $\delta > 2$. There exists a constant $C > 0$ such that*

$$N(\mathcal{D} + V, 0) \le C \int dm(x)\,|V(x)|^{\delta/2}$$

for every nonpositive potential V.

Proof. We can assume that V is continuous and compactly supported. By using the strong resolvent convergence the result can then be easily extended to the wider class of potentials.

We set $-V = F$ and, for $\lambda < 0$, $-\lambda = \kappa^2$. Fix a natural number $n > \delta/2$. As the first step we show that

$$(1) \quad N(\mathcal{D} + V, \lambda)$$
$$\le (n+1)\,\mathrm{Tr}\left(F^{1/2}\sum_{j=1}^{n}(-1)^j\binom{n}{j}(\mathcal{D} + jF + \kappa^2)^{-1}F^{1/2}\right).$$

For a selfadjoint operator $\mathcal{H}$ we define the kth *characteristic number* as

$$\mu_k(\mathcal{H}) = \sup_{\dim L = k-1} \inf_{f \in D(\mathcal{H}),\, f \perp L,\, \|f\|=1} (\mathcal{H}f, f).$$

If $\|f_1\| = \|f_2\| = 1$, then the functions defined by $t \to ((\mathcal{D} - tF)f_i, f_i)$, $t > 0$, $i = 1, 2$, are equicontinuous. Hence $t \to \mu_k(\mathcal{D} - tF)$ defines a continuous function $\mu_k(t)$. Since $F \ge 0$, we have $\mu_k(t+h) < \mu_k(t)$. Using the mini-max principle, we see that

$$N(\mathcal{D} - F, \lambda) = |\{k : \mu_k(1) < \lambda\}| = |\{k : \mu_k(t) = \lambda \text{ for some } 0 < t \le 1\}|.$$

Let η be a function which satisfies the equation

$$(\mathcal{D} - tF)\eta = \lambda\eta = -\kappa^2\eta.$$

Then $(j + t)F\eta = (\mathcal{D} + jF + \kappa^2)\eta$ for $j = 0, 1, \ldots$ So,

$$(\mathcal{D} + jF + \kappa^2)^{-1}F\eta = (j + t)^{-1}\eta.$$

Thus, $\psi = F^{1/2}\eta$ satisfies

$$F^{1/2}(\mathcal{D} + jF + \kappa^2)^{-1}F^{1/2}\psi = (j + t)^{-1}\psi.$$

Therefore, if we set

$$K = F^{1/2}\sum_{j=0}^{n}(-1)^j\binom{n}{j}(\mathcal{D} + jF + \kappa^2)^{-1}F^{1/2},$$

then

$$K\psi = \left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}(j + t)^{-1}\right)\psi.$$

We can write this as

$$K\psi = t^{-1}H(t^{-1})\psi,$$

where

$$H(x) = \sum_{j=0}^{n}\binom{n}{j}(1 + jx)^{-1} = \frac{n!x^n}{(1 + x)(1 + 2x)\ldots(1 + nx)}.$$

By noticing that

$$H(x) = \int_0^\infty e^{-t}(1 - e^{-xt})\,dt,$$

we see that H is a monotonic function. Hence, if t runs from 0 to 1, then $t^{-1}H(t^{-1})$ runs from ∞ to $n!/(n+1)! = (n+1)^{-1}$. Therefore, the number of $t \in (0, 1]$ for which the equation $(\mathcal{D} + tF)\eta = \lambda\eta$ has a solution (counted with multiplicities) is bounded by the number of eigenvalues of K larger

than $(n+1)^{-1}$. Because K is a positive operator, the last number does not exceed $(n+1)\operatorname{Tr}(K)$.

The next step is to show the inequality

$$(2) \quad N(\mathcal{D}-F,0)$$
$$\leq (n+1)\int_0^\infty \operatorname{Tr}\left(F^{1/2}\sum_{j=1}^n (-1)^j \binom{n}{j}\exp(-t(\mathcal{D}+jF))F^{1/2}\right)dt.$$

Formally, (2) is a consequence of (1) and the Laplace transform

$$(\kappa^2+\mathcal{H})^{-1} = -\int_0^\infty \exp(-t(\kappa^2+\mathcal{H}))\,dt.$$

To complete such a formal proof we must show that we can change the order of the trace and the integral. For this purpose, we notice that $F^{1/2}\exp(t\mathcal{D})$ and $\exp(t\mathcal{D})F^{1/2}$ have square integrable kernels (this is an easy consequence of (A)–(F)). The assumption $F \geq 0$ implies that $F^{1/2}\exp(-t(\mathcal{D}+jF))$ and $\exp(-t(\mathcal{D}+jF))F^{1/2}$ are also Hilbert–Schmidt operators, for $j=1,\dots,n$. Therefore, the operator

$$A_j = \exp(-s(\mathcal{D}+jF))F\exp(-(t-s)(\mathcal{D}+jF))$$

is of trace class for any $t > s > 0$, $j = 0,1,\dots$ Lemma 0 and the Markov property lead to

$$A_j f(x) = E^x F(\xi(s))\exp\left(-\int_0^t jF(\xi(r))\,dr\right)f(\xi(t)).$$

By using the "Markov bridge" (described in the preliminaries), we can express the kernel of A_j as

$$A_j(x,y) = \int dP_{x,y,t}(\xi)\,F(\xi(s))\exp\left(-j\int_0^t F(\xi(r))\,dr\right).$$

As we have shown above, $\operatorname{Tr}(A_j) < \infty$. The conditions (A)–(F) imply that this kernel is a continuous function. Hence

$$\operatorname{Tr}(A_j) = \int dx\,A_j(x,x).$$

Now we notice that

$$\operatorname{Tr}(F^{1/2}\exp(-t(\mathcal{D}+jF)F^{1/2}))$$
$$= t^{-1}\int_0^t ds\,\operatorname{Tr}(\exp(-s(\mathcal{D}+jF)F\exp(-(t-s)(\mathcal{D}+jF))).$$

We can interchange the trace and integration if we have the control over trace-norm. We get such a control in the same way as the following final estimate.

Let $R(u) = u(1-\exp(-u))^n$. From the above remarks, we come to

$$(3) \quad N(\mathcal{D}+V,0)$$
$$\leq (n+1)\int_0^\infty dt \int dm(x)\int dP_{x,x,t}(\xi)\,t^{-1}R\left(\int_0^t F(\xi(s))\,ds\right).$$

There exists $a \in \mathbb{R}_+$ such that $R'' > 0$ for $u \in (0,a)$, and $R'' < 0$ for $u \in (a,\infty)$. We define

$$\varrho(u) = \begin{cases} R(u) & \text{for } 0 < u \leq a, \\ R(a) + (u-a)R'(a) & \text{for } u > a. \end{cases}$$

Then $\varrho \sim u^{n+1}$ as $u \to 0$, $\varrho \sim u$ as $u \to \infty$, ϱ is a convex function, and $R(u) \leq \varrho(u)$. So, by the Jensen inequality,

$$R\left(\int_0^t F(\xi(s))\,ds\right) \leq \varrho\left(\int_0^t F(\xi(s))\,ds\right) \leq t^{-1}\int_0^t \varrho(tF(\xi(s)))\,ds.$$

We fix $o \in M$ and a selector s such that $os(x) = x$ for every $x \in M$. We have

$$\int dP_{x,x,t}(\xi)\,f(\xi(s)) = \int dP_{o,o,t}(\xi)\,f(\xi(s)s(x)).$$

The action of G on M is unimodular. So, from (3) we obtain

$$N(\mathcal{D}+V,0) \leq (n+1)\int_0^\infty dt \int dm(x)\int_0^t ds \int dP_{o,o,t}(\xi)\,t^{-2}\varrho(tF(\xi(s)s(x)))$$
$$= (n+1)\int_0^\infty dt \int dP_{o,o,t}(\xi)\int dm(x)\,t^{-1}\varrho(tF(x)).$$

We notice that

$$\int dP_{o,o,t}(\xi) = p(o,o,t) \leq ct^{-\delta/2}.$$

This ends the proof.

The Sobolev inequality. Now we consider the status of the Cwikel–Lieb–Rosenblum inequality in the theory of Dirichlet forms.

COROLLARY 1 (The Sobolev inequality). *Let* $\delta > 2$ *and* $\operatorname{rank}_0(\mathcal{D}) \leq \delta \leq \operatorname{rank}_\infty(\mathcal{D})$. *There exists* $C > 0$ *such that*

$$\|\phi\|_{2\delta/(\delta-2)} \leq C(\mathcal{D}\phi,\phi)^{1/2}$$

for every $\phi \in L^2(M,dm)$.

Proof. By Lemma 1, there exists $c_1 > 0$ such that $((\mathcal{D}+V)\phi,\phi) \geq 0$ for every nonpositive $V \in L^{\delta/2}(M,dm)$ with $\|V\|_{\delta/2} < c_1$ and all $\phi \in L^2(M,dm)$. Thus, there exists $c_2 > 0$ such that

$$\|\phi\|_{2\delta/(\delta-2)}^2 \leq c_2(\mathcal{D}\phi,\phi).$$

The Sobolev inequality plays a key role in the Hardy–Littlewood theory for semigroups (cf. [16]). Let, as above, T_t be the semigroup generated by $\mathcal{D}$. By using the "pivot" of Varopoulos's paper ([16], Theorem 1) and Corollary 1 we can easily prove the following equivalence.

THEOREM 1. *Let $\delta > 2$. The following conditions are equivalent:*

(1) *there exists $c_1 > 0$ such that*

$$\|\phi\|_{2\delta/(\delta-2)} \le c_1 (\mathcal{D}\phi, \phi)^{1/2}$$

for $\phi \in L^2(M, dm)$,

(2) *there exists $c_2 > 0$ such that*

$$\|T_t \phi\|_\infty \le c_2 t^{-\delta/2} \|\phi\|_1$$

for $t > 0$ and $\phi \in L^1(M, dm)$,

(3) *there exists $c_3 > 0$ such that*

$$N(\mathcal{D} + V, 0) \le c_3 \int dm(x) \, |V(x)|^{\delta/2}$$

for every nonpositive potential V.

P r o o f. The equivalence of (1) and (2) is the substance of the Varopoulos paper. For the implication (2)$\Rightarrow$(3), we notice that (2) implies that $\mathrm{rank}_0(\mathcal{D}) \le \delta \le \mathrm{rank}_\infty(\mathcal{D})$ and we can use Lemma 1.

Theorem 1 shows that the C-L-R inequality leads us to the Varapoulos theory. Hence, as corollaries, we get the Hardy–Littlewood estimates for subharmonic functions, existence of Riesz potentials and sharp estimates for time derivatives. For details see [16].

Localization and upper Weyl estimates. Let $\xi(t)$ be the Hunt process considered in the preliminaries. Since any trajectory of ξ is right continuous and has left limits (cf. [1]), we can define the first exit time from a subdomain $U \subset M$ in the following two ways:

The right-first exit time:

$$r_U = \inf\{t : \xi(t) \notin U\}.$$

The left-first exit time:

$$l_U = \inf\{t : \lim_{s \to t_-} \xi(s) \notin U\}.$$

DEFINITION 2. Let T_t be an unexplosive G-invariant symmetric semigroup on $L^2(M, dm)$. Let U be a subdomain of M. The *right-localization* (resp. *left-localization*) of T_t to $L^2(U, dm)$ is the semigroup defined by

$$T_t^+ \phi(x) = E^x \phi(\xi(t)) \mathbf{1}_{\{t < r_U\}} \quad (\text{resp. } T_t^- \phi(x) = E^x \phi(\xi(t)) \mathbf{1}_{\{t < l_U\}})$$

for $\phi \in L^2(U, dm)$. The Markov property proves that T_t^+ (resp. T_t^-) is a semigroup.

Let $\mathcal{D}^+$ (resp. $\mathcal{D}^-$) be the infinitesimal generator of the semigroup T_t^+ (resp. T_t^-). We call it the *right-localization* (resp. *left-localization*) of the infinitesimal generator $\mathcal{D}$ of T_t.

R e m a r k. If $\xi(t)$ is a diffusion, then its trajectories are continuous and there exists only one localization of T_t. This localization has a kernel which is the solution of the Fokker–Planck equation in the domain U with the absorbed boundary. Thus, the localized infinitesimal generator is the same as $\mathcal{D}$ with the Dirichlet boundary condition. The proof of this fact is especially easy for Itô processes.

Just as in the preliminaries we can prove the Feynman–Kac formula for the operator $\mathcal{D}^+ + V$, and repeating the proof of Lemma 1 gives the following local version of the C-L-R inequality.

LEMMA 2 (The local C-L-R inequality). *Let $\delta > 2$ and $\mathrm{rank}_0(\mathcal{D}) \le \delta \le \mathrm{rank}_\infty(\mathcal{D})$. There exists a constant $C > 0$ such that*

$$N(\mathcal{D}^+ + V, 0) \le C \int_U dm(x) \, |V(x)|^{\delta/2}$$

for every nonpositive potential V.

PROPOSITION 1. *Let $\delta > 2$. If $\mathrm{rank}_0(\mathcal{D}) \le \delta \le \mathrm{rank}_\infty(\mathcal{D})$, then also $\mathrm{rank}_0(\mathcal{D}^+) \le \delta \le \mathrm{rank}_\infty(\mathcal{D}^+)$.*

P r o o f. From Lemma 2, we get the Sobolev inequality. Then we use the quoted Varapoulos theorem.

Let $N_U(\lambda)$ denote the number of eigenvalues of $\mathcal{D}^+$ less than $\lambda > 0$. The following upper bound on $N_U(\lambda)$ is an easy corollary of Lemma 2 and Proposition 1.

COROLLARY 2 (The upper Weyl estimate). *Let $\delta = \mathrm{rank}_0(\mathcal{D}) > 2$. Then there exists a constant $C > 0$ such that*

$$N_U(\lambda) \le C \lambda^{\delta/2} |U|_m,$$

where $|U|_m$ denotes the m-measure of U.

As we shall see below, for a large class of Dirichlet forms Corollary 2 gives the asymptotic growth of $N_U(\lambda)$.

R e m a r k. Of course, all the above holds with $\mathcal{D}^-$ in place of $\mathcal{D}^+$.

The Cwikel–Lieb–Rosenblum inequality for optional potentials. We can use the mini-max principle to estimate $N(\mathcal{D} + V, \lambda)$ for any potential V and real λ. Accordingly, for any $\delta > 2$ with $\mathrm{rank}_0(\mathcal{D}) \le \delta \le \mathrm{rank}_\infty(\mathcal{D})$, there exists $C > 0$ such that

$$N(\mathcal{D} + V, \lambda) \le C \int dm(x) \, \mathbf{1}_{\{V(x) \le \lambda\}} (\lambda - V(x))^{\delta/2}.$$

In the case of $M = G$ and the G-action being right translation, we can rewrite the above inequality in a form which associates the upper bound of the C-L-R inequality with the growth of the Haar measure. Let us recall the most important facts concerning the growth of Haar measures.

Let G be a locally compact compactly generated group. Let U be a conditionally compact symmetric neighbourhood of the identity in G. Let $U^n = U \cdot \ldots \cdot U$. We define the growth function of U in G as

$$\gamma_U(n) = |U^n|,$$

where $|\cdot|$ denotes the Haar measure. (By our preliminary assumption (F) we do not need to distinguish between the left and right Haar measures.) Consider a G-invariant metric $d(\cdot, \cdot)$ on G. Let $B_r = \{x : d(x, \text{the identity of } G) \leq r\}$ and $\gamma(r) = |B_r|$. The function $\gamma(r)$ so defined has some well-known properties (cf. [9]):

(a) for any U there exists a constant $C > 0$ such that

$$\frac{1}{C}\gamma_U(n) \leq \gamma(n) \leq C\gamma_U(n),$$

for $n = 1, 2, \ldots$,

(b) for any connected Lie group there exists a constant $C > 0$ such that either

$$\frac{1}{C}e^r \leq \gamma(r) \leq Ce^r$$

(we then say that G has *exponential growth*), or there exists $\alpha = 1, 2, \ldots$ such that

$$\frac{1}{C}r^\alpha \leq \gamma(r) \leq Cr^\alpha$$

(and then G has *polynomial growth* of rank α),

(c) any group of polynomial growth is unimodular.

THEOREM 2. *If G has polynomial growth of rank α, then*

$$N(\mathcal{D} + V, \lambda) \leq C|\{(p,q) : d(p,o)^{2\alpha/\delta} + V(q) \leq \lambda\}|_{m \times m},$$

where $|\cdot|_{m \times m}$ denotes the Haar measure on $G \times G$, o is the identity of G, and δ is as before.

If G is a unimodular group of exponential growth then

$$N(\mathcal{D} + V, \lambda) \leq C\left|\left\{(p,q) : \exp\left(\frac{2}{\delta}d(p,o)\right) + V(q) \leq \lambda\right\}\right|_{m \times m}.$$

P r o o f. This is only a calligraphy. We notice that

$$(\lambda - V(x))^{\delta/2} \leq c\gamma((\lambda - V(x))^{\delta/(2\alpha)}),$$

and then we use the Fubini theorem.

R e m a r k. By defining

$$\gamma(t) = |oB_t|_m,$$

we can rewrite Guivarc'h's theory (of the growth of the Haar measure, cf. [9]) on manifolds with a transitive unimodular G-action. A few technical troubles are of minor importance. Thus, Theorem 2 can be formulated for such manifolds.

The uncertainty principle. As we have seen in Theorem 1, the Cwikel–Lieb–Rosenblum inequality describes the properties of the operator $\mathcal{D}$ rather than those of $\mathcal{D} + V$. This upper bound does not take into consideration growth properties of V more subtle than $L^{\delta/2}$-integrability of $\max(V - \lambda, 0)$. For sharper estimates, we need to examine what properties of the potential influence the eigenvalue asymptotics. In the case of $M = \mathbb{R}^d$ and $\mathcal{D} = -\Delta$, we can use the Fefferman SAK-estimates (cf. [5]) to conclude that only a subdomain of $\{x : V(x) \leq \lambda\}$ influences the number of eigenvalues less than λ (cf. [2]). The next lemma gives us a description of this subdomain.

Suppose that vector fields $X_1, \ldots, X_n$ and their commutators of order $\leq k$ span the tangent space at every point $x \in M$. Let $\mathcal{D} = -\sum_{j=1}^n X_j^2$ (these are best known second order differential operators, cf. [5], [15], [17]). Define

$$X_I = X_{i_1} \cdot \ldots \cdot X_{i_r}$$

for an r-tuple of integers $I = (i_1, \ldots, i_r)$, $1 \leq i_j \leq n$ for $j = 1, \ldots, r$. Let $|I| = r$ for such an r-tuple.

DEFINITION 3. We say that a smooth function V on M is *compatible* with the family of vector fields $\{X_1, \ldots, X_n\}$ if there exists $s \in \mathbb{N}$ such that, for every $x \in M$, there exists $I = (i_1, \ldots, i_r), 0 \leq r \leq s$, for which $X_I V(x) \neq 0$.

LEMMA 3 (The uncertainty principle). *Let V be compatible with the family $\{X_1, \ldots, X_n\}$. Then, for every natural r and all $\varepsilon > 0$, there exists a positive constant $C = C(V, r, \varepsilon)$ such that*

$$((\mathcal{D} + V^2)\phi, \phi) \geq \int dm(x) \left(C \sum_{0 < |I| \leq r} |X_I V(x)|^{2/(|I|+1)} - \varepsilon \right) |\phi(x)|^2$$

for every $\phi \in C_c^\infty(M)$.

P r o o f. Let ψ be a compactly supported smooth function on the real line with $\int_{-\infty}^\infty dt\, |\psi(t)|^2 = 1$ and $\int_{-\infty}^\infty dt\, |\widehat{\psi}(t)|^2 t^2 = a^2$. On the manifold $M \times \mathbb{R}$, we define a family of smooth vector fields $\{Y_1, \ldots, Y_{n+1}\}$ by setting

$$Y_j = \begin{cases} X_j & \text{for } j = 1, \ldots, n, \\ (1/a)V(x)d/dt & \text{for } j = n+1. \end{cases}$$

According to our assumptions, there exists a natural s such that the vector fields $Y_1, \ldots, Y_{n+1}$ and their commutators of order $\leq s$ span the tangent space at every point $(x, t) \in M \times \mathbb{R}$. Thus, we can use the Rothschild–Stein lifting (cf. [15]) to get the estimate

$$(1) \qquad \|(\mathcal{H} + \varepsilon)^{(|I|+1)/2} f\|_2^2 \geq c \|[Y_{i_1}, \ldots, [Y_{i_r}, Y_{n+1}] \ldots] f\|_2^2,$$

where $\mathcal{H} = -\sum_{j=1}^{n+1} Y_j^2$ and $f \in C_c^\infty(M \times \mathbb{R})$.

Notice that

$$[Y_{i_1}, \ldots, [Y_{i_r}, Y_{n+1}] \ldots] = \left(\frac{1}{a} X_I V\right)(x) \frac{d}{dt},$$

and denote by hats the Fourier transform on the added variable. Then (1) implies that

$$\int dm(x) \int_{-\infty}^{\infty} dt\, (\widehat{\mathcal{H}} + \varepsilon)^{|I|+1} \widehat{f}(x,t) \overline{\widehat{f}(x,t)}$$

$$\geq c \int dm(x) \int_{-\infty}^{\infty} dt\, (X_I V(x))^2 t^2 |\widehat{f}(x,t)|^2.$$

Hence, we obtain (cf. [12])

$$(2) \qquad \int dm(x) \int_{-\infty}^{\infty} dt\, (\widehat{\mathcal{H}} + \varepsilon) \widehat{f}(x,t) \overline{\widehat{f}(x,t)}$$

$$\geq c \int dm(x) \int_{-\infty}^{\infty} dt\, |X_I V(x) t|^{2/(|I|+1)} |f(x,t)|^2.$$

Let $\phi \in C_c^\infty(M)$ and $f(x,t) = \phi(x)\psi(t)$. By substituting this in (2), we get

$$\int dm(x) \int_{-\infty}^{\infty} dt\, (|\widehat{\psi}(t)|^2 (\mathcal{D}\phi(x)\overline{\phi(x)} + a^{-2} t^2 V(x)^2 |\phi(x)|^2 + \varepsilon |\phi(x)|^2))$$

$$\geq c \int dm(x) \int_{-\infty}^{\infty} dt\, |X_I V(x) t|^{2/(|I|+1)} |\phi(x)|^2 |\widehat{\psi}(t)|^2,$$

which completes the proof.

R e m a r k s. The label of Lemma 3 is justified by the following easy corollary of the classical Heisenberg principle.

There exists a constant $c > 0$ (depending only on the dimension) such that

$$\int dx\, (-\Delta f(x)\overline{f(x)} + V(x)^2 |f(x)|^2) \geq c \left| \int dx\, |f(x)|^2 \sum_j \partial_j V(x) \right|.$$

For polynomial V, we can choose the constant in Lemma 3 to depend only on the degree of V and on ε (cf. [3]).

An upper bound for $N(\mathcal{D} + V^2, \lambda)$. Let $\mathcal{D}$ be the sum of the squares of vector fields $X_1, \ldots, X_n$ on a unimodular Lie group G. Assume the Hörmander condition ($\text{Lie}\{X_i : i = 1, \ldots, n\}$ equals the Lie algebra of G). Let V be a compatible function on G.

DEFINITION 4. For any $\varepsilon > 0$ and any natural k, we define

$$\text{corr}_{k,\varepsilon}(V, x) = C \sum_{0 < |I| \leq k} |X_I V(x)|^{2/(|I|+1)},$$

where $C = C(\varepsilon, k, V)$ is the maximal constant from Lemma 3. We call it the ε-*correction* of rank k.

As previously $d(\cdot, \cdot)$ denotes an invariant metric on G, and we fix some $\delta > 2$ with $\text{rank}_0(\mathcal{D}) \leq \delta \leq \text{rank}_\infty(\mathcal{D})$.

THEOREM 3. *If G has polynomial growth of rank α, then for any $\lambda > 0$,* $N(\mathcal{D} + V^2, \lambda)$

$$\leq C \inf_{k \in \mathbb{N},\, \varepsilon > 0} |\{(p, q) : d(p, 0)^{2\alpha/\delta} + V(x)^2 + \text{corr}_{k,\varepsilon}(V, x) \leq 2(\lambda + \varepsilon)\}|_{m \times m},$$

where C is a constant which depends only on $\mathcal{D}$ and δ.

If G is of exponential growth, then

$$N(\mathcal{D} + V^2, \lambda) \leq C \inf_{k \in \mathbb{N},\, \varepsilon > 0} |\{(p, q) : \exp((2/\delta) d(p, 0))$$

$$+ V(x)^2 + \text{corr}_{k,\varepsilon}(V, x) \leq 2(\lambda + \varepsilon)\}|_{m \times m}.$$

P r o o f. For $\lambda > 0$ we define

$$\vartheta_\lambda(x) = \begin{cases} 0 & \text{if } \text{corr}_{k,\varepsilon}(V, x) > 2(\lambda + \varepsilon), \\ \frac{1}{2} \text{corr}_{k,\varepsilon}(V, x) - \lambda - \varepsilon & \text{otherwise.} \end{cases}$$

Hence, by using Lemma 3 we obtain

$$((\mathcal{D} + V^2)\phi, \phi) = \left(\left(\tfrac{1}{2}\mathcal{D} + \tfrac{1}{2}V^2 - \vartheta_\lambda\right)\phi, \phi\right) + \left(\left(\tfrac{1}{2}\mathcal{D} + \tfrac{1}{2}V^2 + \vartheta_\lambda\right)\phi, \phi\right)$$

$$\geq \lambda \|\phi\|^2 + \left(\left(\tfrac{1}{2}\mathcal{D} + \tfrac{1}{2}V^2 + \vartheta_\lambda\right)\phi, \phi\right)$$

for $\phi \in C_c^\infty$. By using the mini-max principle,

$$N(\mathcal{D} + V^2, \lambda) \leq N\left(\tfrac{1}{2}\mathcal{D} + \tfrac{1}{2}V^2 + \vartheta_\lambda, 0\right),$$

and Theorem 2 finishes the proof.

An easy consequence is

COROLLARY 3. *Let V be a compatible function. If for every $\lambda > 0$ there exists an r-tuple I (r an arbitrary integer ≥ 0) such that the domain $\{x : |X_I V(x)| < \lambda\}$ has a finite measure, then the operator $\mathcal{D} + V^2$ has a compact resolvent.*

R e m a r k. By using the Varopoulos "analysis on Lie groups" (cf. [17]), we can explain what $\text{rank}_0(\mathcal{D})$ and $\text{rank}_\infty(\mathcal{D})$ are, in the case of $\mathcal{D}$ being

the sum of squares of a Hörmander family on a unimodular Lie group G. According to that paper, $\text{rank}_\infty(\mathcal{D})$ equals the rank of growth of G, if G is of polynomial growth. Moreover, $\text{rank}_\infty(\mathcal{D}) = \infty$ for G of exponential growth. To compute $\text{rank}_0(\mathcal{D})$, denote by K_j the subspace of the Lie algebra of G spanned by the commutators of length $\leq j$. We have $K_1 \subset \ldots \subset K_s$, where K_s is the Lie algebra of G. Let $n_j = \dim(K_j)$, $j = 1, \ldots, s$. Then $\text{rank}_0(\mathcal{D}) = n_1 + 2(n_2 - n_1) + 3(n_3 - n_2) + \ldots + s(n_s - n_{s-1})$. Hence, $\text{rank}_0(\mathcal{D}) \leq 2$ only in the cases $G = \mathbb{R}$, $G = \mathbb{R}^2$.

The Weyl asymptotics for the nilpotent case. Consider vector fields $X_1, \ldots, X_n$ on a euclidian space E with $\dim(E) = m$. Let

$$X_j = \sum_{k=1}^{m} A_{jk}(x)\partial_k,$$

where the A_{jk} are polynomials and $\partial_i A_{jk} = 0$ for $i \geq k$. Let $\mathfrak{G} = \text{Lie}\{X_1, \ldots, X_n\}$. Then $\mathfrak{G}$ is a finite-dimensional nilpotent Lie algebra. Denote the nilpotent Lie group $\exp(\mathfrak{G})$ by $\mathcal{G}$ and define the curve $\gamma(t) = x.\exp tX$, $x \in E$, $X \in \mathfrak{G}$, as the unique curve with $\gamma(0) = x$ and $\dot\gamma(d/dt) = X$. The equivalence between the Campbell-Hausdorff coordinates and the triangular coordinates ensures that the solution of this ordinary differential equation is global and defines an action of $\mathcal{G}$ on E. This action is transitive if and only if the Hörmander condition holds. Since $\mathcal{G}$ is nilpotent, this action is unimodular (relative to the Lebesgue measure on E). Consider the second-order differential operator

$$\mathcal{D} = -\frac{1}{2}\sum_{k=1}^{n} X_k^2.$$

If we assume the Hörmander condition, then $\mathcal{D}$ is a regular Dirichlet form. Let $\{P(\cdot, \cdot, t) : t > 0\}$ be the kernels of the semigroup generated by $\mathcal{D}$. We call $\mathcal{D}$ a *quotient sublaplacian* (this term is justified by the proof of Proposition 2).

PROPOSITION 2. *For any quotient sublaplacian $\mathcal{D}$ there exists $q \in \mathbb{N}$ such that* $\text{rank}_0(\mathcal{D}) = \text{rank}_\infty(\mathcal{D}) = q$.

Proof. If a nilpotent Lie group acts on E, then so does the free nilpotent Lie group with the same step and a suitable number of free generators. Let $\mathcal{F}$ be such a free nilpotent Lie group. The action of $\mathcal{F}$ on E defines (in the natural way) a unitary representation of $\mathcal{F}$ in $L^2(E)$. We denote this representation by π. Choose $\mathcal{F}$ so large that there exists a system of free generators, $\{\mathcal{X}_1, \ldots, \mathcal{X}_n\}$, in its Lie algebra for which $\pi(\mathcal{X}_j) = X_j$, $j = 1, \ldots, n$. Let $\mathcal{L} = -\sum_{j=1}^{n} \mathcal{X}_j^2$ be the sublaplacian on $\mathcal{F}$. Then $\pi(\mathcal{L}) = \mathcal{D}$.

Now, we can use Folland's theory of second-order homogeneous differential operators on stratified nilpotent Lie groups (cf. [7]), according to which $\mathcal{L}$ is the infinitesimal generator of a convolution semigroup $T_t f(x) = p_t * f(x)$, where p_t, $t > 0$, are Schwartz functions. We define a canonical family of dilatations, $\{\gamma_r : r > 0\}$, by putting $\gamma_r(\mathcal{X}_j) = r\mathcal{X}_j$. Thus, $\mathcal{L}$ is a homogeneous differential operator of homogeneous degree 2. Hence,

$$(1) \qquad p_t(x) = t^{-Q/2} p_1(\gamma_{1/\sqrt{t}}(x)),$$

where Q denotes the homogeneous dimension of $\mathcal{F}$.

Let $o \in E$, and $\mathcal{H} \subset \mathcal{G}$ be its isotropy subgroup. We can identify E with the space of right cosets $\mathcal{F}\backslash\mathcal{H}$. Let $\mathfrak{F}$, $\mathfrak{H}$ be the Lie algebras of $\mathcal{F}$, $\mathcal{H}$ respectively. We can decompose $\mathfrak{F}$ as a vector space sum $\mathfrak{F} = \mathfrak{S} \oplus \mathfrak{H}$ in such a way that $\gamma_r(\mathfrak{S}) = \mathfrak{S}$. We identify $\mathcal{F}$ with $\mathfrak{F}$ by the exponential mapping. Since $\mathfrak{S}$ is a selector over the space of right cosets $\mathcal{F}\backslash\mathcal{H}$, it can be identified with E. Any $x \in \mathcal{F}$ can be uniquely represented as $x = h(x)s(x)$ with $h(x) \in \mathcal{H}$ and $s(x) \in \exp(\mathfrak{S})$. Let $\{P(\cdot, \cdot, t) : t > 0\}$ be the kernels of the semigroup generated by $\mathcal{D}$. We have

$$\int ds\, P(o, os, t)f(os) = \pi(p_t)f(o) = \int dh \int ds\, p_t(hs)f(ohs).$$

Because $h \in \mathcal{H} = $ the isotropy subgroup of o, we obtain

$$P(o, o, t) = \int_{\mathcal{H}} dh\, p_t(h).$$

By using (1), we conclude that

$$P(o, o, t) = Ct^{-q/2},$$

where q is the homogeneous dimension of $\mathfrak{S}$.

Since $\text{rank}_0(\mathcal{D}) = \text{rank}_\infty(\mathcal{D})$ for any quotient sublaplacian $\mathcal{D}$, we can denote this number by $\text{rank}(\mathcal{D})$.

As we have seen above, we can construct on E a family of dilatations which is in some connection with $\mathcal{D}$, but $\mathcal{D}$ is a homogeneous operator if and only if we can choose $o \in E$ in such a way that its isotropy subgroup $\mathcal{H}$ is homogeneous.

In the homogeneous case we can easily prove that our upper bound gives the asymptotics for the number of eigenvalues. Let $|\cdot|_\gamma$ denote a homogeneous (with respect to γ_r) norm on $\mathcal{F}$. Because we can identify E with a homogeneous submanifold of $\mathcal{F}$ this norm is defined on E. Let $B(r, os) = \{oz \in E : |zs^{-1}|_\gamma < r\}$. Then there exists a constant $c > 0$ such that the Lebesgue measure of $B(r, os)$ equals cr^q.

Any quotient sublaplacian is the infinitesimal generator of a diffusion semigroup. So, $\mathcal{D}^+ = \mathcal{D}^-$ in any subdomain $U \subset E$. Denote the localized operator by $\mathcal{D}_U$. It is the same as $\mathcal{D}$ with the Dirichlet condition on the

boundary ∂U. As previously, let $N_U(\lambda)$ denote the number of eigenvalues of $\mathcal{D}_U$ less than $\lambda > 0$.

LEMMA 4. *There exists a constant $C > 0$ such that, if $n_U(r)$ denotes the number of balls with radius r (in the sense of the above homogeneous norm on E) which can be packed disjointly inside U, then*

$$n_U(C\lambda^{-1/2}) \leq N_U(\lambda).$$

P r o o f. By recalling the mini-max principle, $N_U(\lambda) \geq N$ if we can find an N-dimensional subspace $H \subset L^2(U)$ such that

$$(\mathcal{D}_U\psi, \psi) \leq \lambda\|\psi\|_2^2$$

for $\psi \in H$. Let ϕ be a smooth function on $\mathfrak{G}$ supported by the unit ball with centre at o. Let $r = c\lambda^{-1/2}$. We can translate $\phi \circ \gamma_r$ to balls with radius r which are contained inside U. Thus, we get a family $\{\psi_1, \ldots, \psi_N\}$ of smooth functions with $N = n_U(r)$. For any $1 \leq i < j \leq N$, the supports of ψ_i and ψ_j are disjoint, and since $\mathcal{D}$ is a homogeneous operator,

$$(\mathcal{D}_U\psi_j, \psi_j) = C\lambda\|\psi_j\|_2^2.$$

We can choose c such that $C = 1$ and the proof is complete.

The following theorem is a consequence of the previous results.

THEOREM 4. *Let $\mathcal{D}$ be a quotient homogeneous sublaplacian on a euclidian space E. Let $\mathrm{rank}(\mathcal{D}) = q > 2$. If a subdomain $U \subset E$ has finite Lebesgue measure, and the measure of ∂U is zero, then*

$$c|U|\lambda^{q/2} \leq N_U(\lambda) \leq C|U|\lambda^{q/2},$$

where $|U|$ denotes the Lebesgue measure of U, and $c, C > 0$ are some constants.

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