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# The stability radius of an operator of Saphar type


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**Abstract.** A bounded linear operator $T$ on a complex Banach space $X$ is called an operator of Saphar type if its kernel is contained in its generalized range $\bigcap_{n=1}^{\infty} T^n(X)$ and $T$ is relatively regular. For $T$ of Saphar type we determine the supremum of all positive numbers $\delta$ such that $T - \lambda I$ is of Saphar type for $|\lambda| < \delta$.


**I. Terminology and introduction.** Throughout this paper let $X$ denote a Banach space over the complex field $\mathbb{C}$ and let $\mathcal{L}(X)$ denote the algebra of all bounded linear operators on $X$. If $T \in \mathcal{L}(X)$, we denote by $N(T)$ the kernel and by $T(X)$ the range of $T$. The *generalized range* of $T$ is defined by

$$T^{\infty}(X) = \bigcap_{n=1}^{\infty} T^n(X).$$

We write $\sigma(T)$ for the spectrum of $T$ and $\varrho(T)$ for the resolvent set $\mathbb{C} \setminus \sigma(T)$. The spectral radius of $T$ is denoted by $r(T)$.

In [6, Theorem 3] T. Kato showed that for $T$ in $\mathcal{L}(X)$ the set

$$\varrho_{\mathrm{K}}(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)(X) \text{ is closed and } N(T - \lambda I) \subseteq (T - \lambda I)^{\infty}(X)\}$$

is an open subset of $\mathbb{C}$. Since $\varrho(T) \subseteq \varrho_{\mathrm{K}}(T)$, the complement $\sigma_{\mathrm{K}}(T) = \mathbb{C} \setminus \varrho_{\mathrm{K}}(T)$ is a compact subset of $\sigma(T)$. We showed in [10, Satz 2] that $\partial\sigma(T) \subseteq \sigma_{\mathrm{K}}(T)$, thus $\sigma_{\mathrm{K}}(T) \neq \emptyset$.

We call $T \in \mathcal{L}(X)$ *relatively regular* if $TST = T$ for some $S \in \mathcal{L}(X)$. In this case $TS$ is a projection on $T(X)$ (hence $T(X)$ is closed), $I - ST$ is a projection on $N(T)$, and we say that $S$ is a *pseudo-inverse* of $T$.

$T$ is called an *operator of Saphar type* if $T$ is relatively regular and $N(T) \subseteq T^{\infty}(X)$. This class of operators has been studied by P. Saphar [9] (see also [2] and [12]). Operators in this class have an important property:

THEOREM 1. *$T \in \mathcal{L}(X)$ is of Saphar type if and only if there is a neighbourhood $U \subseteq \mathbb{C}$ of $0$ and a holomorphic function $F : U \to \mathcal{L}(X)$ such*

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that

$$(T-\lambda I)F(\lambda)(T-\lambda I)=T-\lambda I \quad \textit{for all } \lambda\in U.$$

For a proof see [7, Théorème 2.6] or [11, Theorem 1.4] or [2, Theorem 9, Theorem 11 in §5.2].

If $F$ is a holomorphic function with the property of Theorem 1, then we say that $T-\lambda I$ has holomorphic pseudo-inverse $F$ in $U$. Theorem 1 shows that the set

$$\varrho_{\mathrm{rr}}(T)=\{\lambda\in\mathbb{C}: T-\lambda I \text{ is of Saphar type}\}$$

is open. We also have $\varrho(T)\subseteq\varrho_{\mathrm{rr}}(T)$. Setting $\sigma_{\mathrm{rr}}(T)=\mathbb{C}\setminus\varrho_{\mathrm{rr}}(T)$, we derive

$$\sigma_{\mathrm{K}}(T)\subseteq\sigma_{\mathrm{rr}}(T)\subseteq\sigma(T) \quad\text{and}\quad \varrho(T)\subseteq\varrho_{\mathrm{rr}}(T)\subseteq\varrho_{\mathrm{K}}(T).$$

In general, $\varrho_{\mathrm{K}}(T)\neq\varrho_{\mathrm{rr}}(T)$ (see [12]).

Let $T\in\mathcal{L}(X)$ and $0\in\varrho_{\mathrm{rr}}(T)$, hence $TST=T$ for some $S\in\mathcal{L}(X)$. In the third section of this paper we shall see that then $0\in\varrho_{\mathrm{rr}}(T^n)$ and $T^nS^nT^n=T^n$ for all $n\in\mathbb{N}$ (Proposition 5). For $n\in\mathbb{N}$ put

$$\delta_n(T)=\sup\{r(A)^{-1}: A\in\mathcal{L}(X) \text{ and } T^nAT^n=T^n\}.$$

The aim of this paper is to show that

$$\operatorname{dist}(0,\sigma_{\mathrm{rr}}(T))=\sup_{n\geq1}\delta_n(T)^{1/n}=\lim_{n\to\infty}\delta_n(T)^{1/n}.$$

In the next section we collect some properties of the Kato resolvent set $\varrho_{\mathrm{K}}(T)$. In the final section of this paper we investigate $\varrho_{\mathrm{rr}}(T)$, and we give a proof of the above distance formula.

**II. Properties of $\varrho_{\mathrm{K}}(T)$.** For the rest of this paper we always assume that $T\in\mathcal{L}(X)\setminus\{0\}$. By definition, the *reduced minimum modulus* $\gamma(T)$ of $T$ is given by

$$\gamma(T)=\inf\left\{\frac{\|Tx\|}{d(x,N(T))}: x\in X,\ Tx\neq0\right\}$$

($d(x,N(T))$ denotes the distance of $x$ to $N(T)$). It is well known that $T(X)$ is closed if and only if $\gamma(T)>0$.

PROPOSITION 1. *Suppose that $0\in\varrho_{\mathrm{K}}(T)$. Then*

(a) *$T^n(X)$ is closed for each $n\in\mathbb{N}$ and $T(T^\infty(X))=T^\infty(X)$.*
(b) $\{\lambda\in\mathbb{C}:|\lambda|<\gamma(T)\}\subseteq\varrho_{\mathrm{K}}(T)$.
(c) *$\gamma(T^{m+n})\geq\gamma(T^n)\gamma(T^m)$ for all $n,m\in\mathbb{N}$.*

Proof. (a) [10, Satz 4] shows that $T^n(X)$ is closed for all $n\in\mathbb{N}$. The inclusion $T(T^\infty(X))\subseteq T^\infty(X)$ is obvious. Let $y\in T^\infty(X)$. For every $k\in\mathbb{N}$ there is $x_k\in X$ so that $y=T^kx_k$. Put $z_k=x_1-T^{k-1}x_k$. Then $Tz_k=0$, hence $z_k\in N(T)\subseteq T^{k-1}(X)$. It follows that $x_1\in T^{k-1}(X)$ for all $k$. This gives $y=Tx_1\in T(T^\infty(X))$.



(b) [1, Theorem 1.1]. (c) follows from [4, Lemma 1]. ■

The set of all complex-valued functions which are analytic in some neighbourhood of $\sigma(T)$ is denoted by $\mathcal{H}(T)$. For $f\in\mathcal{H}(T)$, the operator $f(T)$ is defined by the well known analytic calculus.

PROPOSITION 2. *Let $T\in\mathcal{L}(X)$. Then*

(a) *$\sigma_{\mathrm{K}}(f(T))=f(\sigma_{\mathrm{K}}(T))$ for all $f\in\mathcal{H}(T)$.*
(b) *If $C$ is a connected component of $\varrho_{\mathrm{K}}(T)$, then the mappings*

$$\lambda\mapsto\overline{\bigcup_{n=1}^{\infty}N((T-\lambda I)^n)} \quad\textit{and}\quad \lambda\mapsto(T-\lambda I)^\infty(X)$$

*are constant in $C$.*

Proof. (a) [10, Satz 6], (b) [3, Theorem 3]. ■

PROPOSITION 3. *Suppose that $0\in\varrho_{\mathrm{K}}(T)$. Then*

(a) $\lim_{n\to\infty}\gamma(T^n)^{1/n}=\sup_{n\geq1}\gamma(T^n)^{1/n}\leq\min\{|\lambda|:\lambda\in\partial\sigma(T)\}$.
(b) $\{\lambda\in\mathbb{C}:|\lambda|<\lim_{n\to\infty}\gamma(T^n)^{1/n}\}\subseteq\varrho_{\mathrm{K}}(T)$.

Proof. (a) We have $\lim_{n\to\infty}\gamma(T^n)^{1/n}=\sup_{n\geq1}\gamma(T^n)^{1/n}$ by [4, remarks in connection with Lemma 1]. Fix $\mu\in\partial\sigma(T)$ such that $|\mu|=\min\{|\lambda|:\lambda\in\partial\sigma(T)\}$ and assume that $|\mu|<\gamma(T^m)^{1/m}$ for some $m\in\mathbb{N}$, thus $|\mu^m|<\gamma(T^m)$. Since $0\in\varrho_{\mathrm{K}}(T^m)$ by Proposition 2(a), we have $\mu^m\in\varrho_{\mathrm{K}}(T^m)$ (use Proposition 1(b)). Proposition 2(a) implies now that $\mu\in\varrho_{\mathrm{K}}(T)$, a contradiction, since $\mu\in\partial\sigma(T)\subseteq\sigma_{\mathrm{K}}(T)$.

(b) If $|\lambda|<\lim_{n\to\infty}\gamma(T^n)^{1/n}$ then $|\lambda|<\gamma(T^m)^{1/m}$ for some $m\in\mathbb{N}$. The same arguments as above show that $\lambda\in\varrho_{\mathrm{K}}(T)$. ■

If $T\in\mathcal{L}(X)$ and $0\in\varrho_{\mathrm{K}}(T)$ we define

$$\Gamma(T)=\lim_{n\to\infty}\gamma(T^n)^{1/n}\quad(=\sup_{n\geq1}\gamma(T^n)^{1/n}).$$

An immediate consequence of Proposition 3 is

COROLLARY 1. *If $0\in\varrho_{\mathrm{K}}(T)$, then $\Gamma(T)\leq r(T)$.*

**III. Properties of $\varrho_{\mathrm{rr}}(T)$.** Recall that $T\in\mathcal{L}(X)$ is an operator of Saphar type if $T$ is relatively regular and $N(T)\subseteq T^\infty(X)$. For relatively regular operators we have the following basic result.

PROPOSITION 4. *If $T\in\mathcal{L}(X)$ and $TST=T$ for some $S\in\mathcal{L}(X)$, then*

$$\|S\|^{-1}\leq\gamma(T).$$

Proof. [4, Lemma 4]. ■

In [12, Theorem 3] we showed that a spectral mapping theorem for $\sigma_{\mathrm{rr}}(T)$ is valid:

$$f(\sigma_{\mathrm{rr}}(T)) = \sigma_{\mathrm{rr}}(f(T)) \quad \text{for all } f \in \mathcal{H}(T).$$

PROPOSITION 5. *Suppose that $T$ is of Saphar type and $TST = T$ for some $S \in \mathcal{L}(X)$. Then*

(a) *$T^n$ is of Saphar type and $T^nS^nT^n = T^n$ for all $n \in \mathbb{N}$.*
(b) *$(T-\lambda I)(I-\lambda S)^{-1}S(T-\lambda I) = T - \lambda I$ for $|\lambda| < r(S)^{-1}$.*
(c) *$\{\lambda \in \mathbb{C} : |\lambda| < r(S)^{-1}\} \subseteq \varrho_{\mathrm{rr}}(T)$.*

Proof. (a) [12, Proposition 2]. (b) [11, Corollary 1.5]. (c) follows from (b) and Theorem 1. ■

By Theorem 1 we have the following characterization for points in $\varrho_{\mathrm{rr}}(T)$: $\lambda_0 \in \varrho_{\mathrm{rr}}(T)$ if and only if there is a neighbourhood $U$ of $\lambda_0$ and a holomorphic function $F : U \to \mathcal{L}(X)$ such that

$$(T-\lambda I)F(\lambda)(T-\lambda I) = T-\lambda I \quad \text{for all } \lambda \in U.$$

Thus $T - \lambda I$ has *locally* a holomorphic pseudo-inverse on $\varrho_{\mathrm{rr}}(T)$. But we can say more if we use the following result of Shubin [13, p. 161] (see also [5, Theorem 3.9]):

THEOREM 2. *Suppose that $G \subseteq \mathbb{C}$ is open and connected. Then $G \subseteq \varrho_{\mathrm{rr}}(T)$ if and only if there is a holomorphic function $F : G \to \mathcal{L}(X)$ such that*

$$(T-\lambda I)F(\lambda)(T-\lambda I) = T-\lambda I \quad \text{for } \lambda \in G.$$

Theorem 2 shows that $T - \lambda I$ has a *global* holomorphic pseudo-inverse on each connected component of $\varrho_{\mathrm{rr}}(T)$.

In order to facilitate further expressions, we introduce the following notions. Let $T \in \mathcal{L}(X)$ be given and suppose that $0 \in \varrho_{\mathrm{rr}}(T)$ (recall that we then have $0 \in \varrho_{\mathrm{rr}}(T^n)$ for each $n \in \mathbb{N}$, by Proposition 5(a)). We denote the distance $\mathrm{dist}(0, \sigma_{\mathrm{rr}}(T))$ by $d(T)$. Put

$$\delta_n(T) = \sup\{r(A)^{-1} : A \in \mathcal{L}(X),\ T^nAT^n = T^n\} \quad \text{for } n \in \mathbb{N},$$
$$\delta(T) = \sup_{n \ge 1} \delta_n(T)^{1/n}.$$

If $0 \in \varrho_{\mathrm{rr}}(T)$ and $TST = T$ for some $S \in \mathcal{L}(X)$, Propositions 4 and 5 show that $\|S^n\|^{-1} \le \gamma(T^n)$, thus $(\|S^n\|^{1/n})^{-1} \le \gamma(T^n)^{1/n}$ for all $n \in \mathbb{N}$. This gives

$$r(S)^{-1} \le \lim_{n\to\infty} \gamma(T^n)^{1/n} = \Gamma(T).$$

Suppose now that $0 \in \varrho_{\mathrm{rr}}(T)$ and $T^kAT^k = T^k$ for some $A \in \mathcal{L}(X)$ and some $k \in \mathbb{N}$. Then $r(A)^{-1} \le \Gamma(T^k)$ and we see that

$$r(A)^{-1} \le \lim_{n\to\infty} \gamma((T^k)^n)^{1/n} = \lim_{n\to\infty} [\gamma(T^{nk})^{1/nk}]^k = \Gamma(T)^k.$$



Hence $\delta_k(T) \le \Gamma(T)^k$. Thus we have proved the following result (recall that $\Gamma(T) \le r(T)$ by Corollary 1).

PROPOSITION 6. *If $0 \in \varrho_{\mathrm{rr}}(T)$, then $\delta_k(T)^{1/k} \le \delta(T) \le \Gamma(T) \le r(T)$ for each $k \in \mathbb{N}$.*

PROPOSITION 7. *Suppose that $0 \in \varrho_{\mathrm{rr}}(T)$ and put $G = \{\lambda \in \mathbb{C} : |\lambda| < d(T)\}$. Then $G \subseteq \varrho_{\mathrm{rr}}(T)$ and there is a holomorphic function $F : G \to \mathcal{L}(X)$ such that*

$$(T-\lambda I)F(\lambda)(T-\lambda I) = T-\lambda I \quad \text{for all } \lambda \in G.$$

*The function $F$ has the following properties:*

(i) *$(T-\lambda I)^{n+1}F^{(n)}(\lambda)(T-\lambda I)^{n+1} = n!(T-\lambda I)^{n+1}$ for all $\lambda \in G$ and all $n \in \mathbb{N}$.*

(ii) *If $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$ for $|\lambda| < d(T)$, then*

$$d(T) = (\limsup \|A_n\|^{1/n})^{-1}$$

*and*

$$T^{n+1}A_nT^{n+1} = T^{n+1} \quad \text{for each } n = 0, 1, 2, \ldots$$

Proof. The definition of $d(T)$ shows that $G \subseteq \varrho_{\mathrm{rr}}(T)$. The existence of $F$ follows from Theorem 2. For the proof of (i) use induction. If $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$ is the power series expansion around 0, then the radius of convergence $R$ is $R = (\limsup \|A_n\|^{1/n})^{-1}$. Clearly we have $R \ge d(T)$. Assume that $R > d(T)$. Put $H(\lambda) = (T-\lambda I)F(\lambda)(T-\lambda I) - (T-\lambda I)$ for $|\lambda| < R$. Then $H(\lambda) = 0$ for all $|\lambda| < d(T)$, thus $H(\lambda) = 0$ for all $|\lambda| < R$, hence $\{\lambda \in \mathbb{C} : |\lambda| < R\} \subseteq \varrho_{\mathrm{rr}}(T)$, by Theorem 2. This gives $\varrho_{\mathrm{rr}}(T) \cap \sigma_{\mathrm{rr}}(T) \ne \emptyset$, a contradiction. Therefore $R = d(T)$. Since $n!A_n = F^{(n)}(0)$, (i) shows that $T^{n+1}A_nT^{n+1} = T^{n+1}$. ■

COROLLARY 2. *If $0 \in \varrho_{\mathrm{rr}}(T)$, then $\delta(T) \le d(T) \le \Gamma(T)$.*

Proof. Let $|\lambda| < \delta(T)$. Then $|\lambda| < \delta_n(T)^{1/n}$ for some $n \in \mathbb{N}$, thus $|\lambda^n| < \delta_n(T)$. Therefore there is a pseudo-inverse $A$ of $T^n$ with $|\lambda^n| < r(A)^{-1}$. Use Proposition 5(c) to derive $\lambda^n \in \varrho_{\mathrm{rr}}(T^n)$. The spectral mapping theorem for $\sigma_{\mathrm{rr}}(T)$ implies now that $\lambda \in \varrho_{\mathrm{rr}}(T)$. Therefore $\{\lambda \in \mathbb{C} : |\lambda| < \delta(T)\} \subseteq \varrho_{\mathrm{rr}}(T)$. This shows that $\delta(T) \le d(T)$.

It remains to show that $d(T) \le \Gamma(T)$. If $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$ is a holomorphic pseudo-inverse of $T - \lambda I$ for $|\lambda| < d(T)$, then $T^{n+1}A_nT^{n+1} = T^{n+1}$ for all $n \ge 0$. Proposition 4 gives $\|A_n\|^{-1} \le \gamma(T^{n+1})$. Thus

$$(\|A_n\|^{1/n})^{-1} \le \gamma(T^{n+1})^{1/n} = (\gamma(T^{n+1})^{1/(n+1)})^{(n+1)/n} \le \Gamma(T)^{(n+1)/n}.$$

It follows that $d(T) = (\limsup \|A_n\|^{1/n})^{-1} \le \Gamma(T)$. ■

We now state the main result of this paper.

THEOREM 3. *Let $0 \in \varrho_{\mathrm{rr}}(T)$.*

(a) *$\delta(T) = \lim_{n\to\infty} \delta_n(T)^{1/n} = d(T)$.*

(b) *If* $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$ *is a holomorphic pseudo-inverse of* $T - \lambda I$ *for* $|\lambda| < d(T)$, *then*

$$d(T) = (\lim_{n\to\infty} \|A_n\|^{1/n})^{-1} = \lim_{n\to\infty} (r(A_n)^{1/n})^{-1}.$$

Proof. Let $F(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n$ be a holomorphic pseudo-inverse of $T - \lambda I$ for $|\lambda| < d(T)$. Thus $d(T) = (\limsup \|A_n\|^{1/n})^{-1}$. Let $\varepsilon > 0$ be given and $\varepsilon < d(T)$. Put $\eta = d(T) - \varepsilon$. Since $\limsup \|A_n\|^{1/n} = 1/d(T)$, there is an integer $k$ such that

$$\|A_n\|^{1/n} < \frac{1}{d(T)} + \frac{\varepsilon}{d(T)\eta} = \frac{1}{\eta} \quad \text{for all } n \geq k.$$

Since $A_n$ is a pseudo-inverse of $T^{n+1}$ and $\delta(T) \leq d(T)$, we conclude that

$$\eta^n < \|A_n\|^{-1} \leq r(A_n)^{-1} \leq \delta_{n+1}(T) \leq \delta(T)^{n+1} \leq d(T)^{n+1}$$

for all $n \geq k$. This shows that there is an integer $m \geq k$ such that

$$\begin{aligned} d(T) - \varepsilon = \eta &< (\|A_n\|^{1/n})^{-1} \leq (r(A_n)^{1/n})^{-1} \leq \delta_{n+1}(T)^{1/n} \\ &\leq \delta_{n+1}(T)^{1/(n+1)} \sqrt[n]{\delta(T)} \leq \delta(T) \sqrt[n]{\delta(T)} \leq d(T) \sqrt[n]{d(T)} \\ &< d(T) + \varepsilon \quad \text{for } n \geq m. \end{aligned}$$

This gives

$$d(T) = \delta(T) = \lim_{n\to\infty} \delta_n(T)^{1/n} = (\lim_{n\to\infty} \|A_n\|^{1/n})^{-1} = \lim_{n\to\infty} (r(A_n)^{1/n})^{-1}. \blacksquare$$

If $X$ is a complex Hilbert space, then it is well known that $T \in \mathcal{L}(X)$ is relatively regular if and only if $T(X)$ is closed (see [2, p. 12]). Therefore in this case $\varrho_{\mathrm{K}}(T) = \varrho_{\mathrm{rr}}(T)$. Since $\{\lambda \in \mathbb{C} : |\lambda| < \Gamma(T)\} \subseteq \varrho_{\mathrm{K}}(T)$ (Proposition 3(b)), we have an immediate consequence of the last theorem:

COROLLARY 3. *Suppose that* $X$ *is a Hilbert space*, $T \in \mathcal{L}(X)$ *and* $0 \in \varrho_{\mathrm{rr}}(T)$. *Then* $d(T) = \Gamma(T)$.

Remarks. 1. The result of Corollary 3 can be found in [8, Théorème 3.1]. But the proof there contains a gap. Without the use of Theorem 2 it is not clear that if $0 \in \varrho_{\mathrm{rr}}(T)$ then we have a holomorphic pseudo-inverse of $T - \lambda I$ for all $|\lambda| < d(T)$.

2. Suppose that $0 \in \varrho_{\mathrm{K}}(T)$ and $T$ is a semi-Fredholm operator. By [14, Theorem 1], $\mathrm{dist}(0, \sigma_{\mathrm{K}}(T)) = \Gamma(T)$. For more general results for the class of semi-Fredholm operators see [15].

We would like to finish this paper with several questions:

QUESTION 1. If $0 \in \varrho_{\mathrm{rr}}(T)$, must $d(T) = \Gamma(T)$?

QUESTION 2. If $0 \in \varrho_{\mathrm{K}}(T)$, must $\mathrm{dist}(0, \sigma_{\mathrm{K}}(T)) = \Gamma(T)$? Observe that $\mathrm{dist}(0, \sigma_{\mathrm{K}}(T)) \geq \Gamma(T)$ by Proposition 3(b).

QUESTION 3. Does there exist a holomorphic function $F : \varrho_{\mathrm{rr}}(T) \to \mathcal{L}(X)$ with



$$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I \quad \text{for } \lambda \in \varrho_{\mathrm{rr}}(T)$$

and

$$F(\lambda) - F(\mu) = (\lambda - \mu)F(\lambda)F(\mu) \quad \text{for all } \lambda, \mu \in \varrho_{\mathrm{rr}}(T)?$$

Observe that this is *locally* true by Proposition 5(b).

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