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On sequential convergence in weakly compact subsets of Banach spaces

by

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Abstract. We construct an example of a Banach space E such that every weakly compact subset of E is bisequential and E contains a weakly compact subset which cannot be embedded in a Hilbert space equipped with the weak topology. This answers a question of Nyikos.

1. Introduction. Let us recall that a (*uniform*) *Eberlein compactum* is a space homeomorphic to a weakly compact subset of a (Hilbert) Banach space. Equivalently, a compact space K is an Eberlein compactum if K can be embedded in the following subspace of the product $\mathbb{R}^\Gamma$:

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \text{for every } \varepsilon > 0 \text{ the set } \{\gamma : |x(\gamma)| > \varepsilon\} \text{ is finite}\}$$

(see [Ne]). Every Eberlein compactum K is a *Fréchet space*, i.e. given a subset $X \subseteq K$ and a point x in the closure of X there exists a sequence of points of X converging to x. We will consider the following stronger sequential property:

We say that the space X is *bisequential* if for every point x and every ultrafilter $\mathcal{U}$ converging to x there is a sequence (A_n) contained in $\mathcal{U}$ and converging to x (see [Mi, Definition 3.D.1], [Ny1, p. 137]). This property is preserved by arbitrary subsets (see [Mi, Proposition 3.D.3]). Every uniform Eberlein compactum of cardinality smaller than the first uncountable measurable cardinal is bisequential. There are also examples of bisequential Eberlein compacta which are not uniform. Nyikos gave an example of an Eberlein compactum X of size ω_1 which is not bisequential (see [Ny2]; the same construction was used for different purposes by Leiderman and Sokolov [LS, Example 5.3]). He asked if there exists a Banach space such that every weakly compact subset of it is bisequential, but not every weakly compact

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subset is uniform Eberlein compact. In this note we give an affirmative answer to this question.

1.1. EXAMPLE. *There exists a Banach space E such that every weakly compact subset of E is bisequential and E contains a weakly compact subset which cannot be embedded in a Hilbert space equipped with the weak topology.*

Our construction uses another class of compact spaces which is also related to Banach space theory—the class of *Rosenthal compacta*, i.e. compact spaces which can be embedded in the space $B_1(X)$ of the first Baire class functions on a separable completely metrizable space X endowed with the pointwise topology (see [Ne, Section 1] and [Go]). Let us recall that a function $f : X \to \mathbb{R}$ on a separable metrizable space X is of the *first Baire class* if $f^{-1}(U)$ is an F_σ-subset of X for every open $U \subset \mathbb{R}$. All Rosenthal compacta are Fréchet spaces [BFT, Theorem 3F] and Pol refined this result by proving the following:

1.2. THEOREM (Pol). *Every separable Rosenthal compactum is bisequential.*

The proof of this theorem is unpublished, but it is based on ideas similar to the proof of a special case given in [Po, §5, Remark B].

In the next section we will give an example of a nonuniform Eberlein compactum K which can be embedded in a separable Rosenthal compactum and in Sec. 3 we will use this example to construct the Banach space E from 1.1.

2. Nonuniform Eberlein compact space K. Let us set up the notation which we will use in this section.

By 2^n we denote the set of functions from $\{0,1,\ldots,n-1\}$ into $\{0,1\}$, $n \in \omega$, and let $V_s = \{x \in 2^\omega : x|n = s\}$ where $x|n$ is the restriction of a function $x : \omega \to \{0,1\}$ to the set $\{0,1,\ldots,n-1\}$ and $s \in 2^n$. The open-and-closed sets V_s, for $s \in 2^n$, $n \in \omega$, form the canonical base of the Cantor set 2^ω.

Given a set X we denote the family of all subsets of X of cardinality $\le n$ by $[X]^{\le n}$. For a subset A of X let $\chi_A : X \to \{0,1\}$ be the characteristic function of A.

We will construct an Eberlein compact space K with the following properties:

2.1. EXAMPLE. *There exists a nonuniform Eberlein compact space K which can be embedded in a separable Rosenthal compact space L.*

Let us point out that all Eberlein compact spaces of weight less than or equal to continuum are Rosenthal compacta but not all of them can be embedded in separable Rosenthal compacta. Namely, let X be Nyikos' example of an Eberlein compactum of weight ω_1 which is not bisequential (see Sec. 1). From Pol's result 1.2 it follows that X cannot be embedded in a separable Rosenthal compactum. The situation is different for the class of uniform compacta:

2.2. PROPOSITION. *Every uniform Eberlein compactum of weight less than or equal to continuum can be embedded in a separable Rosenthal compactum.*

Proof. Since every uniform Eberlein compactum of weight less than or equal to continuum can be regarded as a subset of the closed unit ball B of the Hilbert space $l_2(2^\omega)$ it is enough to prove our assertion for B. The weak and pointwise topologies coincide on B, therefore we may consider B as a subset of $\mathbb{R}^{2^\omega}$. For every $n \in \omega$ we define the following subsets of $\mathbb{R}^{2^\omega}$:

$$C_n = \Big\{ \sum_{s\in 2^n} t_s \chi_{V_s} : t_s \in \mathbb{R} \text{ and } \sum_{s\in 2^n} t_s^2 \le 1 \Big\},$$
$$D_n = \Big\{ \sum_{s\in 2^n} q_s \chi_{V_s} : q_s \in \mathbb{Q} \text{ and } \sum_{s\in 2^n} q_s^2 \le 1 \Big\},$$

where $\mathbb{Q}$ is the set of rational numbers. One can easily verify that $X = B \cup \bigcup_{n\in\omega} C_n$ is a compact subspace of $B_1(2^\omega)$ and $D = \bigcup_{n\in\omega} D_n$ is a countable dense subset of X. ∎

Now, we proceed with the construction of the compactum K of Example 2.1. Let

$$\mathcal{A} = \bigcup_{n\in\omega} \bigcup_{s\in 2^n} [V_s]^{\le n} \quad \text{and} \quad K = \{\chi_A : A \in \mathcal{A}\} \subseteq \{0,1\}^{2^\omega}.$$

Then K is a compact subset of $c_0(2^\omega)$. We will prove that K is not a uniform Eberlein compactum. By a result of Argyros and Farmaki [AF, Theorem 1.7, Corollary 1.9] (see also [LS, Theorem 4.9]) it is enough to show that for every partition $2^\omega = \bigcup_{i\in\omega} T_i$ there exists an $i \in \omega$ such that T_i contains elements of $\mathcal{A}$ of arbitrarily large finite cardinality. From the Baire Category Theorem it follows that there is an $i \in \omega$ such that T_i is dense in V_s for some $s \in 2^k$, $k \in \omega$. For every $n \in \omega$, $n \ge k$ and $t \in 2^n$ such that $t|k = s$ the set $V_s \cap T_i$ is infinite. Obviously, every $A \subseteq V_s \cap T_i$ of cardinality n belongs to $\mathcal{A}$.

Now, we will define the required separable Rosenthal compactum L containing K. Let

$$\mathcal{B} = \bigcup_{n\in\omega} \bigcup_{s\in 2^n} \bigcup_{k>n} \{V_{t_1} \cup \ldots \cup V_{t_n} : \forall (i \le n)[t_i \in 2^k \text{ and } t_i|n = s]\} \quad \text{and}$$
$$D = \{\chi_B : B \in \mathcal{B}\} \subseteq \{0,1\}^{2^\omega}.$$

One can easily check that the space $L = K \cup D \subsetneq \{0,1\}^{2^\omega} \cap B_1(2^\omega)$ is compact and D is a countable dense subset of L consisting of continuous functions on 2^ω.

3. Construction of the Banach space E. In this section we will use the Rosenthal compactum L from Example 2.1 and the well-known factorization technique of Davis, Figiel, Johnson and Pełczyński [DFJP] to construct a Banach space E with the properties required in Example 1.1. Part of our reasoning follows closely an argument given in [Ma, Remark 6.2] but for the reader's convenience we will include it here.

For a Banach space X we denote by B_X the closed unit ball of X. The weak and weak* topologies in X and X^* are denoted by w and w^*, respectively.

Let K, D and L be the spaces constructed in Section 2. Let $-D = \{-f : f \in D\} \subseteq C(2^\omega)$ and let W be the convex hull of the union $D \cup -D$ in the space $C(2^\omega)$ equipped with the standard supremum norm. Let $\|\cdot\|_n$ be the Minkowski gauge of $2^n W + 2^{-n} B_{C(2^\omega)}$ in $C(2^\omega)$, for $n = 1, 2, \ldots$ For every function $f \in C(2^\omega)$ we define

$$|||f||| = \Big(\sum_{n=1}^{\infty} \|f\|_n^2\Big)^{1/2}.$$

The space $F = \{f \in C(2^\omega) : |||f||| < \infty\}$ endowed with the norm $|||\cdot|||$ is a separable Banach space (see [DFJP, Lemma 1]). We define $E = F^{**}$.

First, we will show that the ball B_E equipped with the weak* topology is a separable Rosenthal compactum. Therefore every weakly compact subset X of E is bisequential. Indeed, without loss of generality we may assume that $X \subseteq B_E$. By the weak compactness of X, the weak and weak* topologies coincide on it. By Pol's result 1.2, (B_E, w^*) is bisequential; hence so is X.

Let $B_1^*(2^\omega)$ be the subspace of $B_1(2^\omega)$ consisting of bounded functions and let $i : B_1^*(2^\omega) \to C(2^\omega)^{**}$ be the linear embedding defined by

$$i(f)(\mu) = \int f\,d\mu \quad \text{for } f \in B_1^*(2^\omega),\ \mu \in C(2^\omega)^*.$$

Obviously, $i|C(2^\omega)$ is the canonical embedding of $C(2^\omega)$ in $C(2^\omega)^{**}$. We will show that $i(W)$ is relatively sequentially compact in $(C(2^\omega)^{**}, w^*)$. Let M be the closure of W in $\mathbb{R}^{2^\omega}$. We have $W \subseteq \mathrm{conv}(-L \cup L)$; therefore [BFT, Theorem 5E] yields that M is a compact subset of $B_1^*(2^\omega)$. From the fact that M is a Fréchet space (see Sec. 1) and from the Lebesgue Dominated Convergence Theorem it follows that the map i embeds M homeomorphically in $(C(2^\omega)^{**}, w^*)$. Since every sequence of points of W has a subsequence convergent in M, it follows that $i(W)$ is relatively sequentially compact in $i(M) \subseteq C(2^\omega)^{**}$. Now, by [DFJP, Lemma 1(xii)] the canonical embedding $F \to F^{**} = E$ maps the ball B_F onto a relatively sequentially compact subset of (E, w^*). From [Ne, Theorem 1.19] it follows that F does not contain an isomorphic copy of l_1 and a theorem of Odell and Rosenthal [Ne, Theorem 1.17] yields that (B_E, w^*) is a separable Rosenthal compactum.

Second, we will verify that the space (E, w) contains a copy of the nonuniform Eberlein compactum K.

Let $j : F \to C(2^\omega)$ be the inclusion. By [DFJP, Lemma 1(iii)] the map $j^{**} : F^{**} = E \to C(2^\omega)^{**}$ is injective. Using the same argument one can prove that $j^{****} : E^{**} \to C(2^\omega)^{****}$ is also one-to-one. We will use the following simple fact (cf. [DFJP, Lemma 1(vii)]):

3.1. Lemma. *Let X and Y be Banach spaces and $T : X \to Y$ be a continuous linear map such that the second conjugate map $T^{**} : X^{**} \to Y^{**}$ is injective. Then $T|B_X$ is a homeomorphic embedding of (B_X, w) in (Y, w).*

Proof. The unit ball $B_{X^{**}}$ is compact in the weak* topology; therefore T^{**} embeds homeomorphically $(B_{X^{**}}, w^*)$ in (Y^{**}, w^*). One can easily obtain the desired conclusion using the canonical embeddings of X and Y in X^{**} and Y^{**}, respectively. ■

By the above lemma it is enough to check that $(j^{**}(B_E), w)$ contains a copy of K. The set $j^{**}(B_E)$ is the weak* closure of $i(B_F)$ in $C(2^\omega)^{**}$ (cf. [DFJP, Lemma 1(iv)]. One can easily compute that $|||f||| < 1$ for $f \in W$, hence $W \subseteq B_F$. It follows that $j^{**}(B_E)$ contains $i(M)$, a copy of M, and therefore also the corresponding copies of $K, L \subseteq M$. It remains to show that $i|K$ is a homeomorphic embedding in $(C(2^\omega)^{**}, w)$. We have $K \subseteq c_0(2^\omega)$. Observe that $i|c_0(2^\omega)$ is an isometry of $c_0(2^\omega)$ equipped with the supremum norm into $C(2^\omega)^{**}$. Hence $(i(K), w)$ can be identified with K considered as a subset of $(c_0(2^\omega), w)$. It is standard that the weak and pointwise topologies coincide on norm bounded sets in $c_0(\Gamma)$.

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On vector spaces and algebras with maximal locally pseudoconvex topologies

by

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Abstract. Let X be a real or complex vector space. We show that the maximal p-convex topology makes X a complete Hausdorff topological vector space. If X has an uncountable dimension, then different p give different topologies. However, if the dimension of X is at most countable, then all these topologies coincide. This leads to an example of a complete locally pseudoconvex space X that is not locally convex, but all of whose separable subspaces are locally convex. We apply these results to topological algebras, considering the problem of uniqueness of a complete topology for semitopological algebras and giving an example of a complete locally convex commutative semitopological algebra without multiplicative linear functionals, but with every separable subalgebra having a total family of such functionals.

Let X be a real or complex vector space. A *p-homogeneous seminorm* on X $(0 < p \le 1)$ is a non-negative function $x \to \|x\|$, $x \in X$, such that

(i) $\|0\| = 0$,
(ii) $\|x + y\| \le \|x\| + \|y\|$ for all $x, y \in X$, and
(iii) $\|\lambda x\| = |\lambda|^p \|x\|$ for all x in X and all scalars λ.

The inequality $(u + v)^p \le u^p + v^p$, $0 < p \le 1$, $u, v \ge 0$, implies that if $\|x\|$ is a p-homogeneous seminorm on X, and $0 < r \le 1$, then $\|x\|^r$ is a pr-homogeneous seminorm on X.

A topological vector space X is said to be *locally pseudoconvex* if its topology is given by means of a family $(\|\cdot\|_\alpha)$ of $p(\alpha)$-homogeneous seminorms, $0 < p(\alpha) \le 1$. For more details on locally pseudoconvex spaces the reader is referred to [2] and [4].

Let X be a vector space and $0 < p \le 1$. The *maximal locally p-convex topology* $\tau^p_{\max}$ on X is the topology given by means of all p-homogeneous seminorms. It is a Hausdorff vector space topology. For $p = 1$ it is the maximal locally convex topology on X. In this case we denote it by $\tau^{\mathrm{LC}}_{\max}$

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