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Volume approximation of convex bodies by polytopes—a constructive method

by

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Abstract. Algorithms are given for constructing a polytope P with n vertices (facets), contained in (or containing) a given convex body K in $\mathbb{R}^d$, so that the ratio of the volumes $|K \setminus P|/|K|$ (or $|P \setminus K|/|K|$) is smaller than $f(d)/n^{2/(d-1)}$.

1. Introduction. This paper deals with constructive approximation of general convex bodies by polytopes, in the volume-difference sense. Specifically, given a convex body (compact, convex set with non-empty interior) K in $\mathbb{R}^d$, we intend to construct a polytope P contained in K (or containing K) so that the quotient of volumes

$$\frac{|K \setminus P|}{|K|} \quad \left(\text{or } \frac{|P \setminus K|}{|K|}\right) \tag{1.1}$$

will be small. (The notation $|A|$ for a measurable subset A of $\mathbb{R}^d$ is used here to denote the k-dimensional volume of A, where k is the dimension of the minimal flat containing A.)

There exists a large body of results concerning approximation of convex bodies by polytopes. We refer the reader to the surveys [5] and [6] by Gruber for information on this subject.

It was proved by Bronshteĭn and Ivanov [2] (cf. also results by Dudley [3] and Betke and Wills [1]) that for any convex body K contained in the Euclidean unit ball B_2^d of $\mathbb{R}^d$ and every sufficiently large positive integer n, there exists a polytope Q, containing K, with at most n vertices, whose distance from K in the Hausdorff metric is less than $c/n^{2/(d-1)}$, where c is an absolute constant. It is easy to check that the proof in [2] provides also

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a polytope P contained in K with the same properties. This easily implies the existence of polytopes P, Q with $P \subset B_2^d \subset Q$, each of them with at most n vertices, such that

$$\frac{|B_2^d \setminus P|}{|B_2^d|} \le C \frac{d}{n^{2/(d-1)}}, \tag{1.2}$$

$$\frac{|Q \setminus B_2^d|}{|B_2^d|} \le C \frac{d}{n^{2/(d-1)}} \tag{1.3}$$

where C is an absolute constant.

A theorem of Macbeath [9] asserts that for a convex body K and for given d and n, $\inf(|K \setminus P_n|/|K|)$ is largest for $K = B_2^d$, where P_n ranges over all polytopes with at most n vertices contained in K. From this it follows that in (1.2), B_2^d can be replaced by any convex body K. On the other hand, an application of the well-known fact that for some ellipsoid E we have $E \subset K \subset dE$, together with the result of [2], enables one to replace B_2^d in (1.3) by any convex body K, with the bigger upper bound $cd^2/n^{2/(d-1)}$.

The method employed by [2] is to find an ε-net on the unit sphere in $\mathbb{R}^d$. Other, probabilistic, methods give the same estimate (cf. [10]).

In this paper we bring a constructive method which enables one to construct polytopes contained in and containing a convex body K in $\mathbb{R}^d$, using a well-defined algorithm in some of the cases, and "almost" well-defined in others. These polytopes give the degree of approximation of K, in the sense of the quotient (1.1), of the same order as (1.2) and (1.3) for the Euclidean ball and for other particular convex bodies. For general convex bodies the method works only for inscribed polytopes and gives the upper bound $cd^3/n^{2/(d-1)}$.

It is worth mentioning that the estimate of the form $f(d)/n^{2/(d-1)}$ is best possible (for the power of n involved) in general. Moreover, recently Gruber [8] proved, for K smooth enough, the asymptotic behaviour in n of $|K \setminus P|$ and $|Q \setminus K|$, where $P \subset K \subset Q$ are the best volume-difference approximating polytopes, P with at most n vertices and Q with at most n facets. The power of n in these estimates is, of course, $-2/(d-1)$. The exact order of growth of $f(d)$ seems to be unknown yet.

In [7] and [8] Gruber gave another algorithm to construct a volume-difference approximating polytope, contained in a convex body of class C^2 with positive curvature; Gruber's algorithm gives the right order of approximation in terms of the number of vertices. It depends on knowing the surface structure of the body.

The notations used in this paper are quite standard. We mention some of them in particular. We have introduced above the notation $|\cdot|$ for volume. The kth standard unit vector in $\mathbb{R}^d$ is denoted by e_k. The Euclidean unit ball in $\mathbb{R}^d$ is B_2^d and more generally, for $1 \le q < \infty$, B_q^d is the unit ball of the normed space ℓ_q^d:

$$B_q^d = \Big\{x \in \mathbb{R}^d : \|x\|_q = \Big(\sum_{i=1}^d |x_i|^q\Big)^{1/q} \le 1\Big\}. \tag{1.4}$$

We set $\chi_d = |B_2^d|$. If K is a convex body in $\mathbb{R}^d$ and $t \in \mathbb{R}$ then

$$K(t) = \{x = (x_1, \ldots, x_d) \in K : x_d = t\} \tag{1.5}$$

is a $(d-1)$-dimensional (if with non-empty relative interior) section of K, perpendicular to the x_d-axis. By a "polytope" we always mean a convex polytope.

Finally, symbols like C, c, c_1, c_2 etc. always denote absolute constants. The same symbol may denote different constants in different paragraphs or even in different parts of the same paragraph. No effort to find the best possible constants or even "good" constants was made here, although the constants obtained by the method are relatively small.

2. Volume approximation of the Euclidean ball. In this section we give a constructive algorithm to produce a convex polytope $P \subset B_2^d$ which has at most n vertices and satisfies

$$\frac{|B_2^d \setminus P|}{|B_2^d|} \le C \frac{d}{n^{2/(d-1)}} \tag{2.1}$$

where C is a constant independent of d. A special property of the polytope thus constructed is its high degree of regularity. The proof of Theorem 2.2 contains the construction algorithm. We shall need the following proposition which may be of some interest for its own sake.

PROPOSITION 2.1. *For every $p \ge 1$ there exists an integer $n_0 = n_0(p)$ such that for every integer $n \ge n_0$ there exists a piecewise linear function g on $[0,1]$, with m nodes, $m \le 2n$, located on the circular arc $y = \sqrt{1-x^2}$, so that*

$$\int_0^1 \{(1-x^2)^{p/2} - g(x)^p\}\, dx \le \frac{c}{n^2} \int_0^1 (1-x^2)^{p/2}\, dx \tag{2.2}$$

where c is a constant independent of p.

Proof. The case $p = 1$ is trivially done by the regular division of the circle. If g is a decreasing function on $[0,1]$ with $g(0) = 1$, $g(1) = 0$, then

$$\int_0^1 \{(1-x^2)^{p/2} - g(x)^p\}\, dx = p \int_0^1 t^{p-1}\{\sqrt{1-t^2} - g^{-1}(t)\}\, dt.$$

Moreover, g^{-1} is piecewise linear with m nodes on the circular arc if and only if so is g. As $\int_0^1 (1-x^2)^{p/2}\,dx \sim c_1/\sqrt{p}$, proving the proposition amounts to showing the existence of a piecewise linear h with m nodes on the circular arc such that

$$(2.3)\qquad p^{3/2}\int_0^1 t^{p-1}\{\sqrt{1-t^2}-h(t)\}\,dt \le \frac{c}{n^2}.$$

Let k be an integer whose feasible values will be determined later. Let $0 = q_0 < q_1 < q_2 < \ldots < q_k < q_{k+1} = p$ be an increasing sequence with members to be determined. Fix the points $Q_j = (t_j, \sqrt{1-t_j^2})$, $j = 0,\ldots,k+1$, on the circular arc by $t_j = 1 - q_j/p$.

We divide each circular arc between Q_j and Q_{j+1} into l_j equal circular arcs, with $l_j = [n/e^j] + 1$. This gives a partition of the positive quadrant of the circle into $m = l_0 + \ldots + l_k$ circular arcs, the end-points of these arcs will be the nodes of the function h.

Let $\varphi_j = \arccos t_j$. By estimating the area of l_j equal circular sectors we get

$$\int_{t_{j+1}}^{t_j} t^{p-1}\{\sqrt{1-t^2}-h(t)\}\,dt \le t_j^{p-1}\int_{t_{j+1}}^{t_j}\{\sqrt{1-t^2}-h(t)\}\,dt$$
$$\le \frac{t_j^{p-1}(\varphi_{j+1}-\varphi_j)^3}{12l_j^2} \le \frac{t_j^{p-1}\varphi_{j+1}^3}{12l_j^2}.$$

We have

$$t_j^{p-1} = \left(1-\frac{q_j}{p}\right)^{p-1} \le e^{1-q_j} \quad\text{and}\quad \varphi_j^3 = \arccos^3\left(1-\frac{q_j}{p}\right) \le c_2\left(\frac{q_j}{p}\right)^{3/2}$$

so, taking into account the values of l_j, we get

$$(2.4)\qquad p^{3/2}\int_0^1 t^{p-1}\{\sqrt{1-t^2}-h(t)\}\,dt$$
$$\le \frac{c_3}{n^2}(q_1^{3/2} + q_2^{3/2}e^{2-q_1} + \ldots + q_k^{3/2}e^{2(k-1)-q_{k-1}} + p^{3/2}e^{2k-q_k}).$$

We now define q_k by

$$(2.5)\qquad q_k = 3k + \tfrac{3}{2}\log p,$$

thus $p^{3/2}e^{3k-q_k} = 1$, we assume that $q_k < p$ and to define q_{k-1} we require that $q_k^{3/2}e^{3(k-1)-q_{k-1}} = 1$, which together with (2.5) gives

$$(2.6)\qquad q_{k-1} = 3(k-1) + \tfrac{3}{2}\log\left(3k + \tfrac{3}{2}\log p\right).$$

The assumption $q_k < p$ implies that $q_{k-1} < q_k$.

This procedure produces a decreasing sequence (q_j) which satisfies

$$(2.7)\qquad q_j^{3/2}e^{3(j-1)-q_{j-1}} = 1 \quad \text{for } 2 \le j \le k$$

and

$$(2.8)\qquad q_1 = 3 + \tfrac{3}{2}\log\left(3\cdot 2 + \tfrac{3}{2}\log\left(3\cdot 3 + \ldots + \left(3(k-1) + \tfrac{3}{2}\log\left(3k + \tfrac{3}{2}\log p\right)\right)\ldots\right)\right).$$

It is standard to see that there exists a constant L (< 7), independent of p, such that q_1 which is obtained by (2.8) satisfies $q_1 \le L$ provided that $k \ge k_0$. The value of k_0 is of the order of $\Phi(\log p)$ where

$$\Phi(x) = \min\{j \in \mathbb{N} : \log^{(j)}(x) \le 1\}$$

($\log^{(j)}$ is the jth iterated logarithm). We may assume that p is such that

$$(2.9)\qquad k_0 \le \tfrac{1}{3}p - \tfrac{1}{2}\log p$$

(see the next Remark). Taking $k = k_0$ in the construction of h, we get, by (2.7), (2.8) and the remark following it,

$$p^{3/2}\int_0^1 t^{p-1}\{\sqrt{1-t^2}-h(t)\}\,dt \le \frac{c_3(L^{3/2}+(e-1)^{-1})}{n^2}.$$

The number of nodes of h is

$$m = 1 + \sum_{j=0}^{k} l_j \le n(1+(e-1)^{-1}) + k + 1$$

so if $n \ge n_0(p) = (1+k_0)/(1-(e-1)^{-1})$ we get $m \le 2n$. ∎

Remarks. a) If p is small, (2.9) may not hold. But by applying the method used above, with an appropriate division of $[0,1]$ into two parts only, and then subdividing each part into n subintervals associated with equal angles, we get

$$p^{3/2}\int_0^1 t^{p-1}\{\sqrt{1-t^2}-h(t)\}\,dt \le \frac{c(\log p)^{3/2}}{n^2}.$$

As this is needed only for a finite number of values p, Proposition 2.1 remains true.

b) The estimate c/n^2 in (2.2) is the best possible. There is a positive constant c such that if g is a piecewise linear function with n nodes on the curve $y = \sqrt{1-x^2}$, $0 \le x \le 1$, then

$$\int_0^1 \{(1-x^2)^{p/2} - g(x)^p\}\,dx \ge \frac{c}{n^2}\int_0^1 (1-x^2)^{p/2}\,dx.$$

The argument is as follows: Let $h = g^{-1}$ and assume that h has n nodes of the form $(t, \sqrt{1-t^2})$, of which m satisfy $1-1/p \le t \le 1$. Since the best approximation is given by the regular division of the circular arc, it follows that

$$\int_{1-1/p}^{1} \{\sqrt{1-t^2} - h(t)\}\, dt \ge \frac{c_1}{m^2} \arccos^3(1-1/p)$$

for some positive constant c_1. Hence

$$p^{3/2} \int_{1-1/p}^{1} t^{p-1}\{\sqrt{1-t^2} - h(t)\}\, dt \ge \frac{p^{3/2} c_2 (1-1/p)^{p-1} p^{-3/2}}{m^2} \ge \frac{c}{n^2}.$$

It is explained in the introduction how the estimate of the following theorem on the degree of approximation of B_2^d by inscribed polytopes can be deduced from the result of [2]. Our proof, however, defines a constructive algorithm which produces the approximating polytope.

THEOREM 2.2. *There exists a constant C such that for every positive integer d and every integer $n \ge n_1(d)$ it is possible to construct a polytope P_n, contained in the Euclidean unit ball B_2^d of $\mathbb{R}^d$, which has at most n vertices and satisfies*

$$\frac{|B_2^d \setminus P_n|}{|B_2^d|} \le C \frac{d}{n^{2/(d-1)}}. \tag{2.10}$$

Moreover, if n is large enough with respect to d, then the polytope P_n can be constructed to have at most n facets ($=(d-1)$-dimensional faces).

Proof. For $\varepsilon > 0$ and a positive integer m, let $\lambda_m(\varepsilon)$ be an integer, as small as possible, such that we can construct an m-dimensional polytope $P \subset B_2^m$ with at most $\lambda_m(\varepsilon)$ vertices satisfying $|B_2^m \setminus P| \le \varepsilon |B_2^m|$. By Proposition 2.1, for a given positive integer k we can find points $t_0 = -1 < t_1 < \ldots < t_k = 1$ so that if g is the piecewise linear function on $[-1, 1]$ with nodes $(t_i, \sqrt{1-t_i^2})$, then

$$\int_{-1}^{1} \{(1-t^2)^{m/2} - g(t)^m\}\, dt \le \frac{c_1}{k^2} \int_{-1}^{1} (1-t^2)^{m/2}\, dt. \tag{2.11}$$

For $1 \le i \le k-1$, $B_2^{m+1}(t_i)$ is a Euclidean ball of radius $g(t_i) = \sqrt{1-t_i^2}$. Given $\delta > 0$ assume that we have constructed polytopes $P_i \subset B_2^{m+1}(t_i)$, $1 \le i \le k-1$, with at most $\lambda_m(\delta)$ vertices, which satisfy

$$|P_i| \ge (1-\delta)|B_2^{m+1}(t_i)| = (1-\delta) g(t_i)^m \chi_m. \tag{2.12}$$

Let P be the $(m+1)$-dimensional polytope defined by

$$P = \operatorname{conv}\Big(\{-e_{m+1}\} \cup \{e_{m+1}\} \cup \bigcup_{i=1}^{k-1} P_i\Big).$$

Clearly, P is a polytope contained in B_2^{m+1} and for $t_i \le t \le t_{i+1}$ we have, by (2.12) and the Brunn–Minkowski theorem,

$$|P(t)| \ge (1-\delta)\chi_m g(t)^m, \tag{2.13}$$

hence

$$\begin{aligned} |B_{m+1} \setminus P| &\le \chi_m \int_{-1}^{1} \{(1-t^2)^{m/2} - (1-\delta) g(t)^m\}\, dt \\ &\le \Big(\frac{c_1}{k^2} + \delta\Big) \chi_m \int_{-1}^{1} (1-t^2)^{m/2}\, dt = \Big(\frac{c_1}{k^2} + \delta\Big) |B_2^{m+1}|. \end{aligned} \tag{2.14}$$

Given $\varepsilon > 0$ let $k = [\sqrt{c_1/\varepsilon}] + 1$. The above polytope P, constructed using this k, has at most $\lambda_m(\delta) k$ vertices. By (2.14) we have $|B_2^{m+1} \setminus P| \le (\varepsilon + \delta)|B_2^{m+1}|$, hence

$$\lambda_{m+1}(\varepsilon + \delta) \le 2\sqrt{\frac{c_1}{\varepsilon}} \lambda_m(\delta) = \frac{c_2}{\sqrt{\varepsilon}} \lambda_m(\delta). \tag{2.15}$$

Let $\eta > 0$ be given. By (2.15) we have

$$\begin{aligned} \lambda_d(\eta) &\le c_2 \Big(\frac{\eta}{d-1}\Big)^{-1/2} \lambda_{d-1}((d-2)\eta/(d-1)) \\ &< c_2^2 \Big(\frac{\eta^2}{(d-1)^2}\Big)^{-1/2} \lambda_{d-2}((d-3)\eta/(d-1)) < \ldots \end{aligned}$$

Using the fact that $\lambda_2(\eta/(d-1)) \le c_1(\eta/(d-1))^{-1/2}$ we get

$$\lambda_d(\eta) \le \frac{c_2^{d-1}(d-1)^{(d-1)/2}}{\eta^{(d-1)/2}}.$$

Set $n = \lambda_d(\eta)$. We have constructed a polytope P_n with at most n vertices such that $|B_2^d \setminus P_n| \le \eta |B_2^d|$ and

$$\eta \le \frac{c_2^2(d-1)}{n^{2/(d-1)}}.$$

This completes the main part of the proof.

For the "moreover" part, assume that n is large enough compared with d, which means that η is small enough. At each step of the construction, taking

$$P = \operatorname{conv}\Big(\bigcup_{i=1}^{k-1} P_i\Big)$$

and neglecting the points $\pm e_{m+1}$ will not change the estimates of the volume difference or the number of vertices. Now, at each step we can have all the sectional polytopes P_i homothetic. Then the number of facets of the $(m+1)$-dimensional polytope P is $\lambda_m(\delta)(k-1)+2$; this is less than the estimate $\lambda_m(\delta)k$ which was obtained for the number of vertices. As the numbers of vertices and edges in the 2-dimensional sections are identical, it follows that the number of facets of P_n is less than the number of its vertices.

Remarks. a) A word is in order concerning the size of $n_1(d)$. In the above process, for $\eta = cd/n^{2/(d-1)}$ we chose ε in the recursive process to be $\varepsilon = \eta/(d-1) \approx c/n^{2/(d-1)}$, and the integer k, for which we applied Proposition 2.1 at each step, was chosen to be $k \approx c/\sqrt{\varepsilon} \approx cn^{1/(d-1)}$. But k, if used in Proposition 2.1, must be at least $n_0(m)$, so we get the requirement

$$n \geq n_0(d)^d \tag{2.16}$$

where $n_0(d)$ is, by the proof of Proposition 2.1, of order $\Phi(\log d)$, that is "almost" a constant. Now, clearly, the size of the estimate $cd/n^{2/(d-1)}$ makes values of n smaller than the estimate (2.16) meaningless for large d.

b) It seems to be unknown at the present moment whether the dependence on the dimension of the right hand side of (2.1) can be improved.

3. Volume approximation of a general convex body. If K is a convex body in $\mathbb{R}^d$, a polytope P with at most n vertices can be inscribed in K and satisfy the same volume-difference estimate (2.1), with B_2^d replaced by K. The constant c there is independent of the body K. This result follows from a result of Macbeath [9], according to which the value of

$$Q(K) = \min_{P_n \subset K} \frac{|K \setminus P_n|}{|K|}$$

is maximal for $K = B_2^d$ (where P_n ranges over all polytopes inscribed in K with at most n vertices). However, this estimate, which is based on successive Steiner symmetrizations which increase $Q(K)$, does not give any algorithm to construct the approximating polytope. In dimension 2 this result (or rather a more accurate upper bound to $Q(K)$) is known as the Sas theorem [11]. It is proved by a method which is "partially constructive" (cf. [4], p. 36).

We bring here a proof of an estimate of the form (2.1) for a general convex body K. This estimate is again of the form $f(d)/n^{2/(d-1)}$ which is best possible in general (cf. [5], [6] and [8]). Our method gives $cd^3/n^{2/(d-1)}$ instead of $cd/n^{2/(d-1)}$. But it provides a "partial algorithm" in the sense that at each of $d-1$ steps we need to approximate only 2-dimensional convex figures (concave functions) by polygons (piecewise linear functions) and the rest is a well-defined procedure, provided that the volumes of sections of K of different dimensions are known.

LEMMA 3.1. *Let f be a non-negative, decreasing concave function on the interval $[a,b]$. For every $p \geq 1$ and positive integer k, there exists a piecewise linear function g on $[a,b]$, with at most k nodes, all of them on the graph of f, which satisfies*

$$\int_a^b \{f(x)^p - g(x)^p\}\,dx \leq \frac{cp^2}{k^2} \int_a^b f(x)^p\,dx \tag{3.1}$$

where c is a constant independent of f or p.

Proof. By changing scale we may assume that $[a,b] = [0,1]$ and $f(0) = 1$. We clearly have

$$f(x)^p - g(x)^p \leq pf(0)\{f(x)-g(x)\} = p\{f(x)-g(x)\}. \tag{3.2}$$

Using the Sas theorem we can find a piecewise linear g with at most k nodes on the graph of f such that

$$\int_0^1 \{f(x)-g(x)\}\,dx \leq \frac{c_1}{k^2}\int_0^1 f(x)\,dx. \tag{3.3}$$

This g satisfies (3.1). In order to show that, we only need, by (3.2) and (3.3), to compare $\int_0^1 f(x)\,dx$ with $\int_0^1 f(x)^p\,dx$. Now, since f is concave and $0 \leq f \leq f(0) = 1$, we have $f(x) \geq 1-x$ on $[0,1]$. So

$$\int_0^1 f(x)^p\,dx \geq \frac{1}{p+1} \geq \frac{1}{p+1}\int_0^1 f(x)\,dx$$

(with a little more effort one can obtain $\int_0^1 f(x)\,dx \leq ((p+1)/2)\int_0^1 f(x)^p\,dx$, which is the best estimate for this comparison). ■

THEOREM 3.2. *There exists a constant C such that for every positive integer d and an integer $n \geq 2d$ and for every convex body K in $\mathbb{R}^d$, it is possible to construct a polytope P_n contained in K, having at most n vertices, which satisfies*

$$\frac{|K \setminus P_n|}{|K|} \leq C\frac{d^3}{n^{2/(d-1)}}. \tag{3.4}$$

Remark. The meaning of "possible to construct" was explained at the beginning of this section.

Proof of Theorem 3.2. For $\varepsilon > 0$ and m a positive integer, let $\lambda_m(\varepsilon)$ be an integer, as small as possible, such that for every m-dimensional convex body K it is possible to construct a polytope $P \subset K$ with at most $\lambda_m(\varepsilon)$ vertices satisfying $|K \setminus P| < \varepsilon|K|$. Let K be an $(m+1)$-dimensional convex body. Define f on the interval $[a,b]$, $a = \min\{t : K(t) \neq \emptyset\}$, $b = \max\{t : K(t) \neq \emptyset\}$, by $f(t) = |K(t)|^{1/m}$. By the Brunn–Minkowski theorem, f is concave. Given a positive integer k we define, using Lemma 3.1, points $t_0 = a < t_1 < \ldots < t_k = b$ such that f and the piecewise linear function g on $[a,b]$, whose nodes are $(t_i, f(t_i))$, satisfy (3.1).

In each of the m-dimensional convex bodies $K(t_i)$ we inscribe a polytope P_i with $|K(t_i) \setminus P_i| < \delta|K(t_i)|$ and with at most $\lambda_m(\delta)$ vertices. An argument identical to the one in the proof of Theorem 2.2 now shows that $P = \mathrm{conv}(\bigcup_{i=0}^k P_i)$ is contained in K and

$$|K \setminus P| \le \left(\frac{cm^2}{k^2} + \delta\right)|K|.$$

This gives

$$\lambda_{m+1}(\varepsilon + \delta) \le 2m\sqrt{\frac{c}{\varepsilon}}\lambda_m(\delta) = \frac{c_1 m}{\sqrt{\varepsilon}}\lambda_m(\delta). \tag{3.5}$$

As before, (3.5) implies

$$\lambda_d(\eta) \le \frac{c_2^{d-1}(d-1)^{(d-1)/2}(d-1)!}{\eta^{(d-1)/2}}$$

and the substitution $\lambda_d(\eta) = n$ gives

$$\eta \le \frac{c_3(d-1)^3}{n^{2/(d-1)}}. \ \blacksquare$$

4. Volume approximation of the unit ball of ℓ_q^d $(1 < q < \infty)$.

The method of approximation in the volume-difference sense, of a general convex body K, which was described in Section 3, can be improved if K is given explicitly and has some nice properties. First, the algorithm may become completely defined. Secondly, the dependence on the dimension of the estimate may be improved and the polytope obtained by this procedure may become more regular if the body K has some regularity properties. In particular, if at each step of the process, the approximating polytopes in the parallel sections are all homothetic, then the final approximating polytope can be made to have not only at most n vertices, but also at most n facets.

In this section we treat as an example the unit balls of the normed spaces ℓ_q^d $(1 < q < \infty)$, i.e. the convex bodies B_q^d. Here all the improvements mentioned above are obtained. In the discussion which follows we fix $1 < q < \infty$. The next lemma is elementary and we omit its proof.

LEMMA 4.1. *For $0 < \alpha < 1$ let D_α be the triangle in $\mathbb{R}^2$ bounded by the lines:* (i) $x = 1$, (ii) *the line through the points* $(1,0)$ *and* $(1-\alpha, (1-(1-\alpha)^q)^{1/q})$, (iii) *the tangent line of the curve* $y = (1-x^q)^{1/q}$ *through the point* $(1-\alpha, (1-(1-\alpha)^q)^{1/q})$. *Then* $|D_\alpha| \le c\alpha^{(q+1)/q}$, *where c is a constant independent of q.*

The following lemma is an extension of Proposition 2.1. We only sketch its proof which is basically the same as the proof of Proposition 2.1, but not as smooth.

LEMMA 4.2. *For every $p \ge 1$ there exists an integer $n_0 = n_0(p)$ such that for every integer $n \ge n_0$ there exists a piecewise linear function g on $[0,1]$, with m nodes, $m \le 2n$, located on the arc $y = (1-x^q)^{1/q}$, so that*

$$\int_0^1 \{(1-x^q)^{p/q} - g(x)^p\}\,dx \le \frac{c}{n^2}\int_0^1 (1-x^q)^{p/q}\,dx \tag{4.1}$$

where c and $n_0(p)$ are independent of q.

Proof. Using the same transformation as in Proposition 2.1 and the fact that

$$\int_0^1 (1-x^q)^{1/q}\,dx \sim cp^{-1/q}$$

we only have to show that

$$p^{(q+1)/q}\int_0^1 t^{p-1}\{(1-t^q)^{1/q} - h(t)\}\,dt \le \frac{c}{n^2} \tag{4.2}$$

for some piecewise linear $h(t)$ with nodes on

$$\Gamma_q = \{(t, (1-t^q)^{1/q}) : 0 \le t \le 1\}.$$

Let an integer k, numbers $0 < q_1 < q_2 < \ldots < q_k < p$, Q_j, t_j and l_j be as in the proof of Proposition 2.1 (to be determined). For $j = 0, 1, \ldots, k$ let E_j be the domain in $\mathbb{R}^2$ which is bounded by Γ_q and the line through the points $(t_{j+1}, (1-t_{j+1}^q)^{1/q})$ and $(t_j, (1-t_j^q)^{1/q})$. Using the Sas theorem we divide the segment of Γ_q with $t_{j+1} < t < t_j$ into l_j subsegments so that if h is the piecewise linear function whose nodes are the end-points of all the subintervals thus constructed, we have

$$\int_{t_{j+1}}^{t_j} \{(1-x^q)^{1/q} - h(t)\}\,dt \le \frac{c_1}{l_j^2}|E_j|. \tag{4.3}$$

Clearly, $E_j \subset D_{t_{j+1}}$ (cf. Lemma 4.1), so we get

$$(4.4)\qquad \int_{t_{j+1}}^{t_j} \{(1-x^q)^{1/q} - h(t)\}\,dt \le \frac{c_1}{l_j^2}\left(\frac{q_j}{p}\right)^{(q+1)/q}.$$

The rest of the proof is exactly the same as in Proposition 2.1, replacing $3/2$ by $(q+1)/q$, of course; all the constants involved are bounded, independent of q. ∎

We substitute Lemma 4.2 for Proposition 2.1 in the proof of Theorem 2.2 to get:

THEOREM 4.3. *There exists a constant C, independent of q, such that for every positive integer $n \ge n_1(d)$ it is possible to construct a polytope $P_n \subset B_q^d$ with at most n vertices satisfying*

$$(4.5)\qquad \frac{|B_q^d \setminus P_n|}{|B_q^d|} \le C\frac{d}{n^{2/(d-1)}}.$$

Moreover, if n is large enough with respect to d, then the polytope P_n can be constructed to have at most n facets.

5. Volume approximation of a convex body by polytopes containing it. The method developed in the previous sections for constructing a polytope contained in a convex body K, which approximates it in the volume-difference sense, does not work in general if we wish to construct an approximating polytope which contains K. The cases for which the method *does* work are those when there exists an orthogonal coordinate system in $\mathbb{R}^d$ and an enumeration of the coordinate axes, say $x_1,\ldots,x_d$, so that for each $2 \le m \le d-1$ all the m-dimensional sections parallel to the $x_1,\ldots,x_m$ axes are homothetic. Then we get the same estimates. We sketch in this section how the process works in these cases. For demonstration we chose once again the unit balls B_q^d of ℓ_q^d, $1 < q < \infty$ (including the Euclidean ball).

PROPOSITION 5.1. *Let $f_q(x) = (1-x^q)^{1/q}$, $1 < q < \infty$. For every $p \ge 1$ and $n \ge n_0(p)$, there exists a piecewise linear function g on $[0,1]$ with $m \le 2n$ nodes such that all the edges of the graph of g are tangent to the graph of f_q, $g(0) = 1$, $g(1) = 0$ and*

$$(5.1)\qquad \int_0^1 \{g(x)^p - f_q(x)^p\}\,dx \le \frac{c}{n^2}\int_0^1 f_q(x)^p\,dx$$

with a constant c independent of q or p.

Proof. We sketch the proof for $q = 2$ and remark on the modification for $q \ne 2$. If we divide a circular arc $0 < \alpha < \pi/2$ into $n > 1$ equal circular arcs to produce a polygon P whose edges are the tangents of B_2^2 at the end-points of the division arcs and the radii bounding the sector $S(\alpha)$ associated with α, then

$$(5.2)\qquad |P \setminus S(\alpha)| = n\left(\tan\frac{\alpha}{2n} - \frac{\alpha}{2n}\right) \le \frac{c_1\alpha^3}{n^2}.$$

We now divide the interval $[0,1]$ exactly as in Proposition 2.1. We construct a piecewise linear function h with the points $(t_j, f_2(t_j))$ as points of tangency of the edges of the graph of h with the graph of f_2, and use (5.2) to estimate

$$p^{3/2}\int_0^1 t^{p-1}\{h(t) - f_2(t)\}\,dt.$$

In the case $q \ne 2$ we imitate the proof of Lemma 4.2 rather than Proposition 2.1. Instead of the Sas theorem we use here known estimates of area-difference approximation of convex figures by polygons containing them (cf. e.g. [4], p. 39). ∎

THEOREM 5.2. *There exists a constant C such that for every $1 < q < \infty$ and every integer $n \ge n_1(d)$, it is possible to construct a polytope P_n containing B_q^d, with at most n facets, such that*

$$(5.3)\qquad \frac{|P_n \setminus B_q^d|}{|B_q^d|} \le C\frac{d}{n^{2/(d-1)}}.$$

Moreover, if n is large enough with respect to d, then the polytope P_n can be constructed to have at most n vertices.

Proof. We repeat the recursive procedure, with a variation. Assume that we can produce the approximating polytope for B_q^m. The m-dimensional sections of B_q^{m+1} are $B_q^{m+1}(t) = f_q(t)B_q^m$. We choose division points (t_i), $i = 1,\ldots,k$, by Proposition 5.1. Let P be the approximating polytope containing B_q^m, with

$$|P \setminus B_q^m| < \delta|B_q^m|.$$

For $i = 1,\ldots,k$ we approximate $B_q^{m+1}(t_i)$ by $f_q(t_i)P$. Each $(m-1)$-face of $f_q(t_i)P$ supports $B_q^{m+1}(t_i)$ and hence B_q^{m+1} at a single point. We extend each such face to an m-dimensional flat supporting B_q^{m+1}. Let $\widetilde{P}$ be the $(m+1)$-dimensional polytope whose m-dimensional faces are contained in these flats (including the supporting flats at $\pm e_{m+1}$). Clearly, $\widetilde{P}$ contains B_q^{m+1}. If g is the function given in Proposition 5.1 (extended symmetrically to $[-1,1]$), then, by homothety of the sections, the m-dimensional sections of $\widetilde{P}$ are $\widetilde{P}(t) = g(t)P$, so

$$(5.4)\qquad |\widetilde{P} \setminus B_q^{m+1}| = |P|\int_{-1}^1 g(x)^m\,dx - |B_q^m|\int_{-1}^1 f_q(x)^m\,dx$$

$$\leq |B_q^m| \int_{-1}^{1} \{g(x)^m - f_q(x)^m\}\,dx + \delta|\widetilde{P}|$$

$$\leq \left(\frac{c}{k^2} + \delta(1+\delta)\right)|B_q^{m+1}|.$$

With notations analogous to those of the previous proofs, (5.4) is written

$$\lambda_{m+1}(\varepsilon + \delta^2 + \delta) \leq \frac{c}{\sqrt{\varepsilon}}\lambda_m(\delta). \tag{5.5}$$

Let $\eta > 0$ and assume $\eta < (2(d-1))^{-1}$. For $\varepsilon > 0$ we define an increasing sequence (η_j), $j = 1, \ldots, d$, by $\eta_1 = 0$, $\eta_2 = \varepsilon$,

$$\eta_{j+1} = \eta_j^2 + \eta_j + \varepsilon, \quad j = 3, \ldots, d-1. \tag{5.6}$$

It is clear from the construction that there is a unique $\varepsilon > 0$ such that $\eta_d = \eta$. By (5.5), (5.6) and the known estimate for $\lambda_2(\cdot)$ (here, for polygons *containing* 2-dimensional figures), we have, with this ε,

$$\lambda_d(\eta) \leq \frac{c}{\sqrt{\varepsilon}}\lambda_{d-1}(\eta_{d-1}) \leq \ldots \leq \frac{c_1^{d-1}}{\varepsilon^{(d-1)/2}}. \tag{5.7}$$

Also,

$$\eta = \sum_{j=1}^{d-1}(\eta_{j+1} - \eta_j) = \varepsilon + \sum_{j=2}^{d-1}(\eta_j^2 + \varepsilon) < (d-2)\eta^2 + (d-1)\varepsilon.$$

By the restriction on η we get $\varepsilon > \eta/2(d-1)$. Together with (5.7) this gives

$$\lambda_d(\eta) < \frac{c_2^{d-1}(d-1)^{(d-1)/2}}{\eta^{(d-1)/2}}, \tag{5.8}$$

which, as in the proof of Theorem 2.2, proves the theorem. Here the assumption $\eta < (2(d-1))^{-1}$ implies the restriction $n > cd^d$ for some constant c, but smaller values of n are not very meaningful in the estimate (5.3). ■

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