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Received January 8, 1993 (3217)

STUDIA MATHEMATICA 110 (3) (1994)

Concerning entire functions in B_0-algebras

by

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Abstract. We construct a non-m-convex non-commutative B_0-algebra on which all entire functions operate. Our example is also a Q-algebra and a radical algebra. It follows that some results true in the commutative case fail in general.

A B_0*-algebra* (an algebra of type B_0) is a topological algebra whose underlying topological vector space is a completely metrizable locally convex space. The topology of a B_0-algebra A can be given by means of a sequence $(|\cdot|_i)$ of seminorms such that

$$|x|_1 \le |x|_2 \le \ldots \quad \text{for all } x \text{ in } A \tag{1}$$

and

$$|xy|_i < C_i |x|_{i+1} |y|_{i+1} \quad \text{for all } x, y \in A,\ i = 1, 2, \ldots, \tag{2}$$

where C_i are positive constants (one can easily have $C_i = 1$ for all i, but here it is more convenient to have inequalities of the form (2)). A B_0-algebra A is said to be *multiplicatively-convex* (m-convex for short) if the seminorms (1) can be chosen so that instead of (2) we have

$$|xy|_i \le |x|_i |y|_i \quad \text{for all } x, y \in A,\ i = 1, 2, \ldots \tag{3}$$

Note that (1) implies that if $\|\cdot\|$ is a continuous seminorm on a B_0-algebra A, then there is an index m and a positive constant C such that

$$\|x\| \le C|x|_m \quad \text{for all } x \text{ in } A. \tag{4}$$

An element x of an algebra A is said to be *quasi-invertible* if there is an element y in A, called a *quasi-inverse* of x, such that $x \circ y = y \circ x = 0$, where $x \circ y = xy + x + y$. This is equivalent to $(x+e)\cdot(y+e) = (y+e)\cdot(x+e) = e$, if A has a unit element e, or to this relation in the unitization A_1 of A, if there is no unit in A. That means that the quasi-inverse of an element x is uniquely determined by x.

1991 *Mathematics Subject Classification*: 46H05, 46H20, 46H30.

The author is supported by the KBN grant No. 2.2007 92 03.

A topological algebra A is said to be a *Q-algebra* if the set of all its quasi-invertible elements is open. If A has a unit, then A is a Q-algebra if and only if the set of all invertible elements of A is open. Clearly the unitization of a Q-algebra without unit is again a Q-algebra.

One can prove that the complexification of a real Q-algebra is a complex Q-algebra. Also, one can easily see that if for an element x of a topological algebra A the series $\sum_{i=1}^{\infty}(-1)^i x^i$ is convergent, then x is quasi-invertible in A with quasi-inverse $\sum_{i=0}^{\infty}(-1)^i x^i$.

Let $\varphi(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n$ be an entire function of a complex variable ζ. We say that φ *operates* on a complex topological algebra A if the series $\sum_{i=1}^{\infty} a_i x^i$ converges for every x in A. If A has a unit element e, we can start the summation from 0, setting $x^0 = e$ for each x in A. The same definition can be given for a real algebra A, provided all coefficients a_i are real numbers.

If A is a real or complex m-convex B_0-algebra then all entire functions (with real coefficients in case of a real algebra) operate on A. This follows immediately from the formula (3) and the estimate

$$\sum_n |a_n x^n|_i \le \sum_n |a_n|\,|x|_i^n, \quad x \in A,\ i = 1, 2, \ldots$$

The main result in [2] gives a partial converse:

THEOREM A. *If A is a commutative complex B_0-algebra, then A is m-convex if and only if all entire functions operate on A.*

The same proof works for real algebras, provided we only consider functions with real coefficients.

It is a long-standing question ([2], Problem 3, see also [5], Problem 13.15, [6], Problem 16.8, and [8], Problem 17) whether Theorem A is also true for non-commutative algebras. In this paper we give a counterexample showing that the condition of commutativity cannot be dropped.

In [2] it was also shown that for every entire function φ there is a commutative non-m-convex algebra A_φ such that φ operates on A_φ. Thus we cannot substantially relax the condition that all entire functions operate on the algebra in question.

Turpin [3] constructed a commutative completely metrizable locally pseudoconvex algebra A with exponent p, $0 < p < 1$, on which all entire functions operate but which is not m-convex. (The definitions are similar to those for B_0-algebras. The only difference is that the seminorms satisfying (1), (2), or (3) are not homogeneous, but p-homogeneous with exponent p, i.e. $|\lambda x| = |\lambda|^p |x|$ for each scalar λ and element x.) Thus the condition of local convexity cannot be relaxed either.

Later the author [7] showed that there is a complete, commutative non-m-convex locally convex algebra on which all entire functions operate. Thus we cannot relax the condition of metrizability. All that means that Theorem A gives the strongest possible result.

Using Theorem A, the author obtained in ([5], Theorem 13.17) the following result:

THEOREM B. *Let A be a commutative complex B_0-algebra with unit which is a Q-algebra. Then A is multiplicatively-convex.*

The same proof gives the result for an algebra without unit, and since the complexification of a Q-algebra is again a Q-algebra, the result is also true for real algebras. It was an open question (see [8], Problem 26) whether Theorem B is true in the non-commutative case. Our example here also provides a negative answer to that question.

Turpin [3] extended Theorem B to the non-metrizable case:

THEOREM C. *Let A be a commutative complex complete locally convex algebra with unit which is a Q-algebra. Then A is a multiplicatively-convex algebra provided the operation of taking inverse $x \to x^{-1}$ is continuous in A.*

Similarly to Theorem B, this result can be extended to algebras without unit (provided the operation of taking quasi-inverse is continuous) and to real algebras. Our example shows that the problem of extending Theorem C to the non-commutative case ([8], Problem 27) has a negative answer.

Using Theorem B, the author proved ([5], Theorem 13.18)

THEOREM D. *If a commutative complex B_0-algebra A has a closed radical* $\operatorname{rad} A$, *then this radical is an m-convex algebra.*

Here again our example shows that the above result fails to be true if A is non-commutative.

For more information on the classes of topological algebras mentioned above the reader is referred to [1] and [4]–[6].

When presenting the above-mentioned example of a pseudoconvex algebra, Turpin used the following lemma given in [2] (see Lemmas 2.1 and 2.2).

LEMMA E. *For any continuous function $v(t) > 0$, $0 \le t < \infty$, such that* $\lim_{t\to\infty} v(t)/t = \infty$, *there exists a continuous function $u(t) > 0$, $0 \le t < \infty$, such that* $\lim_{t\to\infty} u(t)/t = \infty$ *and*

$$u(t_1 + \ldots + t_n) \le 8[u(t_1) + \ldots + u(t_n)] + v(n), \quad 0 \le t_i < \infty.$$

Our construction will also be based upon this lemma. Following Turpin we choose v so that $v(n) = n(\log n)^{1/2}$ for $n \ge 2$. Thus

$$v(n) = r_n n \log n \quad \text{with} \quad \lim_{n\to\infty} r_n = 0 \tag{5}$$

and for the corresponding function u we have

$$\lim_{n\to\infty} u(n)/n = \infty, \tag{6}$$

and

$$u(k_1 + \ldots + k_n) \le 8[u(k_1) + \ldots + u(k_n)] + v(n) \tag{7}$$

for all natural numbers $k_1, \ldots, k_n$, and n.

All results of this paper are corollaries to the following

THEOREM 1. *There exists a non-m-convex B_0-algebra A such that for each x in A,*

$$\lim_{n\to\infty} x^n = 0. \tag{8}$$

Proof. Let $t_1, t_2, \ldots$ be a sequence of variables, and consider the linear span A_0 of all products of the form

$$t_n t_{n+1} \ldots t_{n+k}, \tag{9}$$

where $n \ge 1$, $k \ge 0$. We define on A_0 an associative multiplication by setting $t_i t_j = 0$ whenever $j \ne i+1$. Thus every product $t_{i_1} \ldots t_{i_k}$ is zero except for the case when $i_s = i_1 + s - 1$, i.e. when the product is of the form (9). Every element of A_0 can be written in the form

$$x = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \xi(n,k) t_n \ldots t_{n+k}, \tag{10}$$

where only a finite number of the coefficients $\xi(n,k)$ are different from zero. Define on A_0 a sequence of seminorms setting for an element x of the form (10),

$$|x|_m = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} |\xi(n,k)| \exp(8^m u(k+1)). \tag{11}$$

Clearly these seminorms satisfy condition (1) for each x in A. For x and y of the form (10), with y having coefficients $\eta(n,k)$, by (7) we have

$$\begin{aligned}
|xy|_m &\le \sum_{k,q\ge 0;\, n,p\ge 1} |\xi(n,k)|\,|\eta(p,q)| \exp(8^m u(k+q+2)) \\
&\le \sum_{k,q,n,p} |\xi(n,k)|\,|\eta(p,q)| \exp\{8^m[8u(k+1) + 8u(q+1) + v(2)]\} \\
&\le \exp(8^m v(2)) \sum_{k,n} |\xi(n,k)| \exp(8^{m+1} u(k+1)) \\
&\quad \times \sum_{p,q} |\eta(p,q)| \exp(8^{m+1} u(q+1)) \\
&= \exp(8^m v(2)) |x|_{m+1} |y|_{m+1},
\end{aligned}$$

so that (2) is satisfied with $C_m = \exp(8^m v(2))$.

Denote by A the completion of A_0 in the topology given by the seminorms (11). The algebra A consists of elements of the form (10) with infinite summation, such that all seminorms (11) are finite. Clearly these seminorms satisfy (1) and (2), so that A is a B_0-algebra.

Let x be an arbitrary element of A and let m be a natural number. We have

$$x^m = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \mu(k,n) t_n \ldots t_{n+k},$$

where

$$\begin{aligned}
\mu(k,n) = \sum_{k_1+\ldots+k_m = k} &\xi(n,k_1)\xi(n+k_1+1, k_2-1)\ldots \\
&\ldots \xi(n+k_1+\ldots+k_{m-1}+1, k_m-1).
\end{aligned}$$

Note that among the $m!$ products of elements of the form

$$\xi(n+k_1+\ldots+k_j+1, k_{j+1}-1) t_{n+k_1+\ldots+k_j+1} \ldots t_{n+k_1+\ldots+k_{j+1}}$$

only one product is different from zero. Thus we have

$$\begin{aligned}
|x^m|_j &= \sum_{n,k} |\mu(k,n)| \exp(8^j u(k+1)) \\
&\le \sum_{n,k} \sum_{k_1+\ldots+k_m=k} |\xi(n,k_1)\xi(n+k_1+1,k_2-1)\ldots \\
&\qquad \ldots \xi(n+k_1+\ldots+k_{m-1}+1, k_m-1)| \\
&\qquad \times \exp\{8^j[8u(k_1+1) + 8u(k_2) + \ldots + 8u(k_m) + v(m)]\} \\
&= \exp(8^j v(m)) \sum_{n,k} \sum_{k_1+\ldots+k_m=k} |\xi(n,k_1)| \exp(8^{j+1} u(k_1+1)) \ldots \\
&\qquad \ldots |\xi(n+k_1+\ldots+k_{m-1}+1, k_m-1)| \exp(8^{j+1} u(k_m)).
\end{aligned} \tag{12}$$

On the other hand, we have

$$\begin{aligned}
|x|_{j+1}^m \ge m! \sum_{n,k} \sum_{k_1+\ldots+k_m=k} &|\xi(n,k_1)| \exp(8^{j+1} u(k_1+1)) \ldots \\
&\ldots |\xi(n+k_1+\ldots+k_{m-1}+1, k_m-1)| \exp(8^{j+1} u(k_m)).
\end{aligned} \tag{13}$$

Now (12) and (13) imply

$$|x^m|_j \le \frac{\exp(8^j v(m))}{m!} |x|_{j+1}^m \tag{14}$$

for all x in A and all natural j.

Put $a_{m,j} = \exp(8^j v(m))/m!$. Then (5) implies $a_{m,j} = m^{m 8^j r_m}/m!$. For large m we have $8^j r_m \le 1/2$, and so

$$a_{m,j} \le \left(\frac{m^m}{m!}\right)^{1/2} \frac{1}{(m!)^{1/2}}.$$

But $\lim_m (m^m/m!)^{1/m} = e$, thus $a_{m,j} \le C^m/(m!)^{1/2}$, for large m, where C is a positive constant depending only upon j. Since $\lim_m (CM)^m/(m!)^{1/2} = 0$ for each positive M, it follows that the right hand side of (14) tends to zero as $m \to \infty$ for each fixed $j = 1, 2, \ldots$ This means that $\lim_m x^m = 0$ for each x in A.

Note that $\lim_m a_{m,j} = 0$ implies $a_{m,j} \le C_j$ for all m, where C_j is a positive constant. Thus (14) implies

$$|x^m|_j \le C_j |x|_{j+1}^m \tag{15}$$

for all x in A and all positive integers m and j.

It remains to be shown that A is a non-m-convex algebra. Suppose to the contrary that A is m-convex. Then there is a sequence $(\|\cdot\|_i)$ of seminorms on A satisfying (1) and (3) and giving the same topology as the sequence $(|\cdot|_i)$. Thus, by (4), there is a constant c_1 and a seminorm $\|\cdot\| = \|\cdot\|_i$ such that $c_1|x|_1 \le \|x\|$ for all x in A. Similarly, there is an index k and a positive c_2 such that together with the previous inequality we have

$$c_1|x|_1 \le \|x\| \le c_2|x|_k, \qquad x \in A.$$

By (3), this implies that for any sequence (x_i) in A such that $c_2|x_i|_k \le 1/2$, we have

$$\lim_n |x_1 \ldots x_n|_1 = 0.$$

Put $x_i = \varepsilon t_i$, and choose a positive ε so that $c_2|\varepsilon t_i|_k = c_2\varepsilon \exp(8^k u(1)) \le 1/2$. Then $\lim_n \varepsilon^n |t_1 \ldots t_n|_1 = 0$. But $\varepsilon^n |t_1 \ldots t_n|_1 = \varepsilon^n \exp(8u(n))$, and the right hand term, in view of (6), tends to infinity for each positive ε. The conclusion follows.

As corollaries we obtain the following results.

THEOREM 2. *There exists a non-m-convex algebra A on which all entire functions operate.*

Proof. Let A be the algebra constructed in Theorem 1. Let $\varphi(\zeta) = \sum_n a_n \zeta^n$ be an entire function. Then (15) implies

$$\sum_n |a_n x^n|_j \le C_j \sum_n |a_n|\, |x|_{j+1}^n < \infty$$

for all x in A. Thus all entire functions operate on the algebra A, which is non-m-convex.

If we wish to have in Theorem 2 an algebra with unit element, we just take the unitization A_1 of A. It is a non-m-convex algebra, and every commutative subalgebra of A_1 is m-convex, being the unitization of an m-convex algebra (see the Corollary below). Thus all entire functions operate on A_1.

COROLLARY. *There exists a non-m-convex B_0-algebra with all commutative subalgebras m-convex.*

THEOREM 3. *There exists a non-m-convex B_0-algebra which is a Q-algebra.*

Proof. Let A be the algebra of Theorem 1 and let x be an element in A. Then $\lim_n |2^n x^n|_j = 0$ for every j, and so there is a positive constant C_j such that $|2^n x^n|_j \le C_j$ for all n. This implies

$$\sum_{n=1}^{\infty} |x^n|_j \le \sum_{n=1}^{\infty} \frac{C_j}{2^n} < \infty$$

and the element x has quasi-inverse $\sum_{n=1}^{\infty} (-1)^n x^n$. Thus A is a Q-algebra.

The radical of a non-commutative algebra with unit is the intersection of all its maximal left ideals (equal to the intersection of all its maximal right ideals). Let A_1 be the unitization of the algebra A of Theorem 1, with unit e. Every element of A_1 of the form $\lambda e + x$, where $x \in A$ and λ is a non-zero scalar, is invertible with inverse $\sum_{n=0}^{\infty} (-1)^n \lambda^{-n-1} x^n$. Thus every non-invertible element of A_1 is in A, and so A is the only (two-sided, or one-sided) maximal ideal of A coinciding with its radical. Thus we have

THEOREM 4. *There exists a non-m-convex B_0-algebra with closed radical which is not m-convex.*

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Received January 27, 1994 (3222)

STUDIA MATHEMATICA 110 (3) (1994)

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