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On the characterization of Hardy–Besov spaces on the dyadic group and its applications

by

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Dedicated to Professor C. Watari on the occasion of his sixtieth birthday

Abstract. C. Watari [12] obtained a simple characterization of Lipschitz classes $\mathrm{Lip}^{(p)}\alpha(W)$ $(1\le p\le\infty,\ \alpha>0)$ on the dyadic group using the L^p-modulus of continuity and the best approximation by Walsh polynomials. Onneweer and Weiyi [4] characterized homogeneous Besov spaces $B^{\alpha}_{p,q}$ on locally compact Vilenkin groups, but there are still some gaps to be filled up. Our purpose is to give the characterization of Besov spaces $B^{\alpha}_{p,q}$ by oscillations, atoms and others on the dyadic groups. As applications, we show a strong capacity inequality of the type of the Maz'ya inequality, a weak type estimate for maximal Cesàro means and a sufficient condition of absolute convergence of Walsh–Fourier series.

0. Introduction and notation. The dyadic group, 2^{ω}, is viewed classically as the set of all sequences of 0's and 1's with addition (mod 2) defined pointwise, and is supplied with the usual product topology. Our results are stated in the situation that 2^{ω} is the additive subgroup of the ring of integers in the 2-series field K of formal Laurent series in one variable over $GF(2)$ (see [9]). Such a field K is a particular instance of a local field; that is, a locally compact, totally disconnected, non-discrete, complete field. The results of this paper have extensions to any local field.

We need to set some basic notation. It is taken from [9] where the fundamentals are detailed. For the additive subgroup K^+ of the 2-series field K, we may choose a Haar measure dx. Let $d(\alpha x)=|\alpha|dx$ and call $|\alpha|$ the *valuation* of α. Let $|0|=0$. The mapping $x\mapsto|x|$ has the following properties: $|x|=0\Leftrightarrow x=0$, $|xy|=|x|\cdot|y|$, $|x+y|\le\max(|x|,|y|)$.

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(If $|x| \neq |y|$, then $|x+y| = \max(|x|,|y|)$.) Let $P^0 = \{x \in K : |x| \le 1\}$ and $P^1 = \{x \in K : |x| < 1\}$. K is totally disconnected, hence the value is discrete valued. Thus there is an element of P^1 of maximum value. Let $\wp$ be a fixed element of maximum value. Then an element $x \in K$ is represented as

$$x = \sum_{k=j}^{\infty} a_k \wp^k, \qquad a_k \in GF(2),$$

which can contain a finite number of terms with negative powers of $\wp$. The addition and multiplication of two power series are defined in a natural fashion. The ring of integers $P^0 = \{x = \sum_{k=0}^{\infty} a_k \wp^k\}$ coincides with the dyadic group 2^ω as an additive group.

For E a measurable subset of K, let $|E| = \int_K \Phi_E(x)\,dx$, where Φ_E is the characteristic function of E and dx is Haar measure normalized so $|P^0| = 1$. Then $|P^1| = |\wp| = 2^{-1}$. Let $P^k = \{x \in K : |x| \le 2^{-k}\}$ and Φ_k be its characteristic function.

For $x = x_0 + \sum_{k=j}^{-1} a_k \wp^k$, $a_k \in GF(2)$, $x_0 \in P^0$, set

$$w(\wp^k) = \begin{cases} -1, & k = -1, \\ 1, & k < -1, \end{cases} \qquad w(x_0) = 1.$$

Then w is a character on K^+ that is trivial on P^0, but is non-trivial on P^{-1}. For $x, y \in K$, let $w_y(x) = w(y \cdot x)$. w is constant on cosets of P^0 and if $y \in P^k$ then w_y is constant on cosets of P^{-k}.

We assume that all functions are complex-valued and measurable.

If $f \in L^1(K)$ the Fourier transform of f is the function $\widehat{f}$ defined by

$$\widehat{f}(y) = \int_K f(u) w_y(u)\,du.$$

Then we have $\widehat{\Phi}_k = 2^{-k}\Phi_{-k}$.

The space of test functions, $S(K)$, is the space of finite linear combinations of functions of the form $h + \Phi_k$, $h \in K$, $k \in \mathbb{Z}$. Then $\phi \in S(K)$ if and only if there are integers k, l such that ϕ is constant on cosets of P^k and supported in P^l (see [9, p. 36, Theorem (3.2)]). The space of distributions, $S'(K)$, is the space of continuous linear functionals on $S(K)$.

Let $\{u(n)\}_{n=0}^{\infty}$ be a complete list of distinct coset representatives of P^0 in K^+. We define $u(0) = 0$, $u(1) = \wp^{-1}$ and for $n = b_0 + b_1 \cdot 2 + b_2 \cdot 2^2 + \ldots + b_s \cdot 2^s$ ($b_i = 0$ or 1, $b_s = 1$), $u(n) = u(b_0) + \wp^{-1} u(b_1) + \ldots + \wp^{-s} u(b_s)$. Then $\{w_{u(n)}|_{P^0}\}_{n=0}^{\infty}$ is a complete set of characters on P^0. This is the Walsh–Paley system (see [9, p. 85, Proposition (6.4)]).

The Dirichlet kernels are the functions $D_n(x) = \sum_{k=0}^{n-1} w_k(x)$, $n \ge 1$, $D_0(x) \equiv 0$. If $f \in L^1(P^0)$ the Walsh–Fourier coefficients $\{c_k\}_{k=0}^{\infty} = \{\widehat{f}(u(k))\}_{k=0}^{\infty}$ are given by $c_k = \int_{P^0} f(x) w_k(x)\,dx$. The Walsh–Fourier series is given by $f(x) \sim \sum_{k=0}^{\infty} c_k w_k(x)$. The nth partial sum of the Walsh–Fourier series of f is denoted by $S_n f(x)$ and is defined as $S_n f(x) = \sum_{k=0}^{n-1} c_k w_k(x)$.

If $f \in L^1(P^0)$, $x \in P^0$, $n \ge 0$ then $S_{2^n} f(x) = 2^n \int_{x+P^n} f(t)\,dt$, as follows from the fact that $D_{2^n} = 2^n \Phi_n$.

$S = S(P^0)$ is the collection of test functions on P^0. We have $\phi \in S$ if and only if ϕ is constant on cosets of P^n in P^0 for some $n \ge 0$. If $\phi \in S$ then ϕ is a "polynomial", that is, $\phi = \sum_{k=0}^{2^n-1} \widehat{\phi}(u(k)) w_k(x)$ for some $n \ge 0$. Let $S(j)$ be a subset of S spanned by Φ_i's $(0 \le i \le j)$. $S' = S'(P^0)$ is the space of distributions on P^0. If $f \in S'$ then $f = \sum_{k=0}^{\infty} (f, w_k) w_k$, where (f, w_k) denotes the action of $f \in S'(P^0)$ on $w_k \in S(P^0)$. That is, f is a "formal Walsh–Fourier series". The Fejér kernels, $K_n^\beta(x)$ and $\dot{K}_n^\beta(x)$ $(n \ge 0)$, are the functions

$$K_n^\beta(x) = \frac{1}{A_{n-1}^\beta} \sum_{k=0}^{n-1} A_{n-k-1}^\beta w_k(x), \qquad n \ge 1,$$

$$K_0^\beta \equiv 0, \qquad A_n^\beta = \frac{(\beta+1)\ldots(\beta+n)}{n!} \simeq C(\beta) n^\beta,$$

and

$$\dot{K}_n^\beta(x) = \frac{1}{A_{n-1}^\beta} \sum_{i=0}^{s} b_i A_{2^i}^\beta K_{2^i}^1 \qquad \text{for } n = \sum_{i=0}^{s} b_i 2^i,\ b_s = 1,\ b_i = 0 \text{ or } 1.$$

Let $\sigma_n^\beta f(x) = f * K_n^\beta(x)$ be the Cesàro means of order β of the partial sums of the Walsh–Fourier series of f whenever $f \in S'(P^0)$.

Let $\{\Delta_j\}_{j=0}^{\infty}$ be a family of functions on P^0 satisfying $\Delta_j(x) = 2^j \Phi_j(x) - 2^{j-1}\Phi_{j-1}(x)$ for $j \ge 1$ and $\Delta_0 = \Phi_0$. Since $D_{2^j}(x) = 2^j \Phi_j(x)$, $\Delta_j(x) = D_{2^j}(x) - D_{2^{j-1}}(x)$ for $j \ge 1$.

The *Besov space* $B_{pq}^\alpha(P^0) = B_{pq}^\alpha$ $(0 < p, q \le \infty$ and $-\infty < \alpha < \infty)$ is the collection of all $f \in S'(P^0)$ such that

$$\|f\|_{B_{pq}^\alpha} = \Big(\sum_{j=0}^{\infty} 2^{\alpha q j} \|\Delta_j * f\|_p^q\Big)^{1/q}$$

is finite (modification if $q = \infty$).

C denotes a constant, not always the same one.

1. Characterization by means of difference, oscillations and approximation. To give a characterization of B_{pq}^α by means of difference and oscillations, let

$$\|S_r^\alpha f\|_{pq} = \Big[\sum_{j=0}^{\infty} 2^{\alpha q j} \Big\{ \int_{P^0} \Big(2^j \int_{P^j} |f(x+u) - f(x)|^r\,du\Big)^{p/r} dx \Big\}^{q/p}\Big]^{1/q}$$

and

$$\|D_r^\alpha f\|_{pq} = \Big[\sum_{j=0}^{\infty} 2^{\alpha q j} \|\{S_{2^j}(|f - S_{2^j} f|^r)\}^{1/r}\|_p^q\Big]^{1/q}.$$

Furthermore, to give a characterization by approximation, let

$$\|E^\alpha f\|_{pq} = \Big(\sum_{j=0}^{\infty} 2^{\alpha q j} E_p(2^j, f)^q\Big)^{1/q}, \quad \text{where}$$

$$E_p(2^j, f) = \inf\{\|f - g\|_p : g \in S(j)\}, \quad j = 0, 1, \ldots$$

For $q = \infty$ or $r = \infty$ we have the usual modifications.

The following theorem generalizes and improves [4, Theorem 5(a)] in the inhomogeneous Besov case in the dyadic group setting. The corresponding theorem and its proof for the $\mathbb{R}^n$ case are somewhat complicated (see [11, p. 101, Theorem; p. 105, Theorem; and p. 81, Theorem]).

THEOREM 1. *Let* $0 < p, q \le \infty$ *and* $r \ge 1$. *If* $\alpha > \max(1/p - 1, 0)$, *then*

$$\|f\|_{B_{pq}^\alpha} \approx \|S_r^\alpha f\|_{pq} + \|f\|_p \approx \|D_r^\alpha f\|_{pq} + \|f\|_p \approx \|E^\alpha f\|_{pq} + \|f\|_p.$$

We shall need the following theorems (cf. [11, p. 22, Theorem, and p. 129, Theorem]).

THEOREM A. (i) (Nikol'skiĭ's inequality) *Let* $0 < p \le q \le \infty$. *If* $\phi \in S(j)$, *then* $\|\phi\|_q \le 2^{j(1/p-1/q)}\|\phi\|_p$.

(ii) (Embedding Theorem) *Let* $0 < p \le 1$ *and* $\alpha > 1/p - 1$. *Then* $B_{pq}^\alpha \subset L^1 \subset L^p$. *The inclusion maps are continuous.*

Proof. (i) As $\phi \in S(j)$, we may write $\phi(x) = 2^j \int_{P^0} \phi(y)\Phi_j(x-y)\,dy$. If $0 < p \le 1$, then $|\phi(x)| \le 2^j \sup_y |\phi(y)|^{1-p} \int |\phi(y)|^p dy$. Taking the supremum with respect to x, we have $\|\phi\|_\infty \le 2^{j/p}\|\phi\|_p$. Similarly, if $p \le q$, then $\|\phi\|_q \le \|\phi\|_\infty^{1-p/q}\|\phi\|_p^{p/q} \le 2^{(1/p-1/q)j}\|\phi\|_p$.

(ii) By (i), $\|\Delta_j * f\|_1 \le 2^{(1/p-1)j}\|\Delta_j * f\|_p$. Hence we have $B_{pq}^\alpha \subset B_{1q}^{\alpha-(1/p-1)}$. On the other hand, $B_{1q}^{\alpha-(1/p-1)} \subset B_{11}^0$ is well known (see [11, p. 47, Proposition 2(ii)]). Therefore $\|f\|_1 \le \|f\|_{B_{11}^0} \le \|f\|_{B_{pq}^\alpha}$. ■

We shall use the type of reverse Hölder inequality due to DeVore and Sharpley [1, p. 26, Theorem 4.3] in the dyadic group setting. This plays a vital role in the proof of Theorem 1. Let $0 < \gamma < s \le 1 \le r$ and $\beta = 1/s - 1/r > 0$. For $f \in S'(P^0)$ and $Q = x + P^j$ $(j \ge 0)$ set

$$f_{\beta,\gamma,j}(x) = f_j(x) = \sup_{n \ge j} 2^{n\beta}\{S_{2^n}(|f - S_{2^n} f|^\gamma)(x)\}^{1/\gamma}.$$

THEOREM B. *If* $f \in S'$, *then there exists a constant* C *such that*

$$\|f - S_{2^j} f\|_{L^r(Q)} \le C\|f_{\beta,\gamma,j}\|_{L^s(Q)}.$$

Proof. First, we show

$$[(f - S_{2^j} f)\Phi_Q]^*(t) \le C \int_t^{2^{-j}} f_j^*(u) u^{\beta-1}\,du + Ct^\beta f_j^*(t),$$

$0 < t < 2^{-j-1}$, where Φ_Q and g^* denote the characteristic function of Q and the decreasing rearrangement of g respectively. Let $E = \{u \in Q : f_j(u) > f_j^*(t)\}$, so that $|E| \le t$. Let i be the integer with $2^{-(i+j+1)} \le t < 2^{-(i+j)}$ $(i \ge 1)$. Since $S_{2^k} f$ and $S_{2^{k+1}} f$ are constant on cosets of P^{k+1}, we have, on $x + P^{k+1}$,

$$\begin{aligned}
|S_{2^k} f(u) - S_{2^{k+1}} f(u)|^\gamma &= S_{2^{k+1}}(|S_{2^k} f - S_{2^{k+1}} f|^\gamma)(u)\\
&\le S_{2^{k+1}}(|S_{2^k} f - f|^\gamma)(u) + S_{2^{k+1}}(|S_{2^{k+1}} f - f|^\gamma)(u)\\
&\le 2S_{2^k}(|S_{2^k} f - f|^\gamma)(u) + S_{2^{k+1}}(|S_{2^{k+1}} f - f|^\gamma)(u)\\
&\le 2^{1-k\beta\gamma} \inf_{u \in x+P^k} (f_k(u))^\gamma + 2^{-(k+1)\beta\gamma} \inf_{u \in x+P^{k+1}} (f_{k+1}(u))^\gamma\\
&\le C2^{-k\beta\gamma} \inf_{u \in x+P^{k+1}} (f_k(u))^\gamma.
\end{aligned}$$

Hence, using the monotone property of f_j^* with respect to j and the inequality $\inf_{u \in x+P^{k+1}} f_k(u) \le f_k^*(2^{-(k+1)})$ for $x \in Q \setminus E$, we have,

$$\begin{aligned}
|S_{2^j} f(x) - S_{2^{i+j-1}} f(x)| &\le \sum_{k=j}^{i+j-2} \|S_{2^k} f - S_{2^{k+1}} f\|_{L^\infty(x+P^{k+1})}\\
&\le C \sum_{k=j}^{i+j-2} 2^{-\beta k} \inf_{u \in x+P^{k+1}} f_k(u) \le C \sum_{k=j}^{i+j-2} \int_{2^{-(k+2)}}^{2^{-(k+1)}} f_k^*(v) v^{\beta-1} dv\\
&\le C \int_t^{2^{-(j+1)}} f_k^*(v) v^{\beta-1}\,dv \quad \text{for } x \in Q \setminus E.
\end{aligned}$$

On the other hand, since $S_{2^j} f \to f$ a.e. as $j \to \infty$, for $x \in Q \setminus E$,

$$\begin{aligned}
|S_{2^{i+j-1}} f(x) - f(x)| &\le \sum_{k=i+j-1}^{\infty} |S_{2^k} f(x) - S_{2^{k+1}} f(x)|\\
&\le C \sum_{k=i+j-1}^{\infty} 2^{-\beta k} f_k(x) \le C2^{-\beta(i+j)} f_j^*(t) \le Ct^\beta f_j^*(t).
\end{aligned}$$

Therefore, we have

$$|S_{2^j} f(x) - f(x)| \le C \int_t^{2^{-(j+1)}} f_j^*(v) v^{\beta-1}\,dv + Ct^\beta f_j^*(t) \quad \text{for } x \in Q \setminus E.$$

Since $|E| \le t$, by the property of decreasing rearrangement we have the desired inequality.

Next, taking the L^r-norm ($r \ge 1$) over $[0, 2^{-(j+1)}]$ and using Hardy's inequality, we have

$$\Big(\int_0^{2^{-(j+1)}} |[(f - S_{2^j} f)\Phi_Q]^*(t)|^r dt\Big)^{1/r}$$
$$\le C\Big(\int_0^{2^{-(j+1)}} \Big|\int_t^{2^{-(j+1)}} f_j^*(v)v^{\beta-1}\,dv\Big|^r dt\Big)^{1/r} + C\Big(\int_0^{2^{-(j+1)}} |t^\beta f_j^*(t)|^r dt\Big)^{1/r}$$
$$\le C\Big(\int_0^{2^{-(j+1)}} |t^{1/s} f_j^*(t)|^r \frac{dt}{t}\Big)^{1/r}$$
$$\le C\|f_j\|_{L(s,r)(x+P^{j+1})} \le C\|f_j\|_{L(s,r)(Q)} \le C\|f_j\|_{L^s(Q)},$$

because $L^s = L(s,s) \subset L(s,r)$.

Since $\beta = 1/s - 1/r > 0$ and g^* is decreasing, the first term above is not less than

$$C\Big(\int_0^{2^{-j}} |[(f - S_{2^j} f)\Phi_Q]^*(t)|^r dt\Big)^{1/r} = C\|f - S_{2^j} f\|_{L^r(Q)}.$$

Thus, Theorem B is proved. ■

P r o o f o f T h e o r e m 1. The proof is carried out for the case $0 < p \le 1 \le q < \infty$, the remaining cases are similar. The proof is based on [6, Theorem 1] and contains four steps:

(1.1) $\|S_r^\alpha f\|_{pq} \le C\|D_r^\alpha f\|_{pq}$,

(1.2) $\|D_r^\alpha f\|_{pq} \le C\|D_\gamma^\alpha f\|_{pq}$ for $0 < \gamma < p$ and $\|D_\gamma^\alpha f\|_{pq} \le C\|f\|_{B_{pq}^\alpha}$,

(1.3) $\|f\|_{B_{pq}^\alpha} \le C(\|E^\alpha f\|_{pq} + \|f\|_p)$,

(1.4) $\|E^\alpha f\|_{pq} \le \|S_r^\alpha f\|_{pq}$.

The first step is to prove (1.1). By Minkowski's inequality, we have

$$\Big(2^j \int_{P^j} |f(x+u) - f(x)|^r\,du\Big)^{p/r}$$
$$\le \Big(2^j \int_{P^j} |f(x+u) - S_{2^j} f(x+u)|^r du\Big)^{p/r} + \Big(2^j \int_{P^j} |S_{2^j} f(x) - f(x)|^r du\Big)^{p/r}$$
$$\le \{S_{2^j}(|f - S_{2^j} f|^r)(x)\}^{p/r} + |S_{2^j} f(x) - f(x)|^p.$$

Then

$$\|S_r^\alpha f\|_{pq}^p \le \Big(\sum_{j=0}^\infty 2^{\alpha q j}\Big[\int_{P^0} \{S_{2^j}(|f - S_{2^j}(f)|^r)(x)\}^{p/r}\,dx + \int_{P^0} |S_{2^j} f(x) - f(x)|^p dx\Big]^{q/p}\Big)^{p/q}$$
$$\le \|D_r f\|_{pq}^p + \Big(\sum_{j=0}^\infty 2^{\alpha q j}\|S_{2^j} f - f\|_p^q\Big)^{p/q}.$$

Since $S_{2^j} f \to f$ a.e. as $j \to \infty$,

$$|f - S_{2^j} f| \le \sum_{k=0}^\infty |S_{2^{j+k+1}} f - S_{2^{j+k}} f| \le 2\sum_{k=0}^\infty S_{2^{j+k}}(|f - S_{2^{j+k}} f|).$$

Hence, by the fact that $\ell^p \subset \ell^1$ ($0 < p \le 1$), Minkowski's inequality and then Hölder's inequality, we obtain

$$\Big(\sum_{j=0}^\infty 2^{\alpha q j}\|S_{2^j} f - f\|_p^q\Big)^{p/q} \le C\Big[\sum_{j=0}^\infty 2^{\alpha q j}\Big\|\sum_{k=0}^\infty S_{2^{j+k}}(|f - S_{2^{j+k}} f|)\Big\|_p^q\Big]^{p/q}$$
$$\le C\Big[\sum_{j=0}^\infty 2^{\alpha q j}\Big\{\sum_{k=0}^\infty \|S_{2^{j+k}}(|f - S_{2^{j+k}} f|)\|_p\Big\}^{q/p}\Big]^{p/q}$$
$$\le C\sum_{k=0}^\infty \Big[\sum_{j=0}^\infty 2^{\alpha q j}\|S_{2^{j+k}}(|f - S_{2^{j+k}} f|)\|_p^q\Big]^{p/q}$$
$$\le C\sum_{k=0}^\infty 2^{-\alpha p k}\Big[\sum_{j=0}^\infty 2^{\alpha q j}\|S_{2^j}(|f - S_{2^j} f|)\|_p^q\Big]^{p/q}$$
$$= C\|D_1^\alpha f\|_{pq}^p \le C\|D_r^\alpha f\|_{pq}^p,$$

which proves the desired inequality.

The second step is to prove the two inequalities (1.2).

The crucial point is the first inequality that is a reverse Hölder inequality. We set σ, β and s one after another as

$$\sigma = \frac{1}{p} - \frac{1}{r}, \quad \alpha > \beta > \sigma,$$
$$\beta = \frac{1}{s} - \frac{1}{r}, \quad \gamma < s < p.$$

Using Theorem B, we have

$$\|D_r^\alpha f\|_{pq}^q = \sum_{j=0}^\infty 2^{\alpha qj}\Big\{2^{pj/r}\int_{P^0}\|f-S_{2^j}f\|_{L^r(x+P^j)}^p\,dx\Big\}^{q/p}$$
$$\le C\sum_{j=0}^\infty 2^{\alpha qj}\Big\{2^{pj/r}\int_{P^0}\|f_j\|_{L^s(x+P^j)}^p dx\Big\}^{q/p}$$
$$= C\sum_{j=0}^\infty 2^{\alpha qj}\Big\{2^{pj/r}\int_{P^0}\Big|\int_{x+P^j}\{\sup_{n\ge j}2^{n\beta}S(n,\gamma)(t)\}^s dt\Big|^{p/s}dx\Big\}^{q/p},$$

where $S(n,\gamma)(t)=\{S_{2^n}(|f-S_{2^n}f|^\gamma)(t)\}^{1/\gamma}$. Replacing $\sup_{n\ge j}$ with $\sum_{n=j}^\infty$ and using the fact that $\ell^s\subset\ell^1$ $(0<s<1)$, we have

$$2^{pj/r}\int_{P^0}\Big|\int_{x+P^j}\{\sup_{n\ge j}2^{n\beta}S(n,\gamma)(t)\}^s dt\Big|^{p/s}dx$$
$$= \int_{P^0}\Big|2^j\int_{x+P^j}\{\sup_{n\ge j}2^{(n-j)\beta}S(n,\gamma)(t)\}^s dt\Big|^{p/s}dx$$
$$\le \int_{P^0}\Big|2^j\int_{x+P^j}\Big\{\sum_{n=j}^\infty 2^{(n-j)\beta}S(n,\gamma)(t)\Big\}^s dt\Big|^{p/s}dx$$
$$\le \int_{P^0}\Big|\sum_{n=j}^\infty 2^{j+(n-j)\beta s}\int_{x+P^j}\{S(n,\gamma)(t)\}^s dt\Big|^{p/s}dx.$$

Then, by Minkowski's inequality twice,

$$\|D_r^\alpha f\|_{pq}^q \le C\sum_{j=0}^\infty 2^{\alpha qj}\Big\{\int_{P^0}\Big|\sum_{n=j}^\infty 2^{j+(n-j)\beta s}\int_{x+P^j}\{S(n,\gamma)(t)\}^s dt\Big|^{p/s}dx\Big\}^{q/p}$$
$$\le C\sum_{j=0}^\infty 2^{\alpha qj}\Big[\sum_{n=j}^\infty\Big\{\int_{P^0}\Big(2^{j+(n-j)\beta s}\int_{x+P^j}\{S(n,\gamma)(t)\}^s dt\Big)^{p/s}dx\Big\}^{s/p}\Big]^{q/s}$$
$$= C\sum_{j=0}^\infty 2^{\alpha qj}\Big[\sum_{n=j}^\infty 2^{(n-j)\beta s}\Big\{\int_{P^0}\Big(2^j\int_{x+P^j}\{S(n,\gamma)(t)\}^s dt\Big)^{p/s}dx\Big\}^{s/p}\Big]^{q/s}$$
$$= C\sum_{j=0}^\infty 2^{\alpha qj}\Big[\sum_{m=0}^\infty 2^{m\beta s}\Big\{\int_{P^0}\Big(2^j\int_{x+P^j}\{S(m+j,\gamma)(t)\}^s dt\Big)^{p/s}dx\Big\}^{s/p}\Big]^{q/s}$$
$$\le C\Big(\sum_{m=0}^\infty\Big[\sum_{j=0}^\infty 2^{q(\beta m+\alpha j)}$$
$$\times\Big\{\int_{P^0}\Big(2^j\int_{x+P^j}\{S(m+j,\gamma)(t)\}^s dt\Big)^{p/s}dx\Big\}^{q/p}\Big]^{s/q}\Big)^{q/s}$$

$$= C\Big(\sum_{m=0}^\infty\Big[\sum_{k=m}^\infty 2^{q(\beta m+\alpha(k-m))}$$
$$\times\Big\{\int_{P^0}\Big(2^{k-m}\int_{x+P^{k-m}}\{S(k,\gamma)(t)\}^s dt\Big)^{p/s}dx\Big\}^{q/p}\Big]^{s/q}\Big)^{q/s}.$$

Hence, by the mean convergence theorem for partial sums [13, Theorem 2, or 5, p. 103, Corollary 6] we obtain finally

$$\|D_r^\alpha f\|_{pq}^q \le C\Big(\sum_{m=0}^\infty\Big[\sum_{k=m}^\infty 2^{q(\beta m+\alpha(k-m))}\Big\{\int_{P^0}(S(k,\gamma)(x))^p dx\Big\}^{q/p}\Big]^{s/q}\Big)^{q/s}$$
$$\le C\Big(\sum_{m=0}^\infty 2^{s(\beta-\alpha)m}\Big[\sum_{k=0}^\infty 2^{q\alpha k}\Big\{\int_{P^0}(S(k,\gamma)(x))^p dx\Big\}^{q/p}\Big]^{s/q}\Big)^{q/s}$$
$$= C\|D_\gamma^\alpha f\|_{pq}^q,\qquad \text{since } \alpha>\beta.$$

In order to get the second inequality, we write

$$\|D_\gamma^\alpha f\|_{pq}^q = \sum_{j=0}^\infty 2^{\alpha qj}\Big(\int_{P^0}\Big[2^j\int_{x+P^j}\Big|f(y)-S_{2^j}f(y)\Big|^\gamma dy\Big]^{p/\gamma}dx\Big)^{q/p}$$
$$= \sum_{j=0}^\infty 2^{\alpha qj}\Big(\int_{P^0}\Big[2^j\int_{x+P^j}\Big|\sum_{k=j}^\infty(\Delta_{k+1}*f)(y)\Big|^\gamma dy\Big]^{p/\gamma}dx\Big)^{q/p}.$$

By the mean convergence theorem, the fact that $\ell^p\subset\ell^1$ $(p<1)$ and Hölder's inequality, for $0<\varepsilon<\alpha$, $\|D_\gamma^\alpha f\|_{pq}^q$ is majorized by

$$\sum_{j=0}^\infty 2^{\alpha qj}\Big(\int_{P^0}\Big|\sum_{k=j}^\infty(\Delta_{k+1}*f)(x)\Big|^p dx\Big)^{q/p}$$
$$= \sum_{j=0}^\infty 2^{\alpha qj}\Big(\int_{P^0}\Big|\sum_{k=j}^\infty(\Delta_{k+1}*f)(x)\Big|^p dx\,2^{\varepsilon(k-j)p}2^{-\varepsilon(k-j)p}\Big)^{q/p}$$
$$\le C\sum_{j=0}^\infty 2^{\alpha qj}\sum_{k=j}^\infty\|\Delta_{k+1}*f\|_p^q 2^{\varepsilon(k-j)q} = C\sum_{j=0}^\infty 2^{(\alpha-\varepsilon)qj}\sum_{k=j}^\infty 2^{\varepsilon qk}\|\Delta_{k+1}*f\|_p^q$$
$$= C\sum_{k=0}^\infty 2^{\varepsilon qk}\|\Delta_{k+1}*f\|_p^q\sum_{j=0}^k 2^{(\alpha-\varepsilon)qj} C\sum_{k=0}^\infty 2^{\alpha q(k+1)}\|\Delta_{k+1}*f\|_p^q.$$

Therefore we have

$$\|D_\gamma^\alpha f\|_{pq}\le C\|f\|_{B_{pq}^\alpha}.$$

The third step is to prove (1.3).

This proof is based on [11, p. 81, Theorem]. Since there exists a step function of best approximation for any L^p function, we can choose a sequence $g_j \in S(j)$ such that $\|f-g_j\|_p \le 2E_p(2^j,f)$, $j=0,1,\ldots$ Hence we have $\|\Delta_j * f\|_p^p \le \sum_{k=j}^\infty \|\Delta_j*(g_{k-1}-g_k)\|_p^p$. On the other hand, by Theorem A(i) or [11, p. 26, Theorem], we see that

$$\begin{aligned}\|\Delta_j*(g_{k-1}-g_k)\|_p^p &\le C|P^{-k}|^{1-p}\|\Delta_j\|_p^p\|g_{k-1}-g_k\|_p^p\\ &= C2^{(k-j)(1-p)}\|g_{k-1}-g_k\|_p^p.\end{aligned}$$

Therefore, by Minkowski's inequality, we have

$$\begin{aligned}\|f\|_{B_{pq}^\alpha}^q &= \sum_{j=0}^\infty 2^{\alpha qj}\|\Delta_j*f\|_p^q\\ &\le C\sum_{j=0}^\infty 2^{\alpha qj}\Big(\sum_{k=j}^\infty 2^{(k-j)(1-p)}\|g_{k-1}-g_k\|_p^p\Big)^{q/p}\\ &\le C\sum_{j=0}^\infty 2^{\alpha qj}\Big(\sum_{l=0}^\infty 2^{l(1-p)}\|g_{j+l-1}-g_{j+l}\|_p^p\Big)^{q/p}\\ &\le C\Big[\sum_{l=0}^\infty 2^{l(1-p)}\Big(\sum_{j=0}^\infty 2^{\alpha qj}\|g_{j+l-1}-g_{j+l}\|_p^q\Big)^{p/q}\Big]^{q/p}\\ &\le C\Big[\sum_{l=0}^\infty 2^{l(1-p-\alpha p)}\Big(\sum_{j=0}^\infty 2^{\alpha qj}\|g_{j-1}-g_j\|_p^q\Big)^{p/q}\Big]^{q/p}\\ &\le C\sum_{j=1}^\infty 2^{\alpha qj}\|g_{j-1}-g_j\|_p^q + C\|g_0\|_p^q\\ &\le C\sum_{j=0}^\infty 2^{\alpha qj}\|g_j-f\|_p^q + C\|f\|_p^q, \qquad \text{since } 1-p-\alpha p<0.\end{aligned}$$

Taking the infimum over all such sequences $\{g_j\}$ gives

$$\|f\|_{B_{pq}^\alpha} \le C(\|E^\alpha f\|_{pq}+\|f\|_p).$$

The last step is to prove the inequality (1.4).

By the definition of best approximation, we have

$$\begin{aligned}E_p(2^j,f) \le \|f-S_{2^j}f\|_p &= \Big\{\int_{P^0}\Big|2^j\int_{P^j}(f(x)-f(x+h))\,dh\Big|^p dx\Big\}^{1/p}\\ &\le \Big\{\int_{P^0}\Big(2^j\int_{P^j}|f(x)-f(x+h)|^r\,dh\Big)^{p/r}dx\Big\}^{1/p}.\end{aligned}$$

Then we have the desired inequality:

$$\|E^\alpha f\|_{pq}^q = \sum_{j=0}^\infty 2^{\alpha qj}E_p(2^j,f)^q \le \|S_r^\alpha f\|_{pq}^q.$$

These steps together with Theorem A(ii) complete the proof of Theorem 1. ∎

To obtain a variation of Theorem 1 for the case $r=q=\infty$, we let

$$\|f\|_B = \sup_{j\ge0}2^{\alpha j}\|\Delta_j*f\|_p, \qquad \|f\|_S = \sup_h |h|^{-\alpha}\|f(\cdot+h)-f(\cdot)\|_p,$$
$$\|f\|_D = \sup_{j\ge0}2^{\alpha j}\|f-S_{2^j}f\|_p, \qquad \|f\|_E = \sup_{j\ge0}2^{\alpha j}E_p(2^j,f).$$

The following corollary generalizes the result of [12, Theorem], [4, Theorem 1] and [10, Theorem 3].

COROLLARY. *If $0<p\le\infty$ and $\alpha>\max(1/p-1,0)$, then*

$$\|f\|_B \approx \|f\|_S+\|f\|_p \approx \|f\|_D+\|f\|_p \approx \|f\|_E+\|f\|_p.$$

Proof. The proof is carried out for the case $0<p<1$, the remaining cases are similar. The outline of the proof is to show $\|f\|_S\le C\|f\|_D$, $\|f\|_D\le C\|f\|_B$, $\|f\|_B\le C(\|f\|_E+\|f\|_p)$, and then $\|f\|_E\le C\|f\|_S$.

Since $S_{2^j}f(x)=S_{2^j}f(x+h)$ for $|h|=2^{-j}$, we have

$$\begin{aligned}|h|^{-p\alpha}\|f(\cdot+h)-f(\cdot)\|_p^p &\le 2^{p\alpha j}(\|f(\cdot+h)-S_{2^j}f(\cdot+h)\|_p^p+\|f-S_{2^j}f\|_p^p)\\ &= 2^{1+\alpha pj}\|f-S_{2^j}f\|_p^p \quad \text{if } |h|=2^{-j}.\end{aligned}$$

Thus, we get $\|f\|_S\le C\|f\|_D$.

Next, since $f(x)-S_{2^j}f(x)=\sum_{k=j+1}^\infty \Delta_k*f(x)$ for a.e. x,

$$\|f\|_D \le \sup_{j\ge0}2^{\alpha j}\Big\|\sum_{k=j+1}^\infty \Delta_k*f\Big\|_p \le \sup_{j\ge0}2^{\alpha j}\Big(\sum_{k=j+1}^\infty\|\Delta_k*f\|_p^p\Big)^{1/p}.$$

We shall use Hölder's inequality and then a simple variation of [9, p. 179, Lemma (2.1)]:

Suppose $\alpha>0$ and $\{a_k\}$ is a sequence of non-negative numbers such that $\sup_k 2^{\alpha k}a_k<\infty$. Then $\sup_k 2^{\alpha k}\sum_{j=k+1}^\infty a_j\le C\sup_k 2^{\alpha k}a_k$.

For $0<\varepsilon<\alpha$, we have, by Hölder's inequality,

$$\begin{aligned}\sup_j 2^{\alpha j}\Big(\sum_{k=j+1}^\infty\|\Delta_k*f\|_p^p 2^{\varepsilon pk}2^{-\varepsilon pk}\Big)^{1/p} &\le C\sup_j 2^{(\alpha-\varepsilon)j}\sum_{k=j}^\infty 2^{\varepsilon k}\|\Delta_k*f\|_p\\ &\le C\sup_j 2^{\alpha j}\|\Delta_j*f\|_p = C\|f\|_B.\end{aligned}$$

From the third step of the proof of Theorem 1, we get easily $\|f\|_B\le C(\|f\|_E+\|f\|_p)$.

To prove $\|f\|_E\le C\|f\|_S$, we rewrite [8, p. 359, Lemma 2.1] as follows:

Let $f \in S(n)$. Then for any integer $k \ge 0$, there exists $g \in S(k)$ such that $\|f-g\|_p^p \le C2^k \int_{P^k} \|f(\cdot+h)-f(\cdot)\|_p^p\,dh$.

If $\|f-f_n\|_p \to 0$ as $n\to\infty$, then $E_p(2^k,f_n)\to E_p(2^k,f)$ as $n\to\infty$. Therefore,

$$E_p^p(2^k,f) \le C2^k \int\limits_{P^k} \|f(\cdot+h)-f(\cdot)\|_p^p\,dh$$

for $f\in L^p$. Hence, we have

$$\begin{aligned}\sup_{k\ge 0} 2^{\alpha k}E_p(2^k,f) &\le C\sup_{k\ge0}\Big(2^k\int\limits_{P^k} 2^{\alpha p k}\|f(\cdot+h)-f(\cdot)\|_p^p\,dh\Big)^{1/p}\\ &\le C\sup_{k\ge0}\Big(2^k\int\limits_{P^k} |h|^{-\alpha p}\|f(\cdot+h)-f(\cdot)\|_p^p\,dh\Big)^{1/p}\\ &\le C\sup_h |h|^{-\alpha}\|f(\cdot+h)-f(\cdot)\|_p = C\|f\|_S.\end{aligned}$$

Thus the corollary is proved. ∎

2. Characterization by atoms. We shall show that each $f\in B^\alpha_{pq}(P^0)$ can be decomposed into a sum of atoms.

We define an (α,p)-*atom* $a(x)$ $(-\infty<\alpha<\infty,\ 0<p\le\infty)$ to be a function satisfying, for some point $x_0\in P^0$ and non-negative integer k,

$$\operatorname{supp} a\subset x_0+P^k, \tag{2.1}$$

$$|a(x)|\le C|P^k|^{\alpha-1/p}, \tag{2.2}$$

$$|a(x-y)-a(x)|\le C|P^k|^{\alpha-1/p-l}|y|^l \quad \text{if } |y|<2^{-k}, \tag{2.3}$$

and

$$\int a(x)\,dx=0, \tag{2.4}$$

where $l>\alpha$. The constant function $a(x)\equiv1$ on P^0 is also considered to be an atom. We write a_Q for an atom satisfying (2.1)–(2.4) for a given $Q=x_0+P^k$.

We call $m(x)$ an (α,p)-*molecule* if there exist a non-negative integer k and a point $x_0\in P^0$ such that

$$|m(x)|\le C|P^k|^{\alpha-1/p}(1\vee 2^k|x-x_0|)^{-M}, \tag{2.5}$$

$$|m(x-y)-m(x)|\le C|P^k|^{\alpha-1/p-l}|y|^l(1\vee 2^k|x-x_0|)^{-M}, \quad \text{if } |y|<2^{-k}, \tag{2.6}$$

and

$$\int m(x)\,dx=0, \tag{2.7}$$

where $1\vee f(x)=\max(1,f(x))$, $l>\alpha$ and $M>\max[1/p-\alpha,1]$. We also write m_Q for an (α,p)-molecule satisfying (2.5)–(2.7) for a given coset of P^k in P^0, $Q=x_0+P^k$.

It is easy to check that a non-constant (α,p)-atom is an (α,p)-molecule.

The following decomposition theorem for $B^\alpha_{pq}(P^0)$ is based on comparable $\mathbb{R}^n$-results [2, Theorem 2.6 and Theorem 3.1] and improves [4, Theorem 6] in the dyadic group setting.

THEOREM 2. *Let $-\infty<\alpha<\infty$, $0<p,q\le\infty$.*

(a) *Each $f\in B^\alpha_{pq}$ can be decomposed as follows:*

$$f=\sum_{j=0}^\infty\sum_{|Q|=2^{-j}}\lambda_Q a_Q,$$

where the a_Q's are (α,p)-atoms. The numbers λ_Q satisfy

$$\Big\{\sum_{j=0}^\infty\Big(\sum_{|Q|=2^{-j}}|\lambda_Q|^p\Big)^{q/p}\Big\}^{1/q}=\|f\|_{B^\alpha_{pq}}.$$

(b) *Suppose $f=\sum_{j=0}^\infty\sum_{|Q|=2^{-j}}\lambda_Q m_Q$, where the m_Q's are (α,p)-molecules. Then*

$$\|f\|_{B^\alpha_{pq}}\le C\Big\{\sum_{k=0}^\infty\Big(\sum_{|Q|=2^{-k}}|\lambda_Q|^p\Big)^{q/p}\Big\}^{1/q}.$$

Proof. (a) For each $f\in S'(P^0)$, we have

$$\begin{aligned}f&=\sum_{j=0}^\infty f*\Delta_j*\Delta_j=\sum_{j=0}^\infty\int\limits_{P^0}(f*\Delta_j)(t)\Delta_j(x-t)\,dt\\ &=\sum_{j=0}^\infty\sum_{|Q|=2^{-j}}\int\limits_Q(f*\Delta_j)(t)\Delta_j(x-t)\,dt,\end{aligned}$$

where $\{Q\}$ are cosets of P^j. Since $f*\Delta_j$ and Δ_j are constant functions on each coset Q of P^j,

$$f=\sum_{j=0}^\infty\sum_{|Q|=2^{-j},y\in Q}|Q|(f*\Delta_j)(y)\Delta_j(x-y).$$

For $y\in Q$, define $\lambda_Q=|Q|^{-\alpha+1/p}(f*\Delta_j)(y)$ and $a_Q=|Q|^{\alpha-1/p+1}\Delta_j(x-y)$. For $|Q|=2^{-j}$ $(1\le j)$, we have $\operatorname{supp}a_Q=y+P^{j-1}$, $|a_Q|\le|Q|^{\alpha-1/p+1}\times 2^{j-1}=C|P^{j-1}|^{\alpha-1/p}$ and $\int a_Q(x)\,dx=|Q|^{\alpha-1/p+1}\int\Delta_j(x-t)\,dx=0$. If $|t|\le2^{-j}$, then $|a_Q(x-t)-a_Q(x)|=0$. For $j=0$, $a_Q(x)=|Q|^{\alpha-1/p+1}$ is a constant function on P^0. Thus, a_Q is an (α,p)-atom. Moreover, we have

$$\Big\{\sum_{j=0}^{\infty}\Big(\sum_{|Q|=2^{-j}}|\lambda_Q|^p\Big)^{q/p}\Big\}^{1/q}$$
$$=\Big\{\sum_{j=0}^{\infty}\Big(\sum_{|Q|=2^{-j},y\in Q}[|Q|^{-\alpha+1/p}(f*\Delta_j)(y)|]^p\Big)^{q/p}\Big\}^{1/q}$$
$$=\Big\{\sum_{j=0}^{\infty}\Big(\sum_{|Q|=2^{-j}}|Q|^{-\alpha p}\int_Q|f*\Delta_j(t)|^p\,dt\Big)^{q/p}\Big\}^{1/q}$$
$$=\Big\{\sum_{j=0}^{\infty}2^{\alpha qj}\Big(\int_{P^0}|f*\Delta_j(t)|^p dt\Big)^{q/p}\Big\}^{1/q}=\|f\|_{B^\alpha_{pq}}.$$

(b) To get the norm estimate, we write

$$\|\Delta_j*f\|_p^p=\Big\|\Delta_j*\Big(\sum_{k=0}^{\infty}\sum_{|Q|=2^{-k}}\lambda_Q m_Q\Big)\Big\|_p^p$$
$$\le\Big(\sum_{k=0}^{j}+\sum_{k=j+1}^{\infty}\Big)\sum_{|Q|=2^{-k}}|\lambda_Q|^p\|\Delta_j*m_Q\|_p^p.$$

We shall use the following pointwise inequalities:

$$(2.8)\qquad |\Delta_j*m_Q(x)|\le C2^{-(j-k)l+k(1/p-\alpha)}(1\vee 2^k|x|)^{-M}\qquad\text{if } k\le j-1,$$
$$(2.9)\qquad |\Delta_j*m_Q(x)|\le C2^{(j-k)M+k(1/p-\alpha)}(1\vee 2^j|x|)^{-M}\qquad\text{if } j-1\le k.$$

If we can show (2.8) and (2.9), then since $\ell^p\subset\ell^1$ $(0<p<1)$, we have

$$\|f\|_{B^\alpha_{pq}}^q=\sum_{j=0}^{\infty}2^{\alpha qj}\|\Delta_j*f\|_p^q$$
$$\le C\sum_{j=0}^{\infty}2^{\alpha qj}\Big\{\sum_{k=0}^{j}\sum_{|Q|=2^{-k}}|\lambda_Q|^p2^{-(j-k)lp+k(1-\alpha p)}\int(1\vee 2^k|x|)^{-Mp}\,dx$$
$$+\sum_{k=j+1}^{\infty}\sum_{|Q|=2^{-k}}|\lambda_Q|^p2^{(j-k)Mp+k(1-\alpha p)}\int(1\vee 2^j|x|)^{-Mp}\,dx\Big\}^{q/p}$$
$$=C\sum_{j=0}^{\infty}2^{\alpha qj}\Big\{\Big(\sum_{k=0}^{j}2^{-(j-k)lp+k(1-\alpha p)-k}$$
$$+\sum_{k=j+1}^{\infty}2^{(j-k)Mp+k(1-\alpha p)-j}\Big)\sum_{|Q|=2^{-k}}|\lambda_Q|^p\Big\}^{q/p}$$

$$=C\sum_{j=0}^{\infty}\Big\{\Big(\sum_{k=0}^{j}2^{-(j-k)(l-\alpha)p}+\sum_{k=j+1}^{\infty}2^{(j-k)(Mp-1+p\alpha)}\Big)\sum_{|Q|=2^{-k}}|\lambda_Q|^p\Big\}^{q/p}.$$

Applying Young's inequality, we have

$$\|f\|_{B^\alpha_{pq}}^q\le C\sum_{j=0}^{\infty}\Big(\sum_{|Q|=2^{-j}}|\lambda_Q|^p\Big)^{q/p}\Big\{\sum_{j=0}^{\infty}2^{-j(l-\alpha)p}+\sum_{j=0}^{\infty}2^{-j(Mp-1+p\alpha)}\Big\}^{q/p}$$
$$=C\sum_{j=0}^{\infty}\Big(\sum_{|Q|=2^{-j}}|\lambda_Q|^p\Big)^{q/p},$$

since $l-\alpha>0$ and $Mp-1+p\alpha>0$.

We shall now work towards a proof of (2.8) and (2.9). Consider (2.8) first. By translation, we may assume $x_0=0$ in (2.6). Using the fact that $\int\Delta_j(y)\,dy=0$ and (2.6), we have

$$|\Delta_j*m_Q(x)|=\Big|\int(m_Q(x-t)-m_Q(x))\Delta_j(t)\,dt\Big|$$
$$\le C|P^k|^{(\alpha-1/p-l)}2^{-(j-1)l}(1\vee 2^k|x|)^{-M}\int_{P^{j-1}}|\Delta_j(t)|\,dt$$
$$=C2^{-(j-k)l+k(1/p-\alpha)}(1\vee 2^k|x|)^{-M}.$$

The proof of (2.9) is similar. By (2.7) and (2.5), we have

$$|\Delta_j*m_Q(x)|=\Big|\int m_Q(x-t)(\Delta_j(t)-\Delta_j(x))\,dt\Big|$$
$$\le\int|m_Q(x-t)||\Delta_j(t)-\Delta_j(x)|\,dt$$
$$\le C\Big(\int_{|t|\le 2^{-j}}+\int_{|t|=2^{-j+1}}+\int_{|t|\ge 2^{-j+2}}\Big)2^{k(1/p-\alpha)}$$
$$\times(1\vee 2^k|x-t|)^{-M}|\Delta_j(t)-\Delta_j(x)|\,dt$$
$$=I+II+III.$$

If $|x|\le 2^{-j}$, then $\Delta_j(x)=\Delta_j(t)=2^{j-1}$ in I, $|x-t|=2^{-j+1}$ and $\Delta_j(x)=-\Delta_j(t)=2^{j-1}$ in II, and $|x-t|\ge 2^{-j+2}$ and $\Delta_j(x)=2^{j-1}$ and $\Delta_j(t)=0$ in III. Hence, $I=0$,

$$II=C2^{k(1/p-\alpha)}\int_{|t|=2^{-j+1}}(1\vee 2^{k-j+1})^{-M}|-2^{j-1}-2^{j-1}|\,dt$$
$$=C2^{k(1/p-\alpha)+(j-k)M},$$

and

$$III=C2^{k(1/p-\alpha)}\int_{|t|\ge 2^{-j+2}}(1\vee 2^k|t|)^{-M}2^{j-1}\,dt$$

$$= C2^{k(1/p-\alpha)+j-1} \sum_{n=2}^{j} \int\limits_{|t|=2^{-j+n}} (1 \vee 2^k|t|)^{-M}\, dt$$
$$= C2^{k(1/p-\alpha)+j-1} \sum_{n=2}^{j} 2^{-(k-j+n)M-j+n-1}$$
$$= C2^{k(1/p-\alpha)-(k-j)M} \sum_{n=2}^{j} 2^{n(1-M)}$$
$$= C2^{k(1/p-\alpha)+(j-k)M}, \quad \text{since } M > 1.$$

Therefore we have

$$|\Delta_j * m_Q(x)| \le C2^{k(1/p-\alpha)+(j-k)M} \quad \text{for } |x| \le 2^{-j}.$$

Similarly, we obtain

$$|\Delta_j * m_Q(x)| \le C2^{j(1/p-\alpha)+(j-k)M-nM} \quad \text{for } |x| = 2^{-j+n},\ 1 \le n.$$

Thus we have (2.9). ■

3. Applications. As the first application of Theorem 2 we shall show the strong capacity inequality of the type of the Maz'ya inequality ([3, p. 54, Theorem 1]).

For $0 < p \le \infty$ and a compact subset A of P^0, we set

$$\mathrm{Cap}_{\alpha,p}(A) = \inf\{\|f\|^p_{B^\alpha_{p,p}} : f \ge 1 \text{ on } A,\ f \in B^\alpha_{p,p}(P^0)\}.$$

Basic properties of $\mathrm{Cap}_{\alpha,p}$ will be useful:

$$\mathrm{Cap}_{\alpha,p}(A_1) \le \mathrm{Cap}_{\alpha,p}(A_2), \quad A_1 \subset A_2, \tag{3.1}$$

$$\mathrm{Cap}_{\alpha,p}\Big(\bigcup_i A_i\Big) \le \sum_i \mathrm{Cap}_{\alpha,p}(A_i), \tag{3.2}$$

$$\mathrm{Cap}_{\alpha,p}(\emptyset) = 0, \tag{3.3}$$

$$\mathrm{Cap}_{\alpha,p}(\{x : |f(x)| > \lambda\}) \le \lambda^{-p}\|f\|^p_{B^\alpha_{pp}}. \tag{3.4}$$

THEOREM 3. *If* $0 < p \le \infty$ *and* $\max(1/p-1, 0) < \alpha < \infty$, *then*

$$\int\limits_0^\infty \mathrm{Cap}_{\alpha,p}(\{x \in P^0 : |f(x)| > \lambda\})\lambda^{p-1}\, d\lambda \le C\|f\|^p_{B^\alpha_{pp}}.$$

Proof. Let $f \in B^\alpha_{pp}$. By Theorem 2 we have λ_Q and a_Q such that

$$f = \sum_{j=0}^\infty \sum_{|Q|=2^{-j}} \lambda_Q a_Q \quad \text{and} \quad \Big(\sum_{j=0}^\infty \sum_{|Q|=2^{-j}} |\lambda_Q|^p\Big)^{1/p} \le \|f\|_{B^\alpha_{pp}}.$$

We may assume Q's are mutually disjoint. By (3.2), we have

$$\int\limits_0^\infty \mathrm{Cap}_{\alpha,p}(\{x \in P^0 : |f(x)| > \lambda\})\lambda^{p-1}\, d\lambda$$
$$= \sum_{n=-\infty}^\infty \int\limits_{2^n}^{2^{n+1}} \mathrm{Cap}_{\alpha,p}(\{x \in P^0 : |f(x)| > \lambda\})\lambda^{p-1}\, d\lambda$$
$$\le \sum_{n=-\infty}^\infty 2^{np}\mathrm{Cap}_{\alpha,p}\Big(\Big\{x \in P^0 : \sum_{j=0}^\infty \sum_{|Q|=2^{-j}} |\lambda_Q||a_Q(x)| > 2^n\Big\}\Big)$$
$$\le \sum_{j=0}^\infty \sum_{|Q|=2^{-j}} \sum_{n=-\infty}^\infty 2^{np}\mathrm{Cap}_{\alpha,p}(\{x \in P^0 : |\lambda_Q||a_Q(x)| \ge 2^n\}).$$

If $|Q| = 2^{-j}$, then $n \le \log_2|\lambda_Q| - j(\alpha-1/p) + \log_2 C$ follows from the atom condition (2.2), and

$$\mathrm{Cap}_{\alpha,p}(\{x \in P^0 : |\lambda_Q||a_Q(x)| \ge 2^n\}) \le \mathrm{Cap}_{\alpha,p}(Q) \le \|\Phi_Q\|^p_{B^\alpha_{pp}}.$$

By translation, we may assume $Q = P^j$. Now an easy calculation shows that

$$\|\Phi_Q\|^p_{B^\alpha_{pp}} = \sum_{m=0}^\infty 2^{\alpha pm}\|\Delta_m * \Phi_j\|^p_p$$
$$= \sum_{m=0}^j 2^{\alpha pm}2^{-pj+(m-1)(p-1)} = C2^{(\alpha-1/p)pj},$$

since $\alpha p + p - 1 > 0$. Hence

$$\int\limits_0^\infty \mathrm{Cap}_{\alpha,p}(\{x \in P^0 : |f(x)| > \lambda\})\lambda^{p-1}d\lambda$$
$$\le \sum_{j=0}^\infty \sum_{|Q|=2^{-j}} \sum_{n=-\infty}^{\log_2|\lambda_Q|-j(\alpha-1/p)+\log_2 C} C2^{np+(\alpha p-1)j}$$
$$= C\sum_{j=0}^\infty \sum_{|Q|=2^{-j}} |\lambda_Q|^p = C\|f\|^p_{B^\alpha_{pp}}. \ ■$$

As the second application of Theorem 2 we shall show a weak type estimate for maximal Cesàro means.

We list some properties of $K_n^\beta(x)$ and $\dot{K}_n^\beta(x)$ for $n = \sum_{i=0}^s b_i 2^i$, $b_s = 1$, $b_i = 0$ or 1 (see [15], [5, p. 46]):

(3.5) $$A_{n-1}^{\beta}K_n^{\beta}(x) = \sum_{i=0}^{s} b_i w_{2^s+\ldots+2^i-1}(x)\Big\{-\sum_{k=1}^{2^i-2} kK_k^1(x)A^{\beta-2}_{n-2^s-\ldots-2^i+k+1} - (2^i-1)K^1_{2^i-1}(x)A^{\beta-1}_{n-2^s-\ldots-2^{i+1}-1} + D_{2^i}(x)A^{\beta}_{n-2^s-\ldots-2^{i+1}-1}\Big\},$$

(3.6) $$|nK_n^1(x)| \le 3n\dot{K}_n^1(x) = 3\sum_{i=0}^{s} b_i 2^i K^1_{2^i}(x),$$

(3.7) $$K^1_{2^j}(x) = (2^{j-1}+1/2)\Phi_j(x) + \sum_{r=0}^{j-1} 2^{r-1}\Phi_j(x-\wp^r).$$

Let $a(x)$ be an (α,p)-atom supported by x_0+P^u. Then

(3.8) $$(|a|*\Phi_i)(x) \le C\Phi_{i\wedge u}(x-x_0)2^{-u(\alpha-1/p)-u\vee i},$$

where $i\wedge u = \min[i,u]$ and $i\vee u = \max[i,u]$.

In fact, by (2.2),

$$\begin{aligned}(|a|*\Phi_i)(x) &= \int_{(x+P^i)\cap(x_0+P^u)} |a(t)|\,dt \\ &= \Phi_{i\wedge u}(x-x_0)\int_{(x+P^i)\cap(x_0+P^u)} |a(t)|\,dt \\ &\le C\Phi_{i\wedge u}(x-x_0)|(x+P^i)\cap(x_0+P^u)||P^u|^{\alpha-1/p} \\ &= C\Phi_{i\wedge u}(x-x_0)2^{-u(\alpha-1/p)-u\vee i}.\end{aligned}$$

From (3.7) and (3.8) it will follow that

(3.9) $$\sigma_{2^l}|a|(x) \le C2^{-u(\alpha-1/p+1)}\Big\{(2^{l-1}+2^{-1})\Phi_l(x-x_0) + \sum_{r=0}^{l-1} 2^{r-1}\Phi_l(x-x_0-\wp^r)\Big\} \quad \text{if } l\le u$$

and

(3.10) $$\sigma_{2^l}|a|(x) \le C2^{-u(\alpha-1/p)}\Big\{2\Phi_u(x-x_0) + \sum_{r=0}^{l-1} 2^{r-l-1}\Phi_u(x-x_0-\wp^r)\Big\} \quad \text{if } l > u.$$

We shall use the following obvious estimate.

If $0\le i\le j$ and $0\le k$, then

(3.11) $$\sum_{l=i}^{j} 2^{kl}\Phi_l(x) \le C\frac{\Phi_i(x)}{|x|^k}.$$

THEOREM 4. *Suppose $1/2\le p<1$, $0<q\le\infty$, $\alpha>0$, and $\beta=1/p-1$. If $f\in B^{\alpha}_{pq}$ then for all $\lambda>0$,*

$$|\{x\in P^0 : \sup_n |\sigma_n^{\beta} f(x)| > \lambda\}| \le (C\|f\|_{B^{\alpha}_{pq}}/\lambda)^p.$$

We shall prove below the following lemma.

LEMMA 1. *Suppose $1/2\le p<1$ and $\beta=1/p-1$. If $a_Q(x)$ is a non-constant (α,p)-atom supported in Q then*

$$|\{x\in P^0 : \sup_n |\sigma_n^{\beta} a_Q(x)| > \lambda > 0\}| \le (C|Q|^{\alpha}/\lambda)^p.$$

If $a(x)$ is constant on P^0 then

$$|\{x\in P^0 : \sup_n |\sigma_n^{\beta} a_Q(x)| > \lambda > 0\}| \le (C/\lambda)^p.$$

Theorem 4 follows directly from Lemma 2 whose proof is similar to [7, p. 85, Lemma (1.8)].

LEMMA 2. *Suppose $0<p<1$ and $\{a_Q\}$ is a sequence of (α,p)-atoms such that for each Q and each $\lambda>0$,*

$$|\{x\in P^0 : |a_Q(x)| > \lambda > 0\}| \le (C|Q|^{\alpha}/\lambda)^p.$$

If $\{\lambda_Q\}$ is a sequence such that $\{\sum(|\lambda_Q|^p)^{q/p}\}^{1/q} \le \|f\|_{B^{\alpha}_{pq}}$, then

$$\Big|\Big\{x\in P^0 : \Big|\sum\sum \lambda_Q a_Q\Big| > \lambda\Big\}\Big| \le (C\|f\|_{B^{\alpha}_{pq}}/\lambda)^p.$$

Proof of Lemma 1. This proof is based on [16]. There is no loss in generality if we assume $a(x)$ to be supported in P^u. We have $\widehat{a}(k)=0$ for $0\le k<2^u$ and $S_n a(x) = \sigma_n^{\beta} a(x) = 0$ for $0\le n\le 2^u$. Then we assume $n>2^u$. The proof is carried out for the number $n=\sum_{i=0}^{[\log n]} b_i 2^i$, $b_{[\log n]}=1$, $b_i=0$ or 1 and $1/2<p<1$. We have, from (3.5),

$$|\sigma_n^{\beta} a(x)| \le \frac{C}{n^{\beta}}\sum_{i=u+1}^{[\log n]}\Big\{\sum_{j=1}^{2^i-1}(2^i+j)^{\beta-2} j|\sigma_j a(x)| + 2^{i(\beta-1)}(2^i-1)|\sigma_{2^i-1}a(x)| + 2^{i\beta}|S_{2^i}a(x)|\Big\}$$

$$= \frac{C}{n^\beta}\sum_{i=u+1}^{[\log n]}\sum_{j=1}^{2^u} 2^{i(\beta-2)} j|\sigma_j a(x)| + \frac{C}{n^\beta}\sum_{i=u+1}^{[\log n]}\sum_{j=2^u+1}^{2^i-1} j^{(\beta-2)} j|\sigma_j a(x)|$$
$$+ \frac{C}{n^\beta}\sum_{i=u+1}^{[\log n]} 2^{i(\beta-1)}(2^i-1)|\sigma_{2^i-1}a(x)| + \frac{C}{n^\beta}\sum_{i=u+1}^{[\log n]} 2^{i\beta}|S_{2^i}a(x)|$$
$$= I + II + III + IV.$$

Now, $I = 0$ since $\sigma_j a(x) = 0$ for $1 \le j \le 2^u$. We estimate $II + III$ by making use of (3.6), (3.9), (3.10) and (3.11):

$$II + III$$
$$\le C \sup_{j>2^u} \dot\sigma_j |a|(x) \frac{1}{n^\beta}\Big\{\sum_{i=u+1}^{[\log n]}\sum_{j=2^u+1}^{2^i-1} j^{\beta-1} + \sum_{i=u+1}^{[\log n]} 2^{i\beta}\Big\}$$
$$\le C \sup_{j>2^u} \dot\sigma_j |a|(x)$$
$$\le C \sup_{j>2^u}\Big\{\frac{1}{j}\sum_{i=0}^{u} 2^i\sigma_{2^i}|a|(x) + \frac{1}{j}\sum_{i=u+1}^{[\log j]} 2^i\sigma_{2^i}|a|(x)\Big\}$$
$$\le C2^{-u(\alpha-1/p+2)}\sum_{i=0}^{u}\Big\{2^{2i}\Phi_i(x) + C\sum_{r=0}^{i-1} 2^{r+i}\Phi_i(x-\wp^r)\Big\}$$
$$+ C\sup_{i>u}\sigma_{2^i}|a|(x)$$
$$\le C2^{-u\alpha}\Big\{\sum_{i=0}^{u} 2^{i/p}\Phi_i(x) + \sum_{r=0}^{u-1} 2^{r\beta}\sum_{i=r+1}^{u} 2^i\Phi_i(x-\wp^r)\Big\} + C\sup_{i>u}\sigma_{2^i}|a|(x)$$
$$\le C2^{-u\alpha}\Big\{|x|^{-1/p}\Phi_0(x) + \sum_{r=0}^{u-1} 2^{r\beta}\frac{\Phi_{r+1}(x-\wp^r)}{|x-\wp^r|}\Big\}$$
$$+ C2^{-u(\alpha-1/p)}\Big[\Phi_u(x) + \sup_{i>u}\sum_{r=0}^{i-1} 2^{r-i}\Phi_u(x-\wp^r)\Big].$$

Finally, we estimate IV by (3.8):

$$IV \le \frac{C}{n^\beta}\sum_{i=u+1}^{[\log n]} 2^{i(\beta+1)}(\Phi_i * |a|)(x)$$
$$\le \frac{C}{n^\beta} 2^{-u(\alpha-1/p)}\Phi_u(x)\sum_{i=u+1}^{[\log n]} 2^{i\beta} \le C2^{-u(\alpha-1/p)}\Phi_u(x).$$

Collecting these estimates, we have

$$|\{x : |\sigma_n^\beta a(x)| > \lambda\}|$$
$$\le \Big|\Big\{x : C2^{-u\alpha}|x|^{-1/p}\Phi_0(x) + C2^{-u\alpha}\sum_{r=0}^{u-1} 2^{r\beta}\frac{\Phi_{r+1}(x-\wp^r)}{|x-\wp^r|} > \frac{\lambda}{2}\Big\}\Big|$$
$$+ \Big|\Big\{x : C2^{-u(\alpha-1/p)}\Phi_u(x) + C\sup_{i>u}\sum_{r=0}^{i-1} 2^{r-i}\Phi_u(x-\wp^r) > \frac{\lambda}{2}\Big\}\Big|$$
$$\le \Big|\Big\{x : \frac{C2^{-u\alpha}}{\lambda} > |x|^{1/p}\Big\}\Big|$$
$$+ C\sum_{r=0}^{u-1}\Big|\Big\{x : \frac{C2^{-u\alpha}}{\lambda}2^{r\beta}\Phi_{r+1}(x-\wp^r) > |x-\wp^r|\Big\}\Big|$$
$$+ \Big|\Big\{x : \frac{C2^{-u\alpha}}{\lambda}\Phi_u(x) > 2^{-u/p}\Big\}\Big|$$
$$+ C\sup_{i>u}\sum_{r=0}^{i-1}\Big|\Big\{x : \frac{C2^{-u\alpha}}{\lambda}2^{r-i}\Phi_u(x-\wp^r) > 2^{-u/p}\Big\}\Big|$$
$$\le \Big(\frac{C2^{-u\alpha}}{\lambda}\Big)^p + \Big\{\frac{C2^{-u\alpha}}{\lambda}\sum_{r=0}^{r_0} 2^{r\beta} + \sum_{r=r_0}^{u-1} 2^{-(r+1)}\Big\}$$
$$+ \Big(\frac{C2^{-u\alpha}}{\lambda}\Big)^p \sup_{i>u}\sum_{r=0}^{i-1} 2^{(r-i)p}$$
$$= \Big(\frac{C2^{-u\alpha}}{\lambda}\Big)^p, \quad \text{where } r_0 = \log_2\Big(\frac{C2^{-u\alpha}}{\lambda}\Big)^{-p}.$$

When $p = 1/2$, $I = II = 0$ and we can estimate III and IV in the same way. Thus we proved Lemma 2. ■

Remark. We can see by the same technique of the proof of the theorem that Theorem 4 is valid for $0 < p < 1/2$. However, we shall need a more precise formula for K_n^α and an inductive proof for a range of p.

As the third application of Theorem 1 we shall show a sufficient condition for the absolute convergence of the Walsh–Fourier series. The well known Bernstein–Stechkin criterion says

$$(3.12) \qquad \sum_{j=0}^{\infty} 2^{j/2}E_2(2^j, f) < \infty \quad \text{implies} \quad \sum_{k=0}^{\infty}|(f, w_k)| < \infty,$$

where (f, w_k) denotes the action of $f \in S'(P^0)$ on $w_k \in S(P^0)$ (see [14]).

The following theorem includes [5, p. 64, Theorem 9] whose assumption is given by $\|S_{\infty}^{1/p} f\|_{p,1}$ in our notation.

THEOREM 5. *If* $f \in B_{p,1}^{1/p}$ $(0 < p \le 2)$, *then* $\sum_{k=0}^{\infty} |(f, w_k)| < \infty$.

Proof. From Theorem A(i) and the definition of $B_{p,q}^{\alpha}$, we have $B_{p,1}^{1/p} \subset B_{2,1}^{1/2}$ for $0 < p \le 2$. Thus, we obtain the theorem from (3.12) and Theorem 1. ■

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Weighted Orlicz space integral inequalities for the Hardy–Littlewood maximal operator

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Abstract. Necessary and sufficient conditions are given for the Hardy–Littlewood maximal operator to be bounded on a weighted Orlicz space when the complementary Young function satisfies Δ_2. Such a growth condition is shown to be necessary for any weighted integral inequality to occur. Weak-type conditions are also investigated.

1. Introduction. For an N-function Φ, the Orlicz space $L_\Phi(X, d\mu)$ is the Banach space normed by

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\}.$$

The usual Lebesgue spaces $L^p(X, d\mu)$, $1 < p < \infty$, arise from the N-function $\Phi(x) = x^p/p$. These Lebesgue spaces satisfy a Δ_2 condition; that is,

$$\Phi(2x) \le C\Phi(x).$$

Most papers trying to describe a weighted operator theory in an Orlicz space setting have made Δ_2 assumptions. It is easy to see where these arise. Marcinkiewicz interpolation is one of our most cherished tools. Suppose we start with a sublinear operator T. Then T is of type (∞, ∞) and of weak-type $(1, 1)$ if and only if, for each $\lambda > 0$,

$$|[x : Tf(x) > \lambda]| \le \frac{C}{\lambda} \int_{[x:|f(x)|>\lambda/K]} |f(x)|\,dx$$

where $K = 2\|T\|_\infty$. Writing Φ as

$$\Phi(x) = \int_0^x \phi(t)\,dt,$$

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