The following theorem includes [5, p. 64, Theorem 9] whose assumption is given by $\|S_\infty^{1/p} f\|_{p,1}$ in our notation.

THEOREM 5. *If* $f \in B_{p,1}^{1/p}$ $(0 < p \le 2)$, *then* $\sum_{k=0}^{\infty} |(f, w_k)| < \infty$.

Proof. From Theorem A(i) and the definition of $B_{p,q}^{\alpha}$, we have $B_{p,1}^{1/p} \subset B_{2,1}^{1/2}$ for $0 < p \le 2$. Thus, we obtain the theorem from (3.12) and Theorem 1. ■

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Weighted Orlicz space integral inequalities for the Hardy–Littlewood maximal operator

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Abstract. Necessary and sufficient conditions are given for the Hardy–Littlewood maximal operator to be bounded on a weighted Orlicz space when the complementary Young function satisfies Δ_2. Such a growth condition is shown to be necessary for any weighted integral inequality to occur. Weak-type conditions are also investigated.

1. Introduction. For an N-function Φ, the Orlicz space $L_\Phi(X, d\mu)$ is the Banach space normed by

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \le 1\right\}.$$

The usual Lebesgue spaces $L^p(X, d\mu)$, $1 < p < \infty$, arise from the N-function $\Phi(x) = x^p/p$. These Lebesgue spaces satisfy a Δ_2 condition; that is,

$$\Phi(2x) \le C\Phi(x).$$

Most papers trying to describe a weighted operator theory in an Orlicz space setting have made Δ_2 assumptions. It is easy to see where these arise. Marcinkiewicz interpolation is one of our most cherished tools. Suppose we start with a sublinear operator T. Then T is of type (∞, ∞) and of weak-type $(1, 1)$ if and only if, for each $\lambda > 0$,

$$|[x : Tf(x) > \lambda]| \le \frac{C}{\lambda} \int_{[x:|f(x)|>\lambda/K]} |f(x)|\, dx$$

where $K = 2\|T\|_\infty$. Writing Φ as

$$\Phi(x) = \int_0^x \phi(t)\, dt,$$

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we have

$$\int \Phi(Tf)\,dx = \int \phi(\lambda) \int\limits_{[x:|Tf(x)|>\lambda]} dx\,d\lambda \le C \int\limits_0^\infty \frac{\phi(\lambda)}{\lambda} \int\limits_{[|f|>\lambda/K]} |f(x)|\,dx\,d\lambda$$

$$= C \int |f(x)| \int\limits_0^{K|f(x)|} \phi(\lambda)\,\frac{d\lambda}{\lambda}\,dx \le \int \Phi(C|f(x)|)\,dx$$

provided ϕ satisfies a Dini condition, that is,

$$\int\limits_0^x \frac{\phi(s)}{s}\,ds \le C\phi(x).$$

ϕ Dini is equivalent to a Δ_2 condition on the N-function complementary to Φ. So an operator like the Hardy–Littlewood maximal operator M satisfies an L_Φ integral inequality provided this complementary Δ_2 condition holds. The converse is also true [14].

To avoid Δ_2, one needs to improve this argument, or avoid interpolation. In [3], we characterized weighted L_Φ integral inequalities for generalized Hardy operators, obtaining both weak- and strong-type inequalities with no Δ_2 assumptions. With hopes high, we turned to the Hardy–Littlewood maximal operator.

The weighted theory for the maximal operator,

$$Mf(x) = \sup\left\{\frac{1}{|I|}\int\limits_I |f(y)|\,dy : I \text{ is a cube in } \mathbb{R}^n \text{ containing } x\right\}$$

is quite beautiful. A weight w is a function on $\mathbb{R}^n$ which is positive and finite almost everywhere. w belongs to the Muckenhoupt A_p class if

$$\left(\frac{1}{|I|}\int\limits_I w(x)\,dx\right)\left(\frac{1}{|I|}\int\limits_I w(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C$$

for all cubes I. The maximal operator M is a bounded operator from $L^p(\mathbb{R}^n, w(x)\,dx)$ to $L^p(\mathbb{R}^n, w(x)\,dx)$ if and only if $w \in A_p$, $1 < p < \infty$. See [10].

In [7], Kerman and Torchinsky extended this to Orlicz spaces. They examined the inequality

$$\int\limits_{\mathbb{R}^n} \Phi(Mf(x))w(x)\,dx \le \int\limits_{\mathbb{R}^n} \Phi(C|f(x)|)w(x)\,dx. \tag{1}$$

Assuming both Φ and its complement were in Δ_2, they showed that (1) holds if and only if $w \in A_p$, where $1/p$ is the Boyd upper index of Φ.

Since then, researchers have been chipping away at the Δ_2 assumptions. Bagby [1], Gogatishvili and Pick [6], Pick [11], and Quinsheng [12] have all tackled weak-type problems for the maximal operator. Pick and Quinsheng do strong-type as well. Pick has made the most progress. He showed that (1), even with two weights, forces a local Δ_2 condition on the complement to Φ. This suggests that the Dini condition used for interpolating is actually indispensable. That turns out to be true.

We say an N-function $\Phi \in \Delta_2^{\mathrm{c}}$ if its complement (see Section 2) belongs to Δ_2. Our main result is

THEOREM 1. *Let Φ be an* N-*function and let w be a weight on $\mathbb{R}^n$. Then the following are equivalent:*

(a) *For all f,*

$$\int\limits_{\mathbb{R}^n} \Phi(Mf(x))w(x)\,dx \le \int\limits_{\mathbb{R}^n} \Phi(C|f(x)|)w(x)\,dx.$$

(b) $\Phi \in \Delta_2^{\mathrm{c}}$ *and the weak-type boundedness*

$$\Phi(\lambda)w([x : Mf(x) > \lambda]) \le \int\limits_{\mathbb{R}^n} \Phi(C|f(x)|)w(x)\,dx$$

holds for all f.

(c) $\Phi \in \Delta_2^{\mathrm{c}}$ *and w satisfies the condition*

$$\int\limits_I \Psi\left(\frac{\Phi(\lambda)w(I)}{C\lambda|I|w(x)}\right)w(x)\,dx \le \Phi(\lambda)w(I) < \infty \tag{2}$$

for all cubes I, where Ψ is the complementary Young function to Φ.

This paper is organized into three further sections. In the next section, we describe the Orlicz space theory that we will use. Section 3 gives weak-type results. These are quite general, with two Young functions and four weights. Some interesting consequences are described in the special one Φ, one weight setting. In the last section, we present the strong-type theory and prove Theorem 1.

2. Orlicz spaces. The standard theory of Orlicz spaces can be found in Zygmund [15], Krasnosel'skiĭ and Rutitskiĭ [9], or Rao and Ren [13], and weighted theory is developed in the recent book by Kokilashvili and Krbec [8]. An N-function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative, convex function satisfying

$$\lim_{x\to 0} \frac{\Phi(x)}{x} = 0, \qquad \lim_{x\to\infty} \frac{\Phi(x)}{x} = \infty.$$

Φ has a derivative ϕ which is nondecreasing and nonnegative, $\phi(0) = 0$, and $\phi(\infty) = \infty$, so that

$$\Phi(x) = \int\limits_0^x \phi(t)\,dt,$$

and we can and will take ϕ to be right-continuous.

Set

$$\psi(x) = \inf\{y : \phi(y) \geq x\}$$

and

$$\Psi(x) = \int_0^x \psi(y)\,dy.$$

This is also an N-function, and is known as the complementary Young function to Φ. We have Young's inequality

$$ab \leq \Phi(a) + \Psi(b) \qquad \text{for all } a, b > 0,$$

and Ψ satisfies

$$\Psi(x) = \sup_y\{xy - \Phi(y)\}.$$

If $(X, d\mu)$ is a σ-finite measure space, then the Orlicz space $L_\Phi = L_\Phi(X, d\mu)$ is the Banach space on which we take the Luxemburg norm

$$\|f\|_\Phi = \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1\right\}.$$

The original norm used by Orlicz is

$$\|f\|_\Phi^* = \sup\left\{\left|\int fg\,d\mu\right| : \int \Psi(|g|)\,d\mu \leq 1\right\}.$$

These two norms are equivalent, and one has the Hölder inequality

$$\left|\int fg\,d\mu\right| \leq C\|f\|_\Phi\|g\|_\Psi.$$

The following easy lemma is taken from [3].

LEMMA 2. *Let Φ be an N-function with complementary function Ψ. Let x and $y > 0$. Then*

$$\Phi(x) \leq x\phi(x) \leq \Phi(2x), \tag{3}$$

$$\Phi(x) + \Phi(y) \leq \Phi(x + y) \tag{4}$$

and

$$\Phi\left[\frac{\Psi(x)}{x}\right] \leq \Psi(x). \tag{5}$$

We will also need the inequality

$$x \leq \Phi^{-1}(x)\Psi^{-1}(x) \leq 2x. \tag{6}$$

The left inequality follows immediately from (5), and the other is just Young's inequality.

We say $\Phi \in \Delta_2$ if $\Phi(2x) \leq C\Phi(x)$ for all $x > 0$, and $\Phi \in$ local Δ_2 if this holds for all $x \geq x_0$. $\Phi \in \Delta_2^c$ if the complement Ψ is in Δ_2.

C will always denote a universal constant, and may change in subsequent appearances.

PROPOSITION 3 (Bari and Stechkin [2]). *Let Φ be an N-function with derivative ϕ. Then the following are equivalent:*

(a) *ϕ is Dini, i.e.,*

$$\int_0^x \frac{\phi(s)}{s}\,ds \leq C\phi(x)$$

for all $x > 0$.

(b) *There exists a $\delta > 0$ such that $\phi(\delta x) \leq \frac{1}{2}\phi(x)$ for all $x > 0$.*

(c) *$\Phi \in \Delta_2^c$.*

Proof. (a)⇒(b). Let Ψ be the complement and let $0 < \delta < 1$. We have

$$\int_0^x \frac{\phi(y)}{y}\,dy = \int_0^x \frac{1}{y}\int_0^y d\phi(s)\,dy = \int_0^x d\phi(s)\log\frac{x}{s} \tag{7}$$
$$\geq \int_0^{\delta x} d\phi(s)\log\frac{x}{s} \geq \left(\log\frac{1}{\delta}\right)\phi(\delta x).$$

Choose δ so that $\log\frac{1}{\delta} = 2C$. Then the Dini condition gives

$$2\phi(\delta x) \leq \phi(x).$$

(b)⇒(c). Fix y and put $x > \psi(y)/\delta$. Since $\psi(y) = \inf\{t : \phi(t) \geq y\}$ and ϕ is nondecreasing, we must have $y \leq \phi(\delta x)$. Thus $2y \leq \phi(x)$, or

$$\psi(2y) \leq \psi(\phi(x)) \leq x.$$

Since this holds for every $x > \psi(y)/\delta$, we in fact have

$$\psi(2y) \leq \frac{1}{\delta}\psi(y)$$

and Δ_2 follows on integrating ψ.

(c)⇒(a). Finally, if $\Psi \in \Delta_2$, then

$$\psi(2^n x) \leq C^n\psi(x), \qquad \text{for } n = 0, 1, 2, \ldots,$$

and we can take $C > 1$, obviously. So

$$\int_0^x \frac{\phi(y)}{y}\,dy = \sum_{n=0}^{\infty}\int_{x/C^{n+1}}^{x/C^n} \frac{\phi(y)}{y}\,dy \leq \log C\sum_n \phi\left(\frac{x}{C^n}\right).$$

But

$$\psi\left(2^n\phi\left[\frac{x}{C^n}\right]\right) \leq C^n\psi\left[\phi\left(\frac{x}{C^n}\right)\right] \leq x$$

and by the definition of ψ,

$$2^n\phi\left(\frac{x}{C^n}\right) \leq \phi(x).$$

Hence,

$$\int_0^x \frac{\phi(y)}{y}\,dy \le 2(\log C)\phi(x).$$

We would like to thank our referee for pointing out the reference [2].

There is an important connection between norm inequalities and integral inequalities. This next proposition is from [3].

PROPOSITION 4. *Suppose T is a linear operator acting from a σ-finite measure space $(X, d\mu)$ to a σ-finite measure space $(Y, d\nu)$. Let Φ be an* N-*function and let $L_{\Phi,\varepsilon d\mu}(X)$ be the Orlicz space with the norm*

$$\|f\|_{\Phi,\varepsilon d\mu} = \inf\left\{\lambda > 0 : \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right)\varepsilon\,d\mu(x) \le 1\right\}$$

and let $L_{\Phi,\varepsilon d\nu}(Y)$ be defined similarly. Then

$$\int_Y \Phi(|Tf(y)|)\,d\nu(y) \le \int_X \Phi(C|f(x)|)\,d\mu(x)$$

if and only if

$$\|Tf\|_{\Phi,\varepsilon d\nu} \le C\|f\|_{\Phi,\varepsilon d\mu}$$

for all $\varepsilon > 0$, with C independent of ε.

3. Weak-type integral inequalities. The complete characterization of weak-type weighted L_Φ inequalities for monotone operators on $\mathbb{R}^+$ was given in Theorem 3.1 of [3]. The argument used there can be applied to other operators. For example, it is easy to adapt it to obtain

THEOREM 5. *Let $0 \le \alpha < n$, and let $M_\alpha f(x)$ be the fractional maximal operator on $\mathbb{R}^n$,*

$$M_\alpha f(x) = \sup\left\{|I|^{\alpha/n-1} \int_I |f(y)|\,dy : I \text{ is a cube containing } x\right\}.$$

Let t, u, v, and w be weights on $\mathbb{R}^n$, Φ_1 and Φ_2 be N-*functions with complements Ψ_1 and Ψ_2 respectively. Assume further that $\Phi_2 \circ \Phi_1^{-1}$ is convex. Then weak-type boundedness,*

$$\Phi_2^{-1}\Big[\int_{[M_\alpha f > \lambda]} \Phi_2(\lambda w(x))t(x)\,dx\Big] \le \Phi_1^{-1}\Big[\int_{\mathbb{R}^n} \Phi_1(C|f(x)|u(x))v(x)\,dx\Big], \tag{8}$$

holds if and only if

$$\int_I \Psi_1\left[\frac{\gamma(\lambda, I)}{C\lambda u(y)v(y)}|I|^{\alpha/n-1}\right]v(y)\,dy \le \gamma(\lambda, I) < \infty \tag{9}$$

holds for each cube I, where

$$\gamma(\lambda, I) = \Phi_1 \circ \Phi_2^{-1}\Big[\int_I \Phi_2(\lambda w(y))t(y)\,dy\Big].$$

A couple of remarks about this theorem may be in order. In the Lebesgue setting, and in an expression like $\int(|f(x)|u(x))^p v(x)\,dx$, the weights u and v can obviously be combined. This is not true for general Φ, and that necessitates confronting such unpleasant four-weight inequalities. The convexity assumption in the theorem corresponds to the Riesz triangle $p \le q$ in the Lebesgue setting.

Proof of Theorem 5. The necessity is essentially the argument of [3], with ε chosen so that

$$\int_E \Psi_1\left(\frac{\varepsilon}{uv}\right)\frac{v(x)}{\varepsilon}\,dx = 2C\lambda|I|^{1-\alpha/n}, \qquad E \subset I,$$

and

$$f(x) = \frac{1}{C}\Psi_1\left(\frac{\varepsilon}{uv}\right)\frac{v(x)}{\varepsilon}\chi_E(x).$$

For the sufficiency, let $\Gamma \subset [x : M_\alpha f(x) > \lambda]$ be a compact set. For each $x \in \Gamma$, there exists an open cube I with $|I|^{\alpha/n-1}\int_I |f| > \lambda$, and Γ is covered by finitely many of these cubes. Let I_1 be the largest, and I_{k+1} the largest remaining cube disjoint from $I_1 \cup \ldots \cup I_k$. Let J_k be the cube concentric with I_k but triple its side length. We have

$$\lambda < |I_k|^{\alpha/n-1}\int_{I_k} |f|, \qquad \Gamma \subset \bigcup J_k,$$

and the I_k's are disjoint. Hence,

$$\begin{aligned}
2\gamma(\lambda, J_k) &\le \int_{I_k} |f(x)|\frac{|I_k|^{\alpha/n-1}}{\lambda}2\gamma(\lambda, J_k)\,dx \\
&= \int_{I_k} 2\cdot 3^{n-\alpha}C|f(x)|u(x)\cdot\frac{\gamma(\lambda, J_k)}{C\lambda u(x)v(x)}|J_k|^{\alpha/n-1}v(x)\,dx \\
&\le \int_{I_k} \Phi_1(2\cdot 3^{n-\alpha}C|f(x)|u(x))v(x)\,dx \\
&\quad + \int_{I_k} \Psi_1\left(\frac{\gamma(\lambda, J_k)}{C\lambda u(x)v(x)}|J_k|^{\alpha/n-1}\right)v(x)\,dx \\
&\le \int_{I_k} \Phi_1(2\cdot 3^{n-\alpha}C|f(x)|u(x))v(x)\,dx + \gamma(\lambda, J_k),
\end{aligned}$$

so

$$\gamma(\lambda, J_k) \le \int_{I_k} \Phi_1(2\cdot 3^{n-\alpha}C|f(x)|u(x))v(x)\,dx$$

and thus

$$\int_{J_k} \Phi_2(\lambda w(y))t(y)\,dy \le \Phi_2\circ\Phi_1^{-1}\Big[\int_{I_k} \Phi_1(2\cdot 3^{n-\alpha}C|f(x)|u(x))v(x)\,dx\Big].$$

Summing over k gives

$$\int_{\Gamma} \Phi_2(\lambda w(y))t(y)\,dy \le \sum_k \Phi_2\circ\Phi_1^{-1}\Big[\int_{I_k} \Phi_1(2\cdot 3^{n-\alpha}C|f(x)|u(x))v(x)\,dx\Big].$$

But (4) applies to $\Phi_2\circ\Phi_1^{-1}$, so this last sum is bounded by

$$\Phi_2\circ\Phi_1^{-1}\Big(\sum_k \int_{I_k} \Phi_1(2\cdot 3^{n-\alpha}C|f(x)|u(x))v(x)\,dx\Big)$$
$$\le \Phi_2\circ\Phi_1^{-1}\Big[\int_{\mathbb{R}^n} \Phi_1(2\cdot 3^{n-\alpha}C|f(x)|u(x))v(x)\,dx\Big],$$

and that proves the theorem.

This last result has some history. For the Hardy–Littlewood maximal operator ($\alpha = 0$), Bagby did a one-weight version of this, when $\Phi_1 = \Phi_2$. We say a weight $w \in W_\Phi$ if

$$\Phi(\lambda)w([x : Mf(x) > \lambda]) \le \int_{\mathbb{R}^n} \Phi(C|f(x)|)w(x)\,dx.$$

Bagby, in [1], characterized the weights for which a slightly modified form of this inequality holds. This is the first weighted Orlicz space paper that completely escaped the Δ_2 conditions, and we are quite indebted to this work. Pick extended this to a two-weight setting [11], assuming a doubling condition, which he and Gogatishvili eliminated in [6]. For a further generalization, see [5].

In the Lebesgue setting, for $1 < p < \infty$, it is well known that M is of weak-type (p,p) with respect to w if and only if it is of strong-type (p,p) with respect to w, and this is equivalent to $w \in A_p$. For Orlicz spaces L_Φ, with Φ and its complement Ψ satisfying Δ_2, the Kerman–Torchinsky A_Φ condition and the condition (9) are equivalent. What happens away from Δ_2?

Recall that a weight $w \in A_p$, $1 < p < \infty$, provided

$$\Big(\frac{1}{|I|}\int_I w(x)\,dx\Big)\Big(\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\Big)^{p-1} \le C$$

for all cubes I. The limiting condition as $p \to 1^+$,

$$\frac{1}{|I|}\int_I w(x)\,dx \le C \operatorname*{ess\,inf}_I w(x),$$

is denoted by A_1. The A_p classes are nested, that is, $A_p \subset A_q$ if $p < q$, and the union of these classes is customarily called A_∞. The L_Φ integral inequality for the maximal operator proven in [7] relied heavily on the "reverse Hölder" property of A_∞ weights. It would be nice if A_∞ were necessary beyond Δ_2. Nice, and also true:

THEOREM 6. *We have*

$$\bigcap_{\text{N-functions } \Phi} W_\Phi = A_1, \qquad \bigcup_{\text{N-functions } \Phi} W_\Phi = A_\infty.$$

Proof. Let $w(I)$ denote $\int_I w(x)\,dx$. By Theorem 5, if Φ is an N-function with complement Ψ, then $w \in W_\Phi$ if and only if, for each $\lambda > 0$ and cube I,

$$\int_I \Psi\Big(\frac{\Phi(\lambda)w(I)}{C\lambda|I|w(x)}\Big)w(x)\,dx \le \Phi(\lambda)w(I) < \infty. \tag{10}$$

So assume that $w \in A_1$, and take $\Psi(x) = \int_0^x \psi(y)\,dy$, where $\psi(x) = \inf\{y : \phi(y) \ge x\}$. By (3),

$$\int_I \Psi\Big(\frac{\Phi(\lambda)w(I)}{C\lambda|I|w(x)}\Big)w(x)\,dx \le \Phi(\lambda)\frac{w(I)}{C\lambda|I|}\int_I \psi\Big(\frac{\Phi(\lambda)w(I)}{C\lambda|I|w(x)}\Big)\,dx.$$

Now $w \in A_1$ means

$$\frac{w(I)}{|I|} \le C \operatorname*{ess\,inf}_I w(x).$$

So, with this C,

$$\frac{w(I)}{C|I|w(x)} \le 1 \quad \text{a.e. on } I.$$

Thus

$$\psi\Big(\frac{\Phi(\lambda)w(I)}{C\lambda|I|w(x)}\Big) \le \psi(\phi(\lambda)) \le \lambda$$

and (10) follows, so long as C is taken to be ≥ 1.

Conversely, if $w \notin A_1$, it will suffice to construct an N-function Ψ for which (10) fails, for the λ with $\Phi(\lambda) = 1$. In other words, we will construct such a Ψ for which

$$\int_I \Psi\Big[\frac{w(I)}{C|I|w(x)}\Big]w(x)\,dx \le w(I) \tag{11}$$

must fail for every choice of C. Now $w \notin A_1$ means there are cubes I_k on which

$$\frac{1}{|I_k|}\int_{I_k} w \ge 2k^2 \operatorname*{ess\,inf}_{I_k} w.$$

In particular, if

$$E_k = \left[x \in I_k : w(x) < \frac{1}{k^2|I_k|}\int_{I_k} w\right]$$

then

$$t_k = \frac{|E_k|}{|I_k|} > 0.$$

Set

$$a(k) = \max_{1\le j\le k} \frac{j}{t_j}.$$

This is a nondecreasing sequence which tends to infinity. So we can choose a subsequence $\{k_j\}$ on which $a(k_j)$ is strictly increasing. Define $\psi(t)$ to be continuous, strictly increasing, $\psi(0) = 0$, and $\psi(k_j) = a(k_j)$, and let

$$\Psi(x) = \int_0^x \psi(t)\,dt.$$

Were (11) to hold for Ψ, then, by (3), we must have

$$\frac{1}{|I|}\int_I \psi\left(\frac{w(I)}{2C|I|w}\right) \le 2C. \tag{12}$$

But

$$\frac{1}{|I_{k_j}|}\int_{I_{k_j}} \psi\left(\frac{w(I_{k_j})}{k_j|I_{k_j}|w}\right) \ge \frac{1}{|I_{k_j}|}\int_{E_{k_j}} \psi\left(\frac{w(I_{k_j})}{k_j|I_{k_j}|w}\right)$$
$$\ge \psi(k_j)\frac{|E_{k_j}|}{|I_{k_j}|} = a(k_j)t_{k_j} \ge k_j$$

so that (12) fails.

For the union, clearly $A_\infty \subset \bigcup W_\Phi$. Conversely, if $w \in W_\Phi$, we will show that w must satisfy the fundamental inequality of Coifman and C. Fefferman [4], that there exists a $\beta > 0$ such that, for each cube I,

$$\left|\left[x : w(x) > \frac{\beta\cdot w(I)}{|I|}\right]\right| \ge \frac{1}{2}|I|. \tag{13}$$

(13) implies a reverse Hölder inequality, and so A_∞.

Let

$$E_{t,I} = \left[x \in I : \frac{w(I)}{|I|w(x)} \ge t\right] \quad \text{and} \quad F_{t,I} = I\sim E_{t,I}.$$

By (12), with Ψ the complement and $\psi = \Psi'$,

$$2C \ge \frac{1}{|I|}\int_{E_{t,I}} \psi\left(\frac{w(I)}{2C|I|w}\right) \ge \psi\left(\frac{t}{2C}\right)\frac{|E_{t,I}|}{|I|}$$

and so

$$\frac{|E_{t,I}|}{|I|} \le \frac{2C}{\psi(t/(2C))} \to 0 \quad \text{as } t\to\infty.$$

So we can choose t so large that $|E_{t,I}|/|I| \le 1/2$. Obviously, then, $|F_{t,I}| \ge \frac{1}{2}|I|$, and so (13) holds with $\beta = 1/t$.

4. Strong-type integral inequalities. Pick has shown that a weighted L_Φ integral inequality for the maximal operator M forces the complement Ψ to be in local Δ_2 [11]. A look at his proof shows that this results not from the integral inequality, but from the weaker norm inequality. Since integral inequalities are actually equivalent to a uniform family of norm inequalities, it is not hard to modify Pick's proof to obtain Δ_2.

THEOREM 7. *Let Φ be an* N-*function, and let t, u, v, and w be weights on $\mathbb{R}^n$. Then, in order for*

$$\int \Phi(wMf)t(x)\,dx \le \int \Phi(Cu(x)|f(x)|)v(x)\,dx \tag{14}$$

to hold for all f, we must have $\Phi \in \Delta_2^c$.

Proof. Since these are weights, there exists a constant $K > 0$ such that the set

$$E = [x \in \mathbb{R}^n : K^{-1} \le t(x), u(x), v(x), w(x) \le K]$$

has positive measure. Let x be a point of density of E. Then there exists an $r_0 > 0$ such that

$$|B(r,x)\cap E| \ge \tfrac{1}{2}|B(r,x)|$$

for all $0 < r \le r_0$, where $B(r,x) = [y : |x-y| < r]$. Let

$$B_m = B(2^{-m/n}r_0, x), \quad m = 0,1,2,\ldots,$$

and let $f_m = \chi_{E\cap B_m}$. Pick's construction showed that

$$\int_{(B_0\cap E)\sim B_m} \frac{dy}{|x-y|^n} \ge Cm \tag{15}$$

and

$$Mf_m(y) \ge C|E\cap B_m||x-y|^{-n} \tag{16}$$

when $y \notin B_m$. From (14) and Proposition 4, we have

$$\|wMf_m\|_{\Phi,\varepsilon t} \le C\|uf_m\|_{\Phi,\varepsilon v} \tag{17}$$

for all $\varepsilon > 0$ and m, with C, of course, independent of ε and m. We claim that

$$\|uf_m\|_{\Phi,\varepsilon v} \le \frac{K}{\Phi^{-1}[1/(\varepsilon v(E\cap B_m))]}. \tag{18}$$

For, if this number is λ, then

$$\int \Phi\left(\frac{uf_m}{\lambda}\right)\varepsilon v = \int_{E\cap B_m} \Phi\left(\frac{u}{\lambda}\right)\varepsilon v \le \Phi\left(\frac{K}{\lambda}\right)\varepsilon v(E\cap B_m) = 1,$$

proving the claim.

Let Ψ be the complement of Φ. Using (6), (18) shows that

$$\begin{aligned}\|uf_m\|_{\Phi,\varepsilon v} &\le K\varepsilon v(E\cap B_m)\Psi^{-1}\left[\frac{1}{\varepsilon v(E\cap B_m)}\right]\\ &\le K^2\varepsilon|E\cap B_m|\Psi^{-1}\left[\frac{K}{\varepsilon|E\cap B_m|}\right]\\ &\le K^2\varepsilon|E\cap B_m|\Psi^{-1}\left(\frac{2K}{\varepsilon|B_m|}\right)\\ &= K^2\varepsilon|E\cap B_m|\Psi^{-1}\left(\frac{C2^mK}{\varepsilon|B_0|}\right).\end{aligned}$$

Let

$$g = \Psi^{-1}\left[\frac{1}{\varepsilon t(B_0\cap E)}\right]\chi_{B_0\cap E}.$$

Then $\|g\|_{\Psi,\varepsilon t}\le 1$, and using Hölder's inequality, we have

$$\begin{aligned}\|wMf_m\|_{\Phi,\varepsilon t} &\ge C\int w(Mf_m)g\varepsilon t\\ &\ge CK^{-2}\varepsilon\Psi^{-1}\left[\frac{1}{\varepsilon t(B_0\cap E)}\right]|E\cap B_m|\int_{(B_0\cap E)\sim B_m}\frac{dy}{|x-y|^n}\end{aligned}$$

by (16). Hence, by (15),

$$\begin{aligned}\|wMf_m\|_{\Phi,\varepsilon t} &\ge CmK^{-2}\varepsilon\Psi^{-1}\left[\frac{1}{\varepsilon t(B_0\cap E)}\right]|E\cap B_m|\\ &\ge CmK^{-2}\varepsilon\Psi^{-1}\left[\frac{1}{\varepsilon K|B_0|}\right].\end{aligned}$$

Now, fix $y>0$. Choose ε so that

$$\frac{1}{\varepsilon K|B_0|} = \Psi(y).$$

We have shown that

$$\|wMf_m\|_{\Phi,\varepsilon t} \ge CmK^{-2}\varepsilon|E\cap B_m|y$$

and

$$\|uf_m\|_{\Phi,\varepsilon v} \le K^2\varepsilon|E\cap B_m|\Psi^{-1}(C2^mK^2\Psi(y)).$$

By (17), we have

$$my \le CK^4\Psi^{-1}(C2^mK^2\Psi(y)).$$

Now fix m, chosen so that $m\ge 2CK^4$. For that m, which does not depend on y, we have

$$2y \le \Psi^{-1}(C2^mK^2\Psi(y))$$

or

$$\Psi(2y) \le C2^mK^2\Psi(y),$$

proving the theorem.

We now have most of the ingredients for Theorem 1. To finish off, we will need a series of lemmas, modeled on the Kerman–Torchinsky proof. Henceforth, let $I(w)$ denote the average of w over the cube I,

$$I(w) = \frac{1}{|I|}\int w(x)\,dx.$$

LEMMA 8. *If $w\in W_\Phi$, then there exists a constant C such that*

$$I(w)\phi\left[\frac{1}{C}I\left(\psi\left(\frac{\varepsilon}{w}\right)\right)\right] \le C\varepsilon \tag{19}$$

holds for every cube I and every $\varepsilon>0$.

Proof. Standard arguments using (3) show that (2), and hence $w\in W_\Phi$, is equivalent to

$$\frac{1}{|I|}\int \psi\left[\frac{\Phi(\lambda)I(w)}{C\lambda w(x)}\right]dx \le C\lambda \tag{20}$$

for each cube I and $\lambda>0$.

Given $\varepsilon>0$, since $\Phi(\lambda)/\lambda$ has full range, we can choose λ so that $(\Phi(\lambda)/(C\lambda))I(w)=\varepsilon$. Then (20) says that $I(\psi(\varepsilon/w))\le C\lambda$ or

$$\phi\left[\frac{1}{2C}I\left(\psi\left(\frac{\varepsilon}{w}\right)\right)\right] \le \phi\left(\frac{\lambda}{2}\right) \le 2\frac{\Phi(\lambda)}{\lambda} = 2C\frac{\varepsilon}{I(w)},$$

which is (19) with $2C$ replacing C.

LEMMA 9. *Let $w\in W_\Phi$ with $\Phi\in\Delta_2^c$. Then $v=\psi(\varepsilon/w)$ satisfies a reverse Hölder inequality*

$$I(v^r)^{1/r} \le CI(v) \tag{21}$$

for all cubes I, with $C>0$ and $r>1$ independent of ε.

Proof. Set $E_\alpha = [x \in I : v(x) \le \alpha I(v)]$. We must show that there exists an $\alpha > 0$, independent of ε, with $|E_\alpha| \le \frac{1}{2}|I|$. By (19),

$$\frac{C}{\phi((1/C)I(v))} \ge \frac{1}{|I|}\int_{E_\alpha} \frac{w(x)}{\varepsilon}\,dx.$$

On E_α, $\psi(\varepsilon/w) \le \alpha I(v)$ and so

$$\Psi\left(\frac{\varepsilon}{w}\right) \le \frac{\varepsilon}{w}\alpha I(v).$$

From (6), we get

$$\frac{\varepsilon}{w}\Phi^{-1}\left(\frac{\varepsilon}{w}\alpha I(v)\right) \le 2\frac{\varepsilon}{w}\alpha I(v).$$

Thus

$$\frac{\varepsilon}{w}\alpha I(v) \le \Phi(2\alpha I(v)) \le 2\alpha I(v)\phi(2\alpha I(v)),$$

or

$$\frac{\varepsilon}{w} \le 2\phi(2\alpha I(v)).$$

So

$$\frac{2C}{\phi((1/C)I(v))} \ge \frac{|E_\alpha|}{|I|}\frac{1}{\phi(2\alpha I(v))}.$$

By iterating Proposition 3(b), we can find a $t > 0$ sufficiently small that $\phi(tx) \le \phi(x)/(4C)$. Choose $\alpha = t/(2C)$. Then

$$\frac{|E_\alpha|}{|I|} \le 2C\frac{\phi((t/C)I(v))}{\phi((1/C)I(v))} \le \frac{1}{2}.$$

LEMMA 10. *Let $w \in B_\Phi$ and $\Phi \in \Delta_2^c$. Then there exists an $r > 1$ such that $w \in W_{\Phi_r}$, where Φ_r is the* N-*function with derivative*

$$\Phi_r'(x) = \phi(x^{1/r}).$$

Proof. Let $\phi_r(x) = \phi(x^{1/r})$ and $\Psi_r(x) = x\psi(x)^r$. Choose $r > 1$ so that (21) holds. We claim that there exists a constant C such that, for every $\lambda > 0$,

$$\frac{1}{|I|}\int_I \psi\left(\frac{\Phi_r(\lambda)}{C\lambda w}I(w)\right)^r \le C\lambda.$$

For this, let $C/2$ be the constant in (20) and let K be the reverse Hölder constant in (21). Then, with $\varepsilon = (\Phi_r(\lambda)/(C\lambda))I(w)$, we have

$$\begin{aligned}\frac{1}{|I|}\int_I \psi\left(\frac{\Phi_r(\lambda)}{C\lambda w}I(w)\right)^r &\le K^r\left\{\frac{1}{|I|}\int_I \psi\left(\frac{\Phi_r(\lambda)}{C\lambda w}I(w)\right)\right\}^r\\ &\le K^r\left(\frac{1}{|I|}\int_I \psi\left[\frac{\phi(\lambda^{1/r})}{Cw}I(w)\right]\right)^r\\ &\le K^r\left(\frac{1}{|I|}\int_I \psi\left[\frac{\Phi(2\lambda^{1/r})}{C\lambda^{1/r}w}I(w)\right]\right)^r \le (KC)^r\lambda\end{aligned}$$

by (20), proving the claim with C replaced by $(KC)^r$.

Hence,

$$\int \Psi_r\left(\frac{\Phi_r(\lambda)}{C\lambda w}I(w)\right)w \le \Phi_r(\lambda)w(I),$$

for all cubes I. This is (2), and would give $w \in W_{\Phi_r}$ were we lucky enough to know that Ψ_r is the complement of Φ_r. That, however, need not be the case. Still, the proof of the sufficiency side of Theorem 5 uses only Young's inequality, so it would suffice to show that Φ_r and Ψ_r satisfy Young's inequality. In fact, the proof would carry through verbatim if we just had

$$xy \le \Phi_r(2rx) + \Psi_r(y). \tag{22}$$

We show this. Let r' be the conjugate exponent to r, $1/r + 1/r' = 1$. Using Young's inequality for $\Phi(x)$ and for x^r/r, we get

$$\begin{aligned}xy = x^{1/r'}x^{1/r}y &\le x^{1/r'}(\Phi(x^{1/r}) + \Psi(y))\\ &\le x\phi(x^{1/r}) + x^{1/r'}y\psi(y) \le \Phi_r(2x) + y\left[\frac{x}{r'} + \frac{\psi(y)^r}{r}\right]\end{aligned}$$

and so

$$xy \le r\Phi_r(2x) + \Psi_r(y)$$

and (22) follows from the convexity of Φ_r. This completes the proof of the lemma.

Set

$$k(s) = \sup_{t>0}\frac{\Phi(st)}{\Phi(t)}$$

and

$$\alpha = \lim_{s\to 0^+}\frac{\log k(s)}{\log s}. \tag{23}$$

α is the Orlicz–Maligranda lower index for Φ.

Notice, for $\Phi \in \Delta_2^c$, that by Proposition 3(b), with that δ,

$$\Phi\left(\frac{\delta^n t}{2}\right) \le \left(\frac{\delta}{2}\right)^n\Phi(t)$$

and so, if $\delta^{n+1}/2 < s \le \delta^n/2$,

$$k(s) \le \left(\frac{\delta}{2}\right)^n,$$

but then

$$\frac{\log k(s)}{\log s} \ge \frac{n\log(2/\delta)}{\log 2 + (n+1)\log(1/\delta)}$$

and so, $\Phi \in \Delta_2^c$ forces $\alpha > 1$.

LEMMA 11. *Let α be given by (23) with $\Phi \in \Delta_2^c$, and let $q < \alpha$. Then there exists a constant C depending on q for which*

$$\Phi(st) \le Cs^q\Phi(t)$$

for all $t > 0$ and $0 < s < 1$.

Proof. There exists an s_0 such that

$$\frac{\log k(s)}{\log s} \ge q$$

whenever $0 < s \le s_0$. Thus $k(s) \le s^q$ for such s, and the lemma holds with $C = s_0^{-q}$.

The heart of the argument is contained in our last lemma:

LEMMA 12. *Let $w \in W_\Phi$ and $\Phi \in \Delta_2^c$. Then there exists a $p < \alpha$, with α given by (23), such that $w \in A_p$.*

Proof. As in [7], it will suffice to prove that w is of weak-restricted type (p,p) for some $p < \alpha$, and that requires showing

$$\frac{|E|}{|I|} \le C\left[\frac{w(E)}{w(I)}\right]^{1/p} \tag{24}$$

for all cubes I and measurable sets $E \subset I$. Clearly (24) will hold if it holds whenever $w(E)/w(I) \le \varepsilon_0$. Fix E and put

$$\varepsilon = \frac{w(E)}{w(I)}.$$

Let $\varrho > 1$. Then there exists an s_0 such that

$$\frac{\log k(s)}{\log s} \ge \varrho\alpha$$

for all $0 < s \le s_0$. Hence $1/k(s) \le s^{-\varrho\alpha}$, or

$$s^{\varrho\alpha} \le \sup_t \frac{\Phi(st)}{\Phi(t)},$$

and we can find a t with $s^{\varrho\alpha} \le 2\Phi(st)/\Phi(t)$. Using (3), we have

$$s^{\varrho\alpha-1}\phi\left(\frac{t}{2}\right) \le 4\phi(st).$$

Now

$$\phi(st) = \phi_r((st)^r) \le \frac{\Phi_r(2(st)^r)}{(st)^r}.$$

Likewise,

$$\phi\left(\frac{t}{2}\right) \ge \left(\frac{2}{t}\right)^r \Phi_r\left[\left(\frac{t}{2}\right)^r\right]$$

and so

$$s^{\varrho\alpha-1+r}\Phi_r\left[\left(\frac{t}{2}\right)^r\right] \le 2^{2-r}\Phi_r\left[2^{1+r}s^r\left(\frac{t}{2}\right)^r\right].$$

Thus, there exists a y with

$$s^{\varrho\alpha-1+r}\Phi_r(y) \le 2^{2-r}\Phi_r[2^{1+r}s^r y].$$

Choose s so that $s^{\varrho\alpha-1+r} = \varepsilon$ and set

$$p = \frac{\varrho\alpha - 1 + r}{r}.$$

Notice that $p < \alpha$ if and only if $\varrho\alpha - 1 < r(\alpha - 1)$. Since this holds when $\varrho = 1$, we can choose $\varrho = \varrho(r)$ so that $p < \alpha$. Since $s^r = \varepsilon^{1/p}$, we have

$$\varepsilon\Phi_r(y) \le 2^{2-r}\Phi_r(2^{1+r}\varepsilon^{1/p}y). \tag{25}$$

Let $\lambda = 2^{1+r}\varepsilon^{1/p}y$ and let Ψ_r be the complement of Φ_r. Then

$$\begin{aligned}\frac{yw(I)\Phi_r(\lambda)|E|}{C\lambda|I|} &= \int_I \frac{w(I)\Phi_r(\lambda)}{C\lambda|I|w}\cdot y\chi_E(x)w(x)\,dx \\ &\le \Phi_r(y)w(E) + \int_I \Psi_r\left(\frac{w(I)\Phi_r(\lambda)}{C\lambda|I|w}\right)w \quad \text{by Young's inequality} \\ &\le \Phi_r(y)w(E) + \Phi_r(\lambda)w(I),\end{aligned}$$

since $w \in W_{\Phi_r}$. Dividing by $w(I)$ gives

$$\frac{|E|}{|I|}\cdot\frac{\Phi_r(\lambda)}{C2^{1+r}\varepsilon^{1/p}} \le \varepsilon\Phi_r(y) + \Phi_r(\lambda) \le (1+2^{2-r})\Phi_r(\lambda)$$

by (25), and so

$$\frac{|E|}{|I|} \le C2^{1+r}(1+2^{2-r})\varepsilon^{1/p},$$

proving the lemma.

Now we prove Theorem 1. All that is left to show is the implication (c)⇒(a). Let α be the index given by (23). Fix some $p < \alpha$ for which $w \in A_p$, as in the last lemma, and let $p < q < \alpha$. Since $w \in A_p$,

$$w([x : Mf(x) > \lambda]) \le \lambda^{-p} \int\limits_{[x:2|f(x)|>\lambda]} |f(x)|^p w(x)\,dx,$$

for all $\lambda > 0$. Hence,

$$\begin{aligned}\int \Phi(Mf)w &= \int\limits_0^\infty \phi(\lambda)w([x : Mf(x) > \lambda])\,d\lambda \\ &\le C\int\limits_0^\infty \phi(\lambda)\lambda^{-p} \int\limits_{[x:2|f(x)|>\lambda]} |f(x)|^p w(x)\,dx\,d\lambda \\ &= C\int |f(x)|^p w(x) \int\limits_0^{2|f(x)|} \phi(\lambda)\lambda^{-p}\,d\lambda\,dx \\ &\le C\int |f(x)|^p w(x) \int\limits_0^{2|f(x)|} \Phi(2\lambda)\lambda^{-p-1}\,d\lambda\,dx.\end{aligned}$$

Now, by Lemma 11,

$$\Phi(2\lambda) = \Phi\left(\frac{\lambda}{2|f(x)|}4|f(x)|\right) \le C\left(\frac{\lambda}{2|f(x)|}\right)^q \Phi(4|f(x)|),$$

and so

$$\begin{aligned}\int \Phi(Mf)w &\le C\int |f(x)|^{p-q}\Phi(4|f(x)|)w(x) \int\limits_0^{2|f(x)|} \lambda^{q-p-1}\,d\lambda\,dx \\ &= C\int \Phi(4|f(x)|)w(x)\,dx \le \int \Phi(4C|f(x)|)w(x)\,dx.\end{aligned}$$

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