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Weighted L_{Φ} integral inequalities for operators of Hardy type

by

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Abstract. Necessary and sufficient conditions are given on the weights t , u , v , and w , in order for

$$\Phi_2^{-1}(\int \Phi_2(w(x)|Tf(x)|)t(x) dx) \leq \Phi_1^{-1}(\int \Phi_1(Cu(x)|f(x)|)v(x) dx)$$

to hold when Φ_1 and Φ_2 are N-functions with $\Phi_2 \circ \Phi_1^{-1}$ convex, and T is the Hardy operator or a generalized Hardy operator.

Weak-type characterizations are given for monotone operators and the connection between weak-type and strong-type inequalities is explored.

1. Introduction. In this paper, we will extend some weighted norm inequalities from the Lebesgue setting to the Orlicz space setting. Given a σ -finite measure space $(X, d\mu)$ and an N-function Φ , the Orlicz space $L_{\Phi}(X, d\mu)$ is the Banach space normed by

$$\|f\|_{L_{\Phi}} = \|f\|_{L_{\Phi}(X, d\mu)} = \inf_{\lambda > 0} \left\{ \int_X \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1 \right\}.$$

In this paper, with the exception of Section 2, X will be either $\mathbb{R}^+ = (0, \infty)$ or \mathbb{R}^n , and μ will be defined on the Lebesgue measurable sets.

A weight is a measurable function on X that is positive almost everywhere. For the Lebesgue space, $L^r(X)$, $1 < r < \infty$, which corresponds to the N-function $\Phi(x) = x^r/r$, a weighted norm inequality for an operator T has the form

$$\|Tf\|_{L^q(X, w(x) dx)} \leq C \|f\|_{L^p(X, v(x) dx)}.$$

This has a number of useful equivalent formulations, such as

$$\left(\int |Tf(x)|^q w(x) dx \right)^{1/q} \leq C \left(\int |f(x)|^p v(x) dx \right)^{1/p}$$

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or

$$\left(\int |(Tf)(x)w(x)^{1/q}|^q dx \right)^{1/q} \leq C \left(\int |f(x)v(x)^{1/p}|^p dx \right)^{1/p}$$

or

$$\left(\int |T(f\sigma)|^q w \right)^{1/q} \leq C \left(\int |f|^p \sigma \right)^{1/p}$$

where $\sigma = v^{-1/(p-1)}$. This last was used effectively by E. Sawyer in papers such as [12] and [13].

The Orlicz space versions of these inequalities are

$$(N) \quad \|Tf\|_{L_{\Phi_2}(X, w(x)dx)} \leq C \|f\|_{L_{\Phi_1}(X, v(x)dx)},$$

$$(O) \quad \Phi_2^{-1} \left(\int \Phi_2(|Tf|)w \right) \leq \Phi_1^{-1} \left(\int \Phi_1(C|f|)v \right),$$

$$(I) \quad \Phi_2^{-1} \left(\int \Phi_2(w|Tf|) \right) \leq \Phi_1^{-1} \left(\int \Phi_1(Cv|f|) \right)$$

and

$$(S) \quad \Phi_2^{-1} \left(\int \Phi_2(|T(fv)|)w \right) \leq \Phi_1^{-1} \left(\int \Phi_1(C|f|)v \right),$$

the norm, outer, inner, and Sawyer formulations. Unfortunately, *none* of these are equivalent, and it is not clear which is the correct approach to a weighted Orlicz space problem. (O), (I), and (S) can be effectively combined by allowing four weights t , u , v , and w , and considering the weighted integral inequality

$$(1.1) \quad \Phi_2^{-1} \left(\int \Phi_2(w(x)|Tf(x)|)t(x) dx \right) \leq \Phi_1^{-1} \left(\int \Phi_1(Cu(x)|f(x)|)v(x) dx \right),$$

and this reduces to (O), (I), or (S) by taking the appropriate weight to be identically one. We will seek criteria on t , u , v , and w for (1.1) to hold. Characterizing (1.1) does not characterize (N). We will describe how these relate in Section 2, but for now, we will simply point out that weighted *integral* inequalities and weighted *norm* inequalities are different. We are solving integral inequalities in this paper.

The first weighted L_{Φ} inequality was given by R. Kerman and A. Torchinsky in 1982 [6]. They showed that (O) holds for $T = M$, the Hardy–Littlewood maximal operator, when $v = w$, $\Phi_1 = \Phi_2 = \Phi$, $\Phi \in \Delta_2$ together with its complementary N-function Ψ , if and only if w belongs to the Muckenhoupt A_p class, with $1/p$ the Boyd upper index of Φ .

Quite recently, S. S. Kazarian did the inner version of the Kerman–Torchinsky theorem [5], and it is interesting that the techniques and the results are somewhat different from the outer theory.

A_p weights have enough flexibility that proofs relying on them can be fuzzy. Moving away from A_p means grappling with inherent Orlicz space problems. Thus, until quite recently, characterizations of weights did not

exist in the L_{Φ} setting for such elementary operators as the Hardy antidifferentiation operator.

The Hardy operator I is given by

$$If(x) = \int_0^x f(t) dt$$

and its adjoint I^* by

$$I^*f(x) = \int_x^{\infty} f(t) dt.$$

The grandfather of the (two) weighted norm inequalities is

THEOREM 1.2. *Let v and w be weights on \mathbb{R}^+ and let $1 < p \leq q < \infty$. Then*

$$(1.3) \quad \left(\int_0^{\infty} |If(x)|^q w(x) dx \right)^{1/q} \leq \left(\int_0^{\infty} |f(x)|^p v(x) dx \right)^{1/p}$$

if and only if

$$(1.4) \quad [Iv^{-1/(p-1)}(x)]^{1-1/p} I^*w(x)^{1/q} \leq C \quad \text{for each } x > 0.$$

(See [1], [3], [9], [15], and [16].)

L. Quinsheng in [10] made substantial progress in the study of L_{Φ} inequalities for I . He did an outer version of (1.3) in which Φ_1 and Φ_2 together with their complementary functions Ψ_1 and Ψ_2 satisfy the Δ_2 condition and the two Orlicz spaces correspond to $p \leq q$.

In the same paper, Quinsheng adapted Sawyer’s two-weight proof for the Hardy–Littlewood maximal operator to an Orlicz setting, to obtain an (S) version. This is perhaps the least natural of the various formulations in this context, and Sawyer’s proof does not seem to adapt to any of the others.

Heinig and Maligranda extended Quinsheng’s Hardy operator result to the four-weight setting of (1.1) and dispensed with one of the Δ_2 conditions in [4].

With this recent progress in mind, it seems reasonable to characterize (1.1) for the Hardy operator without any Δ_2 assumptions. We do this for the generalized Hardy operators, first treated in [2].

DEFINITION 1.5. *A generalized Hardy operator, or GHO, is an operator*

$$Tf(x) = \int_0^x k(x, y)f(y) dy,$$

where the kernel k is nonnegative, nondecreasing in x , nonincreasing in y , and satisfies the triangle inequality

$$(1.6) \quad k(x, y) \leq D[k(x, z) + k(z, y)], \quad y < z < x.$$

These include the Hardy operator, $k \equiv 1$, and the Riemann–Liouville fractional integral operators $k(x, y) = (x - y)^\alpha$, $\alpha > 0$. Weighted norm inequalities for such operators have been studied in [2], [8], [14].

The main result of this paper is

THEOREM 1.7. *Let T be a GHO with kernel k . Let t, u, v , and w be weights on \mathbb{R}^+ , Φ_1 and Φ_2 N-functions having complements Ψ_1 and Ψ_2 respectively, and with $\Phi_2 \circ \Phi_1^{-1}$ convex. Then (1.1) holds if and only if there exists $C > 0$, independent of $\lambda, x > 0$, such that*

$$(1.8) \quad \int_0^x \Psi_1 \left(\frac{\alpha(\lambda, x)k(x, y)}{C\lambda u(y)v(y)} \right) v(y) dy \leq \alpha(\lambda, x) < \infty$$

and

$$(1.9) \quad \int_0^x \Psi_1 \left(\frac{\beta(\lambda, x)}{C\lambda u(y)v(y)} \right) v(y) dy \leq \beta(\lambda, x) < \infty,$$

where

$$(1.10) \quad \alpha(\lambda, x) = \Phi_1 \circ \Phi_2^{-1} \left(\int_x^\infty \Phi_2(\lambda w(y))t(y) dy \right)$$

and

$$(1.11) \quad \beta(\lambda, x) = \Phi_1 \circ \Phi_2^{-1} \left(\int_x^\infty \Phi_2(\lambda w(y)k(y, x))t(y) dy \right).$$

The Orlicz space machinery is described in the next section. Section 3 is devoted to weak-type L_Φ inequalities, i.e.,

$$(1.12) \quad \Phi_2^{-1} \left(\int_{\{|Tf|>\lambda\}} \Phi_2(\lambda w(x))t(x) dx \right) \leq \Phi_1^{-1} \left(\int \Phi_1(Cu(x)|f(x)|)v(x) dx \right).$$

We characterize (1.12) for the general class of monotone operators.

(1.1) is obviously stronger than (1.12). It is somewhat surprising that, for the Hardy operator at least, (1.12) implies (1.1). This result, coupled with the weak-type theory, gives Theorem 1.7 for $T = I$. We present these results, and also the connection between weak-type and strong-type inequalities for GHO's in Section 4. Finally, the proofs for GHO's are presented in Section 5.

2. Orlicz spaces. The standard theory of Orlicz spaces can be found in Zygmund [17], Krasnosel'skiĭ and Rutitskiĭ [7], or Rao and Ren [11]. An N-function $\Phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nonnegative, convex function satisfying

$$\lim_{x \rightarrow 0^+} \frac{\Phi(x)}{x} = 0, \quad \lim_{x \rightarrow \infty} \frac{\Phi(x)}{x} = \infty.$$

Such a Φ has a derivative, φ , which is nondecreasing and nonnegative, with $\varphi(0+) = 0$ and $\varphi(\infty) = \infty$, so that

$$\Phi(x) = \int_0^x \varphi(t) dt,$$

and we can and will take φ to be right-continuous. The Young function complementary to Φ is given by

$$\Psi(x) = \sup_y \{xy - \Phi(y)\}.$$

This is also an N-function, and we have Young's inequality

$$ab \leq \Phi(a) + \Psi(b) \quad \text{for all } a, b > 0.$$

If $(X, d\mu)$ is a σ -finite measure space, then the Orlicz space $L_\Phi = L_\Phi(X, d\mu)$ is the Banach space normed by either

$$\|f\|_\Phi^* = \sup \left\{ \left| \int fg d\mu \right| : \int \Psi(|g|) \leq 1 \right\}$$

or

$$\|f\|_\Phi = \inf \left\{ \lambda : \int_X \Phi \left(\frac{|f(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\}.$$

These two norms are equivalent, and there is a Hölder inequality

$$\left| \int fg d\mu \right| \leq C \|f\|_\Phi \|g\|_\Psi.$$

(C can be taken as $\Phi(1) + \Psi(1)$.) Hölder's inequality seems like a completely indispensable tool for weighted norm inequalities, yet we will use this only once.

There are a few easy inequalities which we will need, and which we list in the following

LEMMA 2.1. *Let Φ be an N-function with complementary Young function Ψ . Let $x, y > 0$. Then*

$$(2.2) \quad \Phi(x) \leq x\varphi(x) \leq \Phi(2x),$$

$$(2.3) \quad \Phi(x) + \Phi(y) \leq \Phi(x + y)$$

and

$$(2.4) \quad \Phi \left(\frac{\Psi(x)}{x} \right) \leq \Psi(x).$$

Proof. (2.2) and (2.3) follow easily from $\Phi(x) = \int_0^x \varphi(t) dt$, φ nondecreasing. For (2.4), fix a y with $xy - \Phi(y) \geq 0$. Let $h = y - \Phi(y)/x$. Now $\Phi(t)/t$ is nondecreasing (by (2.2)) and since $h \leq y$, we have

$$\frac{\Phi(h)}{h} \leq \frac{\Phi(y)}{y} \leq x$$

which means that

$$\Phi(h) \leq hx = xy - \Phi(y) \leq \Psi(x).$$

Taking the supremum over all such y 's gives (2.4).

For many applications, it helps to have a constructive definition of the complementary N-function Ψ . This can be done by setting

$$\psi(x) = \inf\{y : \varphi(y) \geq x\} \quad \text{and} \quad \Psi(x) = \int_0^x \psi(y) dy.$$

This and the standard complement are essentially equivalent.

Now we turn to a comparison between norm and integral inequalities. Norm inequalities are weaker than integral inequalities. In fact, integral inequalities are equivalent to the uniform boundedness of a family of norm inequalities, as evidenced by

PROPOSITION 2.5. *Suppose that $(X, d\mu)$ and $(Y, d\nu)$ are σ -finite measure spaces and that T is a linear operator mapping measurable functions on X to measurable functions on Y . Let Φ be an N-function and, given $\varepsilon > 0$, let $L_{\Phi, \varepsilon d\mu}(X)$ be the Orlicz space with the norm*

$$\|f\|_{\Phi, \varepsilon d\mu} = \inf \left\{ \lambda : \int_X \Phi \left(\frac{|f(x)|}{\lambda} \right) \varepsilon d\mu(x) \leq 1 \right\},$$

with $L_{\Phi, \varepsilon d\nu}(Y)$ defined similarly. Then

$$(2.6) \quad \int_Y \Phi(|Tf(y)|) d\nu(y) \leq \int_X \Phi(C|f(x)|) d\mu(x)$$

if and only if

$$\|Tf\|_{\Phi, \varepsilon d\nu} \leq C \|f\|_{\Phi, \varepsilon d\mu}$$

holds for all $\varepsilon > 0$, with C independent of ε .

Proof. Suppose the norms are bounded. Fix an f not identically zero and put

$$\varepsilon = \left(\int \Phi(|f|) d\mu \right)^{-1}.$$

Since $\|f\|_{\Phi, \varepsilon d\mu} \leq 1$, $\|Tf\|_{\Phi, \varepsilon d\nu} \leq C$. Thus,

$$\int \Phi \left(\frac{|Tf|}{C} \right) d\nu \leq \frac{1}{\varepsilon} = \int \Phi(|f|) d\mu.$$

Replacing f by Cf gives (2.6).

Conversely, fix an f and let $\alpha = \|f\|_{\Phi, \varepsilon d\mu}$. Then $\int \Phi(|f|/\alpha) \varepsilon d\mu \leq 1$, and so

$$\int \Phi \left(\frac{|Tf|}{C\alpha} \right) \varepsilon d\nu = \varepsilon \int \Phi \left(\left| T \left(\frac{f}{C\alpha} \right) \right| \right) d\nu \leq \varepsilon \int \Phi \left(\frac{|f|}{\alpha} \right) d\mu \quad \text{by (2.6)}$$

$$\leq 1,$$

which shows that

$$\|Tf\|_{\Phi, \varepsilon d\nu} \leq C\alpha.$$

A consequence of this is a duality inequality which we will use later.

COROLLARY 2.7. *Let $(X, d\mu)$, $(Y, d\nu)$, T , and Φ be as in Proposition 2.5, Φ having complement Ψ . Let u and v be weights on X , and t and w be weights on Y , and let T^* denote the adjoint operator to T . Then*

$$(2.8) \quad \int \Phi(w(y)|T(f(y))|)t(y) dy \leq \int \Phi(Cu(x)|f(x)|)v(x) dx$$

holds if and only if

$$(2.9) \quad \int \Psi((uv)^{-1}T^*g(x))v(x) dx \leq \int \Psi(C(wt)^{-1}g(y))t(y) dy.$$

Proof. Suppose (2.8) holds. Let $Sf(x) = (uv)^{-1}T^*(wtf)(x)$. (2.9) is clearly equivalent to

$$\int \Psi(|Sf(x)|)v(x) dx \leq \int \Psi(C|f(y)|)t(y) dy$$

and this will hold provided

$$\|Sf\|_{\Psi, \varepsilon v} \leq C \|f\|_{\Psi, \varepsilon t}$$

for each $\varepsilon > 0$. But

$$\|Sf\|_{\Psi, \varepsilon v} \leq C \sup \left\{ \varepsilon \left| \int (Sf)gv \right| : \|g\|_{\Phi, \varepsilon v} \leq 1 \right\}.$$

For such a g ,

$$\varepsilon \left| \int (Sf)gv \right| = \varepsilon \left| \int T^*(wtf) \cdot g/u \right| = \left| \int f(wT(g/u))\varepsilon t \right|$$

$$\leq C \|f\|_{\Psi, \varepsilon t} \|wT(g/u)\|_{\Phi, \varepsilon t}.$$

So we need

$$\|wT(g/u)\|_{\Phi, \varepsilon t} \leq C \|g\|_{\Phi, \varepsilon v},$$

which, by Proposition 2.5, holds if and only if

$$\int \Phi(w|T(g/u)|)t \leq \int \Phi(Cg)v,$$

an inequality obviously equivalent to (2.8).

The other direction is similar.

There, by the way, was our promised Hölder inequality. If you blinked, you probably missed it!

3. Weak-type inequalities. An operator $Tf(x) = \int_{\mathbb{R}^+} k(x, y)f(y) dy$, $x \in \mathbb{R}^+$, is called *monotone* on \mathbb{R}^+ if its kernel $k(x, y)$ is nonnegative and either increasing or decreasing in x .

The complete characterization of weak-type weighted L_Φ inequalities for monotone operators on \mathbb{R}^+ is given by the following

THEOREM 3.1. *Suppose T is a monotone operator on \mathbb{R}^+ with kernel k . Let t, u, v , and w be weights on \mathbb{R}^+ and let Φ_1 and Φ_2 be N -functions with complements Ψ_1 and Ψ_2 respectively. If k is nondecreasing in x , then the weak-type boundedness (1.12) holds if and only if*

$$(3.2) \quad \int \Psi_1 \left(\frac{\alpha(\lambda, x)k(x, y)}{C\lambda u(y)v(y)} \right) v(y) dy \leq \alpha(\lambda, x) < \infty,$$

where α is given by (1.10).

If, instead, k is nonincreasing in x , then (1.12) holds if and only if

$$(3.3) \quad \int \Psi_1 \left(\frac{\alpha^*(\lambda, x)k(x, y)}{C\lambda u(y)v(y)} \right) v(y) dy \leq \alpha^*(\lambda, x) < \infty,$$

where

$$(3.4) \quad \alpha^*(\lambda, x) = \Phi_1 \circ \Phi_2^{-1} \left(\int_0^x \Phi_2(\lambda w(y))t(y) dy \right).$$

Proof. We do the nondecreasing case. The other is similar. Suppose (1.12) holds and fix $x > 0$. Since u and v are weights, they are positive almost everywhere, and so

$$\Psi_1 \left(\frac{k(x, y)}{u(y)v(y)} \right) v(y) < \infty \quad \text{y-a.e.}$$

If we fix x , we can thus find sets $E_n \subset \{y : k(x, y) > 0\}$ with

$$\int_{E_n} \Psi_1 \left(\frac{k(x, y)}{u(y)v(y)} \right) v(y) dy < \infty$$

and with E_n increasing to the support set $\{y : k(x, y) > 0\}$. Fix such an $E = E_n$. Consider

$$\int_E \Psi_1 \left(\frac{\varepsilon k(x, y)}{u(y)v(y)} \right) \frac{v(y)}{\varepsilon} dy.$$

Now $\Psi_1(t)/t$ is nondecreasing, and has full range \mathbb{R}^+ , so this integral is a continuous, increasing function of ε , with range \mathbb{R}^+ . So we can choose $\varepsilon > 0$ for which

$$\int_E \Psi_1 \left(\frac{\varepsilon k(x, y)}{u(y)v(y)} \right) \frac{v(y)}{\varepsilon} dy = 2C\lambda.$$

Set

$$f(y) = \frac{1}{C} \cdot \Psi_1 \left(\frac{\varepsilon k(x, y)}{u(y)v(y)} \right) \frac{v(y)}{\varepsilon k(x, y)} \chi_E(y).$$

Then for $s \geq x$,

$$Tf(s) \geq Tf(x) = 2\lambda > \lambda,$$

and so, by (1.12),

$$\begin{aligned} \alpha(\lambda, x) &\leq \Phi_1 \circ \Phi_2^{-1} \left(\int_{\{s: Tf(s) > \lambda\}} \Phi_2(\lambda w(s))t(s) ds \right) \leq \int \Phi_1(Cuf)v \\ &= \int_E \Phi_1 \left(\frac{u(y)v(y)}{\varepsilon k(x, y)} \right) \Psi_1 \left(\frac{\varepsilon k(x, y)}{u(y)v(y)} \right) v(y) dy \\ &\leq \int_E \Psi_1 \left(\frac{\varepsilon k(x, y)}{u(y)v(y)} \right) v(y) dy \quad \text{by (2.4)} \\ &= 2C\lambda\varepsilon. \end{aligned}$$

In particular, $\alpha(\lambda, x) < \infty$. Also, by (2.2), with ψ_1 the derivative of Ψ_1 , we obtain

$$\begin{aligned} J &= \int_E \psi_1 \left(\frac{\alpha(\lambda, x)k(x, y)}{4C\lambda u(y)v(y)} \right) \frac{k(x, y)}{u(y)} dy \\ &\leq \int_E \psi_1 \left(\frac{\varepsilon k(x, y)}{2u(y)v(y)} \right) \frac{k(x, y)}{u(y)} dy \leq 2 \int_E \Psi_1 \left(\frac{\varepsilon k(x, y)}{u(y)v(y)} \right) \frac{v(y)}{\varepsilon} dy = 4C\lambda. \end{aligned}$$

But, again by (2.2),

$$J \geq \frac{4C\lambda}{\alpha(\lambda, x)} \int_E \Psi_1 \left(\frac{\alpha(\lambda, x)k(x, y)}{4C\lambda u(y)v(y)} \right) v(y) dy.$$

Combining these, and letting $n \rightarrow \infty$, gives (3.2).

Conversely, fix $f \geq 0$ and $\lambda > 0$. Put $E_\lambda = \{x : Tf(x) > \lambda\}$. We can assume without loss of generality that $E_\lambda \neq \emptyset$. The monotonicity of Tf assures us that E_λ is an interval of the form (β, ∞) (or $[\beta, \infty)$). Let $\gamma > \beta$. Then $Tf(\gamma) > \lambda$, and so

$$\begin{aligned} 2\alpha(\lambda, \gamma) &\leq \int 2Cf(y)u(y) \cdot \frac{\alpha(\lambda, \gamma)k(\gamma, y)}{C\lambda u(y)v(y)} \cdot v(y) dy \\ &\leq \int \Phi_1(2Cuf)v + \int \Psi_1 \left(\frac{\alpha(\lambda, \gamma)k(\gamma, y)}{C\lambda u(y)v(y)} \right) v(y) dy \quad \text{by Young's inequality,} \\ &\leq \int \Phi_1(2Cuf)v + \alpha(\lambda, \gamma) \quad \text{by (3.2).} \end{aligned}$$

Hence,

$$\alpha(\lambda, \gamma) \leq \int \Phi_1(2Cuf)v.$$

But as $\gamma \rightarrow \beta$, $\alpha(\lambda, \gamma) \rightarrow \Phi_1 \circ \Phi_2^{-1}(\int_{E_\lambda} \Phi_2(\lambda w(y))t(y) dy)$, so that (1.12) holds.

4. The connection between weak- and strong-type inequalities. For the Hardy operator $I f(x) = \int_0^x f(y) dy$, weak-type boundedness and strong-type boundedness are completely equivalent when $\Phi_2 \circ \Phi_1^{-1}$ is convex.

THEOREM 4.1. *Let Φ_1 and Φ_2 be N-functions with $\Phi_2 \circ \Phi_1^{-1}$ convex. Let t, u, v , and w be weights on \mathbb{R}^+ . Then strong-type boundedness for the Hardy operator*

$$(4.2) \quad \Phi_2^{-1} \left(\int \Phi_2(w(x)|I(f(x))|)t(x) dx \right) \leq \Phi_1^{-1} \left(\int \Phi_1(Cu(x)|f(x)|)v(x) dx \right)$$

holds if and only if weak-type boundedness

$$(4.3) \quad \Phi_2^{-1} \left(\int_{\{x:|If(x)|>\lambda\}} \Phi_2(\lambda w(x))t(x) dx \right) \leq \Phi_1^{-1} \left(\int \Phi_1(Cu(x)|f(x)|)v(x) dx \right)$$

holds.

Proof. One direction is trivial. To prove that (4.3) implies (4.2), take $f \geq 0$. Choose $\{x_n\}$ so that $I f(x_n) = 2^n$. Put $I_n = [x_{n-1}, x_n)$ and $f_n = f \chi_{I_n}$. Then

$$\int \Phi_2(w I f) t \leq \sum_n \int_{I_n} \Phi_2(2^n w(x)) t(x) dx.$$

Now if $x \in I_n$, then

$$I(8f_{n-1})(x) \geq 8 \int_0^{x_{n-1}} f_{n-1}(y) dy = 8 \int_{x_{n-2}}^{x_{n-1}} f(y) dy = 2^{n+1} > 2^n$$

and so

$$I_n \subset \{x : I(8f_{n-1})(x) > 2^n\}.$$

Thus, by (4.3),

$$\int_{I_n} \Phi_2(2^n w(x)) t(x) dx \leq \Phi_2 \circ \Phi_1^{-1} \left(\int \Phi_1(8Cf_{n-1}(x)u(x))v(x) dx \right).$$

Applying (2.3) to $\Phi_2 \circ \Phi_1^{-1}$ gives

$$\int \Phi_2(w I f) t \leq \Phi_2 \circ \Phi_1^{-1} \left(\sum_n \int \Phi_1(8Cu_{f_{n-1}})v \right) = \Phi_2 \circ \Phi_1^{-1} \left(\int \Phi_1(8Cu f) v \right),$$

proving the theorem.

Remark 4.4. Similarly, weak-type boundedness and strong-type boundedness are equivalent for the dual Hardy operator $I^* f(x) = \int_x^\infty f(y) dy$.

For GHO's, this result is too strong. Still, if weak-type boundedness holds, as well as a dual condition, then one can obtain strong-type boundedness. Now duality is essentially a *norm* property, and deducing from an L_Φ integral inequality another dual *integral* inequality seems to be quite problematical. We can do it when $\Phi_1 = \Phi_2$, as in Corollary 2.7. This yields the weak-type connection:

THEOREM 4.5. *Let T be a GHO with adjoint T^* . Let Φ and Ψ be complementary N-functions, and let t, u, v , and w be weights on \mathbb{R}^+ . Then the strong-type boundedness*

$$(4.6) \quad \int \Phi(w(x)|T(f(x))|)t(x) dx \leq \int \Phi(Cu(x)|f(x)|)v(x) dx$$

holds if and only if both of these weak-type inequalities hold:

$$(4.7) \quad \int_{\{x:|Tf(x)|>\lambda\}} \Phi(\lambda w(x))t(x) dx \leq \int \Phi(Cu(x)|f(x)|)v(x) dx$$

and

$$(4.8) \quad \int_{\{x:|T^*f(x)|>\lambda\}} \Psi \left(\frac{\lambda}{u(x)v(x)} \right) v(x) dx \leq \int \Psi \left(\frac{C|f(x)|}{w(x)t(x)} \right) t(x) dx.$$

We prove Theorem 4.5 in the next section.

5. Proofs of the GHO theorems. Our first result follows immediately from the weak-type characterizations and Theorem 4.1.

THEOREM 5.1. *Let Φ_1 and Φ_2 be N-functions with $\Phi_2 \circ \Phi_1^{-1}$ convex. Let t, u, v , and w be weights on \mathbb{R}^+ . Then for the Hardy operator I ,*

$$\Phi_2^{-1} \left(\int \Phi_2(w(x)|I(f(x))|)t(x) dx \right) \leq \Phi_1^{-1} \left(\int \Phi_1(Cu(x)|f(x)|)v(x) dx \right)$$

holds if and only if for each λ and $x > 0$,

$$\int_0^x \Psi_1 \left(\frac{\alpha(\lambda, x)}{C\lambda u(y)v(y)} \right) v(y) dy \leq \alpha(\lambda, x) < \infty$$

where $\alpha(\lambda, x)$ is given by (1.10).

In the proof of Theorem 1.7, we will need a slight generalization of this, which we state as

LEMMA 5.2. *Let Φ_1 and Φ_2 be N-functions with $\Phi_2 \circ \Phi_1^{-1}$ convex. Let t, u, v , and w be weights on \mathbb{R}^+ , and let $g(x)$ be a strictly increasing function*

on \mathbb{R}^+ with a differentiable inverse, and with $g(0) = 0$, and $g(\infty) = \infty$. Let I_g be the operator

$$I_g f(x) = \int_0^{g(x)} f(y) dy.$$

Then

$$\Phi_2^{-1} \left(\int \Phi_2(w(x)|I_g(f(x))|)t(x) dx \right) \leq \Phi_1^{-1} \left(\int \Phi_1(Cu(x)|f(x)|)v(x) dx \right)$$

holds if and only if

$$\int_0^x \Psi_1 \left(\frac{\alpha(\lambda, g^{-1}(x))}{C\lambda u(y)v(y)} \right) v(y) dy \leq \alpha(\lambda, g^{-1}(x)) < \infty$$

for all x and $\lambda > 0$.

Proof. We have

$$\int \Phi_2(w|I_g f|)t = \int_0^\infty \Phi_2(w(g^{-1}(y))|If(y)|)t(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) dy,$$

and the weighted L_Φ integral inequality asserted by the lemma will hold, by Theorem 5.1, if and only if

$$\int_0^x \Psi_1 \left(\frac{\alpha_g(\lambda, x)}{C\lambda u(y)v(y)} \right) v(y) dy \leq \alpha_g(\lambda, x) < \infty$$

where

$$\begin{aligned} \alpha_g(\lambda, x) &= \Phi_1 \circ \Phi_2^{-1} \left(\int_x^\infty \Phi_2(\lambda w(g^{-1}(y)))t(g^{-1}(y))dg^{-1}(y) \right) \\ &= \Phi_1 \circ \Phi_2^{-1} \left(\int_{g^{-1}(x)}^\infty \Phi_2(\lambda w(y))t(y) dy \right) = \alpha(\lambda, g^{-1}(x)), \end{aligned}$$

proving the lemma.

Now we prove Theorem 1.7. For the necessity, (1.8) follows from the weak-type boundedness and Theorem 3.1. For (1.9), take $E \subset [0, x]$ and $F \subset [x, \infty)$ with

$$\beta_F(\lambda, x) = \Phi_1 \circ \Phi_2^{-1} \left(\int_F \Phi_2(\lambda w(y)k(y, x))t(y) dy \right) < \infty$$

and

$$\int_E \Psi_1 \left(\frac{\beta_F(\lambda, x)}{C\lambda u(y)v(y)} \right) v(y) dy < \infty.$$

As in Theorem 3.1, we can choose an $\varepsilon > 0$ such that

$$(5.3) \quad \int_E \Psi_1 \left(\frac{\varepsilon \beta_F(\lambda, x)}{C\lambda u(y)v(y)} \right) \frac{v(y)}{\varepsilon} dy = \beta_F(\lambda, x).$$

Take

$$f(y) = \Psi_1 \left(\frac{\varepsilon \beta_F(\lambda, x)}{C\lambda u(y)v(y)} \right) \frac{\lambda}{\varepsilon \beta_F(\lambda, x)} v(y) \chi_E(y).$$

Then

$$\begin{aligned} \int \Phi_1(Cuf)v &= \int_E \Phi_1 \left(\Psi_1 \left(\frac{\varepsilon \beta_F(\lambda, x)}{C\lambda u(y)v(y)} \right) \frac{C\lambda uv}{\varepsilon \beta_F(\lambda, x)} \right) v(y) dy \\ &\leq \int_E \Psi_1 \left(\frac{\varepsilon \beta_F(\lambda, x)}{C\lambda u(y)v(y)} \right) v(y) dy \quad \text{by (2.4)} \\ &= \varepsilon \beta_F(\lambda, x) \quad \text{by (5.3)}. \end{aligned}$$

Hence,

$$\varepsilon \beta_F(\lambda, x) \geq \Phi_1 \circ \Phi_2^{-1} \left(\int_x^\infty \Phi_2(\lambda w(y)Tf(y))t(y) dy \right).$$

Now for $y \geq x$,

$$Tf(y) = \int_0^x k(y, s)f(s) ds \geq k(y, x) \int_0^x f(s) ds = \lambda k(y, x),$$

so we have

$$\varepsilon \beta_F(\lambda, x) \geq \Phi_1 \circ \Phi_2^{-1} \left(\int_x^\infty \Phi_2(\lambda w(y)k(y, x))t(y) dy \right) = \beta(\lambda, x).$$

In particular, $\beta(\lambda, x) < \infty$, and so we can take $F = (x, \infty)$. Then $\beta_F = \beta$, and we see that $\varepsilon \geq 1$. Since the integral in (5.3) increases in ε , (5.3) must still hold when we replace ε by 1,

$$\int_E \Psi_1 \left(\frac{\beta_F(\lambda, x)}{C\lambda u(y)v(y)} \right) v(y) dy \leq \beta(\lambda, x)$$

and (1.9) follows as $E \rightarrow (0, x)$.

For the sufficiency, we will use a variant of a method of Martín-Reyes and Sawyer [8]. Fix $f \geq 0$, f bounded with compact support. The conditions (1.8) and (1.9) will still hold if we replace the kernel $k(x, y)$ by a smaller kernel. Since k is monotone in x , we can approximate k from below by kernels continuous in x , and then salvage (1.1) by Monotone Convergence. So we can assume, without loss of generality, that k is continuous in x . In that case, $Tf(x)$ is an increasing and continuous function, so we can choose

a sequence $\{x_n\}$ with $Tf(x_n) = (D+1)^n$, with D the GHO constant from (1.6) and n running from $-\infty$ to some N , which might be $+\infty$. Now

$$\begin{aligned} (D+1)^{n+1} &= (D+1)^2((D+1)^n - D(D+1)^{n-1}) \\ &= (D+1)^2 \left(\int_0^{x_n} k(x_n, y) f(y) dy - D \int_0^{x_{n-1}} k(x_{n-1}, y) f(y) dy \right) \\ &= (D+1)^2 \left(\int_0^{x_{n-1}} (k(x_n, y) - Dk(x_{n-1}, y)) f(y) dy \right. \\ &\quad \left. + \int_{x_{n-1}}^{x_n} k(x_n, y) f(y) dy \right). \end{aligned}$$

Let $I_n = (x_{n-1}, x_n]$. Then, by (1.6), we have

$$(5.4) \quad (D+1)^{n+1} \leq (D+1)^2 Dk(x_n, x_{n-1}) \int_0^{x_{n-1}} f(y) dy + (D+1)^2 \int_{I_n} k(x_n, y) f(y) dy.$$

Set

$$A = \left\{ n : (D+1)^2 Dk(x_n, x_{n-1}) \int_0^{x_{n-1}} f(y) dy < \frac{(D+1)^{n+1}}{2} \right\}$$

and

$$B = \{n \leq N : n \notin A\}.$$

We have

$$\begin{aligned} \int \Phi_2(wTf) &\leq \sum_{n \in A} \int_{I_{n+1}} \Phi_2((D+1)^{n+1} w(x)) t(x) dx \\ &\quad + \sum_{n \in B} \int_{I_{n+1}} \Phi_2((D+1)^{n+1} w(x)) t(x) dx. \end{aligned}$$

By (5.4), when $n \in A$,

$$\int_{I_n} k(x_n, y) f dy > \frac{(D+1)^{n-1}}{2}$$

so if $f_n = f \chi_{I_n}$, we have

$$Tf_n(x_n) > \frac{1}{2}(D+1)^{n-1}$$

for $n \in A$. Thus for $x \in I_{n+1}$,

$$T(2(D+1)^2 f_n)(x) > (D+1)^{n+1}$$

and so

$$\sum_{n \in A} \leq \sum_n \int_{\{x: T(2(D+1)^2 f_n)(x) > (D+1)^{n+1}\}} \Phi_2((D+1)^{n+1} w(x)) t(x) dx.$$

Since (1.8) implies the weak-type boundedness (1.12) for T ,

$$\begin{aligned} \sum_{n \in A} &\leq \sum_n \Phi_2 \circ \Phi_1^{-1} \left(\int \Phi_1(2C(D+1)^2 f_n(x) u(x)) v(x) dx \right) \\ &\leq \Phi_2 \circ \Phi_1^{-1} \left(\sum_n \int \Phi_1(2C(D+1)^2 f_n(x) u(x)) v(x) dx \right) \quad \text{by (2.3)} \\ &= \Phi_2 \circ \Phi_1^{-1} \left(\int \Phi_1(2C(D+1)^2 f_n(x) u(x)) v(x) dx \right). \end{aligned}$$

It will suffice to get a similar estimate for $\sum_{n \in B}$. Set

$$W(x) = 2D(D+1)^2 k(x_n, x_{n-1}) w(x) \quad \text{for } x \in I_{n+1}$$

and

$$h(x) = x_{n-1} \quad \text{for } x \in I_{n+1}.$$

Then

$$\sum_{n \in B} \leq \int \Phi_2 \left(W(x) \int_0^{h(x)} f(y) dy \right) t(x) dx.$$

Let $g(x)$ be strictly increasing with a differentiable inverse, and with $g(x_{n+1}) = x_{n-1}$. There exist such g 's converging monotonically to h , and so, thanks to the Monotone Convergence Theorem, it will suffice to show

$$\int \Phi_2(W(x) I_g f(x)) t(x) dx \leq \Phi_2 \circ \Phi_1^{-1} \left(\int \Phi_1(Cu f) v \right)$$

with I_g as in Lemma 5.2. This will hold provided

$$(5.5) \quad \int_0^x \Psi_1 \left(\frac{\alpha_W(\lambda, g^{-1}(x))}{C\lambda u(y)v(y)} \right) v(y) dy \leq \alpha_W(\lambda, g^{-1}(x)) < \infty$$

where

$$\alpha_W(\lambda, x) = \Phi_2 \circ \Phi_1^{-1} \left(\int_x^\infty \Phi_2(\lambda W(y)) t(y) dy \right).$$

As usual, $\Psi_1(\alpha/(C\lambda uv))/\alpha$ increases in α , so it will suffice to show (5.5) with α_W replaced by something larger. But

$$\alpha_W(\lambda, g^{-1}(x)) = \Phi_2 \circ \Phi_1^{-1} \left(\int_{g^{-1}(x)}^\infty \Phi_2(\lambda W(y)) t(y) dy \right)$$

and so, if $g^{-1}(x) \in I_{n+1}$,

$$\begin{aligned} \alpha_W(\lambda, g^{-1}(x)) &= \Phi_2 \circ \Phi_1^{-1} \left(\sum_{k \geq n} \int_{I_{k+1}} \Phi_2(\lambda 2D(D+1)^2 k(x_k, x_{k-1}) w(y)) t(y) dy \right) \\ &\leq \Phi_2 \circ \Phi_1^{-1} \left(\sum_{k \geq n} \int_{I_{k+1}} \Phi_2(\lambda 2D(D+1)^2 k(y, x_{k-1}) w(y)) t(y) dy \right). \end{aligned}$$

But $x_{n-1} = g(x_{n+1}) \geq g(g^{-1}(x)) = x$ and so

$$\begin{aligned} \alpha_W(\lambda, g^{-1}(x)) &= \Phi_2 \circ \Phi_1^{-1} \left(\int_x^\infty \Phi_2(\lambda 2D(D+1)^2 k(y, x) w(y)) t(y) dy \right) \\ &= \beta(2\lambda D(D+1)^2, x), \end{aligned}$$

whence (5.5) will hold provided

$$\int_0^x \Psi_1 \left(\frac{\beta(2\lambda D(D+1)^2, x)}{C\lambda u(y)v(y)} \right) v(y) dy \leq \beta(2\lambda D(D+1)^2, x) < \infty,$$

which is (1.9) with C replaced by $C/(2D(D+1)^2)$. This completes the proof of Theorem 1.7.

Proof of Theorem 4.5. The necessity is easy, since (4.6) forces the dual condition (2.9). For the sufficiency, we follow the previous proof. One of the estimates was handled by the weak-type boundedness (4.7); the other reduced to showing

$$(5.6) \quad \int \Phi \left(W(x) \int_0^{g(x)} f(y) dy \right) t(x) dx \leq \Phi_2 \circ \Phi_1^{-1} \left(\int \Phi_1(Cu f)v \right)$$

where

$$W(x) = 2D(D+1)^2 k(x_n, x_{n-1}) w(x) \quad \text{for } x \in I_{n+1}$$

and g was continuous, strictly increasing, with $g(x_{n+1}) = x_{n-1}$. Let $W_g(y) = W(g^{-1}(y))$ and $t_g(y) = t(g^{-1}(y))(d/dy)g^{-1}(y)$. Then, as in Lemma 5.2, the left side of (5.6) is

$$\int \Phi(W_g(x) I f(x)) t_g(x) dx$$

so proving (5.6) is equivalent to showing

$$(5.7) \quad \int \Psi((uv)^{-1} |I^* f(x)|) v(x) dx \leq \int \Psi(C(W_g t_g)^{-1} |f(x)|) t_g(x) dx,$$

by Corollary 2.7, where I^* is the adjoint to the Hardy operator,

$$I^* f(x) = \int_x^\infty f(y) dy.$$

By Remark 4.4, (5.7) holds provided the corresponding weak-type inequality holds, which, by (3.3), means that

$$(5.8) \quad \int_x^\infty \tilde{\Phi} \left(\frac{\alpha^*(\lambda, x)}{C\lambda} W_g(y) \right) t_g(y) dy \leq \alpha^*(\lambda, x) < \infty,$$

where

$$\alpha^*(\lambda, x) = \int_0^x \Psi \left(\frac{\lambda}{u(y)v(y)} \right) v(y) dy$$

and $\tilde{\Phi}$ is the minimal complement to Ψ . So we have to show that (4.8) implies (5.8). By (3.3), (4.8) is equivalent to

$$(5.9) \quad \int_x^\infty \tilde{\Phi} \left(\frac{\alpha^*(\lambda, x)}{C\lambda} k(y, x) w(y) \right) t(y) dy \leq \alpha^*(\lambda, x) < \infty.$$

After a change of variable, the left side of (5.8) is

$$(5.10) \quad \int_{g^{-1}(x)}^\infty \tilde{\Phi} \left(\frac{\alpha^*(\lambda, x)}{C\lambda} W(y) \right) t(y) dy \leq \alpha^*(\lambda, x) < \infty$$

and, exactly as in the previous proof, when $g^{-1}(x) \in I_{n+1}$, this is controlled by

$$\begin{aligned} \sum_{k \geq n} \int_{I_{k+1}} \tilde{\Phi} \left(\frac{\alpha^*(\lambda, x)}{C\lambda} 2D(D+1)^2 k(x_k, x_{k-1}) w(y) \right) t(y) dy \\ \leq \int_{x_{n-1}}^\infty \tilde{\Phi} \left(\frac{\alpha^*(\lambda, x)}{C\lambda} 2D(D+1)^2 k(y, x_{n-1}) w(y) \right) t(y) dy \\ \leq \int_x^\infty \tilde{\Phi} \left(\frac{\alpha^*(\lambda, x)}{C\lambda} 2D(D+1)^2 k(y, x) w(y) \right) t(y) dy \end{aligned}$$

since, once again, $x_{n-1} \geq x$. Choosing C appropriately in (5.10), we obtain (5.8) from (5.9).

Unlike the case of the Hardy operator, weak-type boundedness for a GHO need not imply strong-type boundedness. This can be seen by a simple example. Take $\Phi_1(x) = \Phi_2(x) = x^2/2$ and $k(x, y) = (x - y)^{1/2}$. Let $u(x) = w(x) = 1$, $v(x) = e^x$ and $t(x) = x^{-2}$. Then

$$\beta(\lambda, x) = \frac{1}{2} \int_x^\infty \lambda^2 (y - x) y^{-2} dy = \infty$$

so (1.9) fails, and with it goes the strong-type boundedness. On the other hand, (1.8) easily reduces to

$$\int_x^\infty y^{-2} dy \int_0^x (y-x)e^{-y} dy \leq C$$

which clearly holds for all $x > 0$. So the weak-type boundedness holds.

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On group extensions of 2-fold simple ergodic actions

by

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Abstract. Compact group extensions of 2-fold simple actions of locally compact second countable amenable groups are considered. It is shown what the elements of the centralizer of such a system look like. It is also proved that each factor of such a system is determined by a compact subgroup in the centralizer of a normal factor.

1. Introduction. In this paper we describe the centralizer and the structure of factors for ergodic group extensions of a 2-fold simple action of a locally compact second countable amenable group on a standard Borel space.

Our method is an adaptation of the methods developed by Lemańczyk and Mentzen in [5], [7], [4] for \mathbb{Z} -actions and consists in a description of the ergodic joinings of these actions. We show that ergodic joinings are relatively independent extensions of certain isomorphisms between normal natural factors. For that we will need the ergodic theorem for our general case (for proofs we refer to [1] and Krengel's book [3]). For ergodic \mathbb{Z} -actions the form of the elements of the centralizer for group extensions of a discrete spectrum transformation was found by D. Newton [8] in the abelian case and by M. K. Mentzen [7] in general case. Here we generalize Mentzen's result to arbitrary locally compact second countable group actions. We also generalize the main result of [5], [7] describing factors in terms of compact subgroups in the centralizers of normal natural factors (for related results see [2], [4], [9]).

2. Definitions and theorems. Let \mathfrak{R} be a locally compact second countable group and (X, \mathcal{B}, μ) be a standard Borel space. We will say that \mathfrak{R} acts on (X, \mathcal{B}, μ) if there exists a Borel map from $X \times \mathfrak{R}$ to X (we denote it by $(x, t) \mapsto xt$) such that

- (i) $x(t_1 t_2) = (x t_1) t_2$ for all $t_1, t_2 \in \mathfrak{R}$ and a.a. $x \in X$,
- (ii) $x e = x$ for a.a. $x \in X$.