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Integral operators on weighted amalgams

by

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Abstract. For large classes of indices, we characterize the weights u, v for which the Hardy operator is bounded from $\ell^{\bar q}(L_v^{\bar p})$ into $\ell^q(L_u^p)$. For more general operators of Hardy type, norm inequalities are proved which extend to weighted amalgams known estimates in weighted L^p-spaces. Amalgams of the form $\ell^q(L_w^p)$, $1 < p, q < \infty$, $q \neq p$, $w \in A_p$, are also considered and sufficient conditions for the boundedness of the Hardy–Littlewood maximal operator and local maximal operator in these spaces are obtained.

1. Introduction. The weighted amalgam on the real line with weight w is the space $\ell^q(L_w^p)$, $1 \le p, q \le \infty$, consisting of functions which are locally in the weighted Lebesgue space L_w^p where the integrals over intervals $[n, n+1]$ form an ℓ^q sequence. The norm

$$\|f\|_{p,w,q} = \Big\{ \sum_{n\in\mathbb{Z}} \Big(\int_n^{n+1} w(x)|f(x)|^p\,dx \Big)^{q/p} \Big\}^{1/q}, \tag{1.1}$$

with the usual convention applying when p or q are infinite, makes $\ell^q(L_w^p)$ into a Banach space.

Amalgams arise naturally in harmonic analysis and were introduced by N. Wiener in 1926. For a systematic study of these spaces and their role in Fourier analysis we refer to [7] and [9]. In this paper we study operators of Hardy type and the Hardy–Littlewood maximal function and extend to weighted amalgams some of the well known mapping properties in weighted L^p-spaces (cf. [1], [4], [13]). Specifically we characterize in Section 2 the weight functions u and v for which the Hardy operator (and its dual) is bounded from $\ell^{\bar q}(L_v^{\bar p})$ into $\ell^q(L_u^p)$, for large classes of indices. The more

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general Hardy-type operators

$$(Kf)(x)=\int_{-\infty}^{x} k(x,y)f(y)\,dy,\qquad (K^*f)(t)=\int_{t}^{\infty} k(x,t)f(x)\,dx$$

are considered in Section 3 and conditions on the weight functions u, v in terms of the kernel $k(x,y)$ are given which imply their boundedness between weighted amalgams. This extends to amalgams corresponding L^p results of [1], while the case $k=1$ reduces to that of the Hardy operators discussed in Section 2.

The significance of the Hardy–Littlewood maximal operator in harmonic analysis is well documented ([8]). If

$$(Mf)(x)=\sup_{h>0}\frac{1}{2h}\int_{x-h}^{x+h}|f(y)|\,dy,\qquad (M_1f)(x)=\sup_{\substack{|I|\le 1\\ x\in I}}\frac{1}{|I|}\int_I |f(y)|\,dy$$

are the maximal and local maximal operators on $\mathbb{R}$, then conditions on the weight function w are given which imply their boundedness on $\ell^q(L^p_w)$, $1<p\le q<\infty$, $q\ne p$. These results are contained in Section 4. For the Hardy–Littlewood operator M, the conditions on w are somewhat stronger than the A_p condition (Theorems 4.2, 4.4, 4.5). For the local maximal function M_1, the condition $w\in A_p$ is, however, sufficient for the boundedness of $M_1:\ell^q(L^p_w)\to\ell^q(L^p_w)$, $1<p<q<\infty$ (Theorem 4.7).

Throughout $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{N}$ denote the real line, the positive real numbers, the integers and natural numbers respectively. The conjugate index p' of $p\in\mathbb{R}_+$ is given by $p'=p/(p-1)$, $p\ne 1$ and similarly for $\overline{p}$, q, $\overline{q}$ etc. Positive constants are denoted by $A, B, C,\ldots$ (sometimes with subscripts). χ_E is the characteristic function of the set E and we also write $\chi_{[n,n+1]}=\chi_n$. Inequalities (such as (1.2) below) are interpreted to mean that if the right side is finite so is the left side and the inequality holds.

We conclude this section by stating some known results. The following inclusion relations are easily established (cf. [7]):

PROPOSITION 1.1. *If $1\le q_1\le q_2\le\infty$ and $1\le p\le\infty$, then $\ell^{q_1}(L^p_w)\subset\ell^{q_2}(L^p_w)$, and if $1\le p_1\le p_2\le\infty$, $1\le q\le\infty$, then $\ell^q(L^{p_2}_w)\subset\ell^q(L^{p_1}_w)$.*

THEOREM 1.2 ([14, Theorem 7.1]). *Suppose $1<p,\ q<\infty$ and μ, ν are nonnegative regular Borel measures on $\mathbb{R}$. Then there exists a constant $B>0$ such that*

$$\Big\{\int_{\mathbb{R}}\Big(\int_{(-\infty,x]} g\,d\mu\Big)^q d\nu(x)\Big\}^{1/q}\le B\Big\{\int_{\mathbb{R}} g^p\,d\mu\Big\}^{1/p},\tag{1.2}$$

for all nonnegative $g\in L^p_\mu$, if and only if

(i) *in case $p\le q$,*

$$B_1\equiv\sup_{y\in\mathbb{R}}\Big(\int_{[y,\infty)}d\nu\Big)^{1/q}\Big(\int_{(-\infty,y]}d\mu\Big)^{1/p'}<\infty;$$

(ii) *in case $q<p$,*

$$B_2\equiv\Big\{\int_{\mathbb{R}}\Big(\int_{[y,\infty)}d\nu\Big)^{r/q}\Big(\int_{(-\infty,y]}d\mu\Big)^{r/q'}d\mu(y)\Big\}^{1/r}<\infty,$$

where $1/r=1/q-1/p$.

Furthermore, if B is the smallest constant such that (1.2) *holds, then $B\sim B_i$ for $i=1,2$.*

Here and in the sequel $A\sim B$ means that there exist positive constants C_1, C_2 such that $C_1\le A/B\le C_2$.

The following corollary of Theorem 1.2 will be used in our study:

COROLLARY 1.3. *Suppose $1<p,q<\infty$ and $u_k\ge 0$, $v_k>0$, $k\in\mathbb{Z}$. Then there exists a constant $B>0$ such that*

$$\Big\{\sum_{n\in\mathbb{Z}}u_n\Big(\sum_{k=-\infty}^{n}a_k\Big)^q\Big\}^{1/q}\le B\Big(\sum_{n\in\mathbb{Z}}v_na_n^p\Big)^{1/p},\tag{1.3}$$

for all nonnegative sequences $\{a_k\}\in\ell^p_{\{v_n\}}$, if and only if

(i) *in case $1<p\le q<\infty$,*

$$B_1\equiv\sup_{m\in\mathbb{Z}}\Big(\sum_{n=m}^{\infty}u_n\Big)^{1/q}\Big(\sum_{n=-\infty}^{m}v_n^{1-p'}\Big)^{1/p'}<\infty;$$

(ii) *in case $1<q<p<\infty$,*

$$B_2\equiv\Big\{\sum_{m\in\mathbb{Z}}\Big(\sum_{n=m}^{\infty}u_n\Big)^{r/q}\Big(\sum_{n=-\infty}^{m}v_n^{1-p'}\Big)^{r/q'}v_m^{1-p'}\Big\}^{1/r}<\infty,$$

where $1/r=1/q-1/p$.

Furthermore, if B is the smallest constant such that (1.3) *holds, then $B\sim B_i$ for $i=1,2$.*

Proof. Let δ_k denote the Dirac measure concentrated at $k\in\mathbb{Z}$ and define $\mu=\sum_{k\in\mathbb{Z}}v_k^{1-p'}\delta_k$ and $\nu=\sum_{k\in\mathbb{Z}}u_k\delta_k$. Then the expressions B_i, $i=1,2$, in Theorem 1.2 reduce to those in Corollary 1.3 above. Moreover, the Hardy inequality (1.2) becomes

$$\Big\{\sum_{n\in\mathbb{Z}}u_n\Big(\sum_{k=-\infty}^{n}v_k^{1-p'}g(k)\Big)^q\Big\}^{1/q}\le B\Big(\sum_{n\in\mathbb{Z}}v_n^{1-p'}g(n)^p\Big)^{1/p}$$

for all nonnegative $g \in L^p_\mu = \ell^p_{\{v_n^{1-p'}\}}$. The result then follows if we set $a_k = v_k^{1-p'} g(k)$, for then $v_k^{1-p'} g(k)^p = a_k^p v_k$.

Note that Corollary 1.3 still holds if $\mathbb{Z}$ is replaced by $\mathbb{N}$. For this latter case the result was also obtained by a different method by G. Bennett ([3]).

2. The Hardy operator in amalgams. In this section we consider the operators P and P^* defined by

$$(Pf)(x) = \int_{-\infty}^{x} f, \quad (P^*f)(x) = \int_{x}^{\infty} f.$$

Let T be either P or P^* and u and v weight functions, that is, $u > 0$, $v > 0$ a.e. with $u \in L^1_{\mathrm{loc}}$ and $v^{1-\overline{p}'} \in L^1_{\mathrm{loc}}$, where $\overline{p} \in (1,\infty)$ is some index. Then conditions on the weight functions are given which are shown to be equivalent to the boundedness of $T : \ell^{\overline{q}}(L^{\overline{p}}_v) \to \ell^q(L^p_u)$ for certain ranges of the indices $p, \overline{p}, q, \overline{q}$.

THEOREM 2.1. *Suppose u and v are weight functions and $1 < p, \overline{p} < \infty$, $1 < \overline{q} \le q$. Then there is a constant $B > 0$ such that*

$$\|Pf\|_{p,u,q} \le B\|f\|_{\overline{p},v,\overline{q}} \quad \text{for all } f \in \ell^{\overline{q}}(L^{\overline{p}}_v) \tag{2.1}$$

if and only if

(a) *in case $\overline{p} \le p$,*

$$C_1 \equiv \sup_{m\in\mathbb{Z}} \Big\{ \sum_{n=m}^{\infty} \Big(\int_n^{n+1} u \Big)^{q/p} \Big\}^{1/q} \Big\{ \sum_{n=-\infty}^{m-1} \Big(\int_n^{n+1} v^{1-\overline{p}'} \Big)^{\overline{q}'/\overline{p}'} \Big\}^{1/\overline{q}'} < \infty$$

and

$$C_2 \equiv \sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \Big(\int_{n+\alpha}^{n+1} u \Big)^{1/p} \Big(\int_n^{n+\alpha} v^{1-\overline{p}'} \Big)^{1/\overline{p}'} < \infty;$$

(b) *in case $1 < p < \overline{p}$, $C_1 < \infty$ and*

$$C_3 \equiv \sup_{n\in\mathbb{Z}} \Big\{ \int_n^{n+1} \Big(\int_t^{n+1} u \Big)^{s/p} \Big(\int_n^t v^{1-\overline{p}'} \Big)^{s/p'} v(t)^{1-p'}\, dt \Big\}^{1/s} < \infty,$$

where $1/s = 1/p - 1/\overline{p}$.

Proof. *Sufficiency.* Since $|Pf| \le P(|f|)$ we may assume without loss of generality that $f \ge 0$. Moreover, $C_1 < \infty$ implies that $(Pf)(x)$ exists for every $x \in \mathbb{R}$. In order to estimate $\|Pf\|_{p,u,q}$ note that since

$$(Pf)(x) = \int_{-\infty}^{n} f\,dt + \int_n^x f\,dt \quad \text{for } x \in [n, n+1],$$

we have

$$\Big(\int_n^{n+1} |(Pf)(x)|^p u(x)\,dx \Big)^{1/p} \le U_n^{1/p} (Pf)(n) + T_n,$$

where

$$U_n = \int_n^{n+1} u \quad \text{and} \quad T_n = \Big\{ \int_n^{n+1} u(x) \Big(\int_n^x f \Big)^p dx \Big\}^{1/p}.$$

Let $a_k = \int_{k-1}^k f$. Then $(Pf)(n) = \sum_{k=-\infty}^n a_k$ and therefore

$$\|Pf\|_{p,u,q} \le \Big\{ \sum_{n\in\mathbb{Z}} U_n^{q/p} \Big(\sum_{k=-\infty}^{n} a_k \Big)^q \Big\}^{1/q} + \Big(\sum_{n\in\mathbb{Z}} T_n^q \Big)^{1/q} = I_1 + I_2.$$

Since $\overline{q} \le q$, Corollary 1.3(i) implies that

$$I_1 \le B \Big(\sum_{n\in\mathbb{Z}} V_n a_n^{\overline{q}} \Big)^{1/\overline{q}} \tag{2.2}$$

for any positive sequence $\{V_n\}$ satisfying

$$B_1 \equiv \sup_{m\in\mathbb{Z}} \Big(\sum_{n=m}^{\infty} U_n^{q/p} \Big)^{1/q} \Big(\sum_{n=-\infty}^{m} V_n^{1-\overline{q}'} \Big)^{1/\overline{q}'} < \infty.$$

By Hölder's inequality

$$a_n \le \Big(\int_{n-1}^{n} f^{\overline{p}} v \Big)^{1/\overline{p}} \Big(\int_{n-1}^{n} v^{1-\overline{p}'} \Big)^{1/\overline{p}'}.$$

If we choose $V_n = (\int_{n-1}^n v^{1-\overline{p}'})^{-\overline{q}/\overline{p}'}$, $n \in \mathbb{Z}$, then on substituting, the right side of (2.2) is dominated by $\|f\|_{\overline{p},v,\overline{q}}$ and with this choice $B_1 = C_1 < \infty$.

To estimate I_2, observe that Theorem 1.2 applied with measures $\mu = v^{1-\overline{p}'}\chi_n$, $\nu = u\chi_n$, and $f = gv^{1-\overline{p}'}$ yields

$$T_n \le C \Big(\int_n^{n+1} v f^{\overline{p}} \Big)^{1/\overline{p}}$$

under the following conditions:

(a) in case $1 < \overline{p} \le p$, if and only if $C_2 < \infty$, where $C \sim C_2$;

(b) in case $p < \overline{p}$, if and only if $C_3 < \infty$, where $C \sim C_3$. Therefore $I_2 \le C\|f\|_{\overline{p},v,q} \le C\|f\|_{\overline{p},v,\overline{q}}$, where the last inequality follows from Proposition 1.1, since $\overline{q} \le q$. From these two estimates of I_1 and I_2 the sufficiency of the theorem follows.

Necessity. Let U_n and V_n be defined as above. For any nonnegative sequence $\{a_k\}$ define $f=\sum_{k\in\mathbb{Z}} a_k v^{1-\overline{p}'}\chi_k$. Then, for $n\le x<n+1$,

$$(Pf)(x)=\sum_{k=-\infty}^{n} A_k + a_n\int_n^x v^{1-\overline{p}'},$$

where $A_k=a_{k-1}\int_{k-1}^k v^{1-\overline{p}'}$. This implies that

$$\Big(\int_n^{n+1}(Pf)(x)^p u(x)\,dx\Big)^{1/p}\ge\Big(\sum_{k=-\infty}^{n}A_k\Big)U_n^{1/p}$$

and therefore

$$\|Pf\|_{p,u,q}\ge\Big\{\sum_{n\in\mathbb{Z}}U_n^{q/p}\Big(\sum_{k=-\infty}^{n}A_k\Big)^q\Big\}^{1/q}.$$

On the other hand, an easy calculation shows that $\|f\|_{\overline{p},v,\overline{q}} = \{\sum_{n\in\mathbb{Z}} A_n^{\overline{q}}V_n\}^{1/\overline{q}}$ so that (2.1) implies that

$$\Big\{\sum_{n\in\mathbb{Z}}U_n^{q/p}\Big(\sum_{k=-\infty}^{n}A_k\Big)^q\Big\}^{1/q}\le B\Big(\sum_{n\in\mathbb{Z}}A_n^{\overline{q}}V_n\Big)^{1/\overline{q}}$$

for all nonnegative sequences $\{A_k\}\in\ell^{\overline{q}}_{\{V_n\}}$. Hence by Corollary 1.3, $C_1<\infty$ for $1<p,\overline{p}<\infty$, since $1<\overline{q}\le q<\infty$.

It remains to show that (2.1) implies $C_2<\infty$ if $\overline{p}\le p$, and $C_3<\infty$ if $\overline{p}>p$. Choose $f=g\chi_m$, where $g\ge0$ and $m\in\mathbb{Z}$ is fixed. Then by (2.1),

$$\Big\{\int_m^{m+1}\Big(\int_m^x g\Big)^p u(x)\,dx\Big\}^{1/p}\le\|Pf\|_{p,u,q}\le\|f\|_{\overline{p},v,\overline{q}}=B\Big(\int_m^{m+1}g^{\overline{p}}v\Big)^{1/\overline{p}}$$

for all $m\in\mathbb{Z}$ with B independent of m. Hence, by Theorem 1.2 applied with the appropriate measures (see also [2, Lemma 1.1], [13]) it follows that $C_2<\infty$ whenever $\overline{p}\le p$ and in case $\overline{p}>p$ that $C_3<\infty$. This completes the proof of the theorem.

In case $1<q<\overline{q}$ the embedding property of the involved sequence spaces differs from that given in Theorem 2.1. The result is given next.

PROPOSITION 2.2. *Suppose u and v are weight functions*, $1<p,\ \overline{p}<\infty$, $1<q<\overline{q}$, $1/r=1/q-1/\overline{q}$. *Define*

$$C_1'\equiv\Big\{\sum_{k\in\mathbb{Z}}\Big[\sum_{n=k}^{\infty}\Big(\int_n^{n+1}u\Big)^{q/p}\Big]^{r/q}\Big[\sum_{n=-\infty}^{k}\Big(\int_{n-1}^{n}v^{1-\overline{p}'}\Big)^{\overline{q}'/\overline{p}'}\Big]^{r/q'}\times\Big(\int_{n-1}^{n}v^{1-\overline{p}'}\Big)^{\overline{q}'/\overline{p}'}\Big\}^{1/r}.$$

(i) *For* $1<\overline{p}\le p$, *let*

$$D_n\equiv\sup_{\alpha\in(0,1)}\Big(\int_{n+\alpha}^{n+1}u\Big)^{1/p}\Big(\int_n^{n+\alpha}v^{1-\overline{p}'}\Big)^{1/\overline{p}'}.$$

(ii) *For* $1<p<\overline{p}$, *let*

$$D_n'\equiv\Big\{\int_n^{n+1}\Big(\int_t^{n+1}u\Big)^{s/p}\Big(\int_n^t v^{1-\overline{p}'}\Big)^{s/p'}v(t)^{1-\overline{p}'}\,dt\Big\}^{1/r},$$

where $1/s=1/p-1/\overline{p}$. *Then there is a constant B such that* (2.1) *holds if* $C_1'<\infty$ *and* $\{D_n\}\in\ell^r$ (*respectively* $C_1'<\infty$ *and* $\{D_n'\}\in\ell^r$) *in case* $1<\overline{p}<p$ (*respectively in case* $1<p<\overline{p}$).

Conversely, $C_1'<\infty$ *is necessary for* (2.1) *whenever* $1<p,\overline{p}<\infty$. *Also*, $\sup_n D_n<\infty$ (*respectively* $\sup_n D_n'<\infty$) *is necessary for* (2.1) *in case* $1<\overline{p}\le p$ (*respectively, in case* $1<p<\overline{p}$).

Proof. Again, without loss of generality we assume $f\ge0$. With the notation as in the proof of Theorem 2.1 it follows as before that

$$\|Pf\|_{p,u,q}\le\Big\{\sum_{n\in\mathbb{Z}}U_n^{q/p}\Big(\sum_{k=-\infty}^{n}a_k\Big)^q\Big\}^{1/q}+\Big(\sum_{n\in\mathbb{Z}}T_n^q\Big)^{1/q}=I_1+I_2.$$

Since $q<\overline{q}$, Corollary 1.3(ii) yields

$$I_1\le B\Big(\sum_{n\in\mathbb{Z}}V_n a_n^{\overline{q}}\Big)^{1/\overline{q}}\tag{2.3}$$

if and only if

$$B_2\equiv\Big\{\sum_{m\in\mathbb{Z}}\Big(\sum_{n=m}^{\infty}U_n^{q/p}\Big)^{r/q}\Big(\sum_{n=-\infty}^{m}V_n^{1-\overline{q}'}\Big)^{r/q'}V_m^{1-\overline{q}'}\Big\}^{1/r}<\infty.$$

As in the proof of Theorem 2.1, if we choose $V_n=(\int_{n-1}^n v^{1-\overline{p}'})^{-\overline{q}/\overline{p}'}$, the right side of (2.3) is dominated by $\|f\|_{\overline{p},v,\overline{q}}$. Moreover, with this choice of V_n, $B_2=C_1'<\infty$ for $1<p,\ \overline{p}<\infty$.

(i) If $1<\overline{p}\le p$, then by Theorem 1.2(i) with measures $\mu=v^{1-\overline{p}'}\chi_n$, $\nu=u\chi_n$ and $f=gv^{1-\overline{p}}$,

$$\Big\{\int_n^{n+1}\Big(\int_n^x f\Big)^p u(x)\,dx\Big\}^{1/p}\le C_n\Big(\int_n^{n+1}f^{\overline{p}}v\Big)^{1/\overline{p}}$$

if $C_n\sim D_n$. Therefore by Hölder's inequality with index $\gamma=\overline{q}/q$,

$$\Big(\sum_{n\in\mathbb{Z}}T_n^q\Big)^{1/q}\le A\Big\{\sum_{n\in\mathbb{Z}}D_n^q\Big(\int_n^{n+1}f^{\overline{p}}v\Big)^{q/\overline{p}}\Big\}^{1/q}\le A\|f\|_{\overline{p},v,\overline{q}}\Big(\sum_{n\in\mathbb{Z}}D_n^{q\gamma'}\Big)^{1/(q\gamma')}.$$

Since $q\gamma'=r$ and $\{D_n\}\in\ell^r$ the result follows.

(ii) If $1 < p < \overline{p}$ the argument is the same as in (i), only now we apply Theorem 1.2(ii) instead of 1.2(i). Hence the sufficiency assertions follow.

The necessity of the condition $C_1' < \infty$ is established in the same way as the necessity of $C_1 < \infty$ in Theorem 2.1, only now part (ii) of Corollary 1.3 is applied instead of part (i).

To prove the necessity of $\sup_n D_n < \infty$ (respectively $\sup_n D_n' < \infty$) we note that if $\|Pf\|_{p,u,q} \le B\|f\|_{\overline{p},v,\overline{q}}$ holds for all $f \in \ell^{\overline{q}}(L_v^{\overline{p}})$ with some $B > 0$, then the inequality holds in particular for all f in the subspace $\ell^q(L_v^{\overline{p}})$ of $\ell^{\overline{q}}(L_v^{\overline{p}})$ since $q < \overline{q}$ (cf. Proposition 1.1). Theorem 2.1 is then applicable with $\overline{q} = q$ and this proves the assertions.

The preceding results are now applied to show that the dual operator P^* of P coincides on the amalgam space with $Q : g \to \int_t^\infty g$. A classical duality argument then yields the corresponding conditions for the boundedness of Q.

THEOREM 2.3. *Suppose* $1 < p, \overline{p} < \infty$, $1 < \overline{q} \le q < \infty$ *and* u, v *are weight functions. Let* C_1, C_2, *and* C_3 *be as in Theorem* 2.1. *Then the operator* Q *is bounded from* $\ell^{q'}(L^{p'}_{u^{1-p'}})$ *into* $\ell^{\overline{q}'}(L^{\overline{p}'}_{v^{1-\overline{p}'}})$ *if and only if*

(i) $C_1 < \infty$ *and* $C_2 < \infty$ *in case* $\overline{p} \le p$,

(ii) $C_1 < \infty$ *and* $C_3 < \infty$ *in case* $p < \overline{p}$.

Proof. We shall prove the assertion for $\overline{p} \le p$ only since the remaining case can be dealt with in a similar way.

(a) Suppose C_1 and C_2 are finite. By Theorem 2.1, P is bounded from $\ell^{\overline{q}}(L_v^{\overline{p}})$ into $\ell^q(L_u^p)$. Since $(L_u^p)^* = L^{p'}_{u^{1-p'}}$ for $1 < p < \infty$, $u > 0$ a.e., and $(\ell^q)^* = \ell^{q'}$ for $1 < q < \infty$, the dual of $\ell^q(L_u^p)$ is $\ell^{q'}(L^{p'}_{u^{1-p'}})$ (cf. [9]). Similarly $[\ell^{\overline{q}}(L_v^{\overline{p}})]^* = \ell^{\overline{q}'}(L^{\overline{p}'}_{v^{1-\overline{p}'}})$. As a consequence, the dual operator P^* of P is bounded from $\ell^{q'}(L^{p'}_{u^{1-p'}})$ into $\ell^{\overline{q}'}(L^{\overline{p}'}_{v^{1-\overline{p}'}})$. The sufficiency part of the theorem will thus be proved if we show that $P^* = Q$. To do this, let $f \ge 0$, $f \in \ell^{\overline{q}}(L_v^{\overline{p}})$, $g \ge 0$, $g \in \ell^{q'}(L^{p'}_{u^{1-p'}})$ and define $f_N = f\chi_{(-N,N)}$, $N \in \mathbb{N}$, $g_S = g\chi_{(-S,S)}$, $S \in \mathbb{N}$. In view of the local integrability of $v^{1-\overline{p}'}$ and u, it is easy to see that $f_N \in L^1$ and $g_S \in L^1$. Therefore

$$\int_{\mathbb{R}} (Pf_N)(x)g_S(x)\,dx = \int_{\mathbb{R}} (Qg_S)(t)f_N(t)\,dt \tag{2.4}$$

for all $N, S \in \mathbb{N}$. The sequences $(Pf_N)(x)g_S(x)$ and $(Qg_S)(t)f_N(t)$ are both increasing with N and S. Moreover, the former converges to $(Pf)(x)g(x)$ a.e. and the left integral of (2.4) is uniformly bounded by

$$\int_{\mathbb{R}} (Pf)(x)g(x)\,dx \le \|P\| \cdot \|f\|_{\overline{p},v,\overline{q}} \|g\|_{p',u^{1-p'},q'}\,.$$

It follows from the B. Levi theorem that

$$\lim_{\substack{N\to\infty\\ S\to\infty}} (Qg_S)(t)f_N(t) = (Qg)(t)f(t)$$

a.e. and in L^1. In particular, this means that $(Qg)(t) = \int_t^\infty g$ exists for each $t \in \mathbb{R}$ and

$$\int_{\mathbb{R}} (Pf)g = \int_{\mathbb{R}} f(Qg)\,.$$

By Tonelli's theorem this identity remains valid for all $f \in \ell^{\overline{q}}(L_v^{\overline{p}})$ and $g \in \ell^{q'}(L^{p'}_{u^{1-p'}})$, which yields $P^* = Q$.

(b) Suppose now that $Q : g \to \int_t^\infty g$ is bounded from $\ell^{q'}(L^{p'}_{u^{1-p'}})$ into $\ell^{\overline{q}'}(L^{\overline{p}'}_{v^{1-\overline{p}'}})$. Then (2.4) and arguments quite similar to those above show that $Q^* = P$. Hence P is bounded from $\ell^{\overline{q}}(L_v^{\overline{p}})$ into $\ell^q(L_u^p)$. The necessity part of Theorem 2.1 then shows that $C_1 < \infty$ and $C_2 < \infty$.

Remark 2.4. 1. The following reformulation of Theorem 2.3 may be more manageable:

THEOREM 2.3$'$. *Suppose* u *and* v *are weight functions and* $1 < p, \overline{p} < \infty$, $1 < \overline{q} \le q < \infty$. *Then there is a constant* $B > 0$ *such that* $\|P^*f\|_{p,u,q} \le B\|f\|_{\overline{p},v,\overline{q}}$ *if and only if*

(a) *in case* $1 < \overline{p} \le p$,

$$C_1^* \equiv \sup_{m\in\mathbb{Z}} \Big\{ \sum_{n=-\infty}^{m-1} \Big(\int_n^{n+1} u \Big)^{q/p} \Big\}^{1/q} \Big\{ \sum_{n=m}^{\infty} \Big(\int_n^{n+1} v^{1-\overline{p}'} \Big)^{\overline{q}'/\overline{p}'} \Big\}^{1/\overline{q}'} < \infty$$

and

$$C_2^* \equiv \sup_{n\in\mathbb{Z}} \sup_{0<\alpha<1} \Big(\int_n^{n+\alpha} u \Big)^{1/p} \Big(\int_{n+\alpha}^{n+1} v^{1-\overline{p}'} \Big)^{1/\overline{p}'} < \infty;$$

(b) *in case* $1 < p < \overline{p}$, $C_1^* < \infty$ *and*

$$C_3^* \equiv \sup_{n\in\mathbb{Z}} \Big\{ \int_n^{n+1} \Big(\int_n^t u \Big)^{s/p} \Big(\int_t^{n+1} v^{1-\overline{p}'} \Big)^{s/p'} u(t)\,dt \Big\}^{1/s} < \infty$$

where $1/s = 1/p - 1/\overline{p}$.

2. In case $1 < q < \overline{q} < \infty$, similar arguments can be used to obtain the dual analogue of Proposition 2.2. We omit the details.

3. Variants of Theorems 2.1 and 2.3 hold for amalgams of weighted Lebesgue spaces over $(0,\infty)$ and sequence spaces over $\mathbb{N}$. Of course if $p = q$, $\overline{p} = \overline{q}$ they reduce to the classical weighted Hardy inequalities [4], [11], [13].

We conclude this section by applying Theorem 2.1 to obtain the classical Hardy inequality in amalgams.

COROLLARY 2.5. *If $1 < p, q < \infty$ and $f \geq 0$, then*

$$\Big\{\sum_{n=0}^{\infty}\Big[\int_n^{n+1}\Big(\frac{1}{x}\int_0^x f\Big)^p dx\Big]^{q/p}\Big\}^{1/q} \leq B\Big\{\sum_{n=0}^{\infty}\Big(\int_n^{n+1} f^p\Big)^{q/p}\Big\}^{1/q}.$$

Proof. Let f be supported on $\mathbb{R}_+$ and set $p = \overline{p}$, $q = \overline{q}$, $u(x) = x^{-p}$, $v(x) = 1$ in Theorem 2.1. Then $C_1 \leq \sup_{m>0} m^{1/q'}(\sum_{n=m}^{\infty} n^{-q})^{1/q} < \infty$. Also, the integrals in the definition of C_2 are dominated by $(\alpha + n)^{-1} \times (1-\alpha)^{1/p}\alpha^{1/p'}$, which is bounded by 1 if $n \geq 1$. If $n = 0$, the integrals are dominated by $(p-1)^{-1/p}$. Hence the result follows.

3. Integral operators of Hardy type. We now consider the operator K defined by

$$(Kf)(x) = \int_{-\infty}^{x} k(x,y)f(y)\,dy, \qquad f \geq 0,$$

and its dual K^*. Here we assume that the kernel $k(x,y) \geq 0$ is nonincreasing in the first variable, nondecreasing in the second and defined on the set $\Delta = \{(x,y) \in \mathbb{R}^2 : y < x\}$. As in Section 2 we consider weight functions u and v on $\mathbb{R}$, that is, functions which satisfy $u > 0$, $v > 0$, $u \in L^1_{\text{loc}}$ and $v^{1-\overline{p}} \in L^1_{\text{loc}}$ for some $\overline{p} \in (1,\infty)$. In our results conditions on these weight functions u and v are given which imply the boundedness of K and K^* on amalgam spaces with weights u, v, thereby extending the corresponding L^p results given in [1].

THEOREM 3.1. *Let $1 < \overline{p} \leq p < \infty$, $1 < \overline{q} \leq q < \infty$. Suppose u and v are weight functions and K is the operator with kernel k defined above. If there exist numbers $\beta, \gamma \in [0,1]$ independent of $n \in \mathbb{Z}$ such that*

$$\text{(3.1)}\qquad D_1 \equiv \sup_{m\in\mathbb{Z}}\Big\{\sum_{n=m+1}^{\infty} k(n,m)^{\gamma q}\Big(\int_n^{n+1} u\Big)^{q/p}\Big\}^{1/q} \times\Big\{\sum_{n=-\infty}^{m} k(m+1,n)^{\overline{q}'(1-\gamma)}\Big(\int_{n-1}^{n} v^{1-\overline{p}'}\Big)^{\overline{q}'/\overline{p}'}\Big\}^{1/\overline{q}'}$$

and

$$\text{(3.2)}\qquad D_2 \equiv \sup_{n\in\mathbb{Z}}\sup_{\alpha\in(n-1,n+1)}\Big\{\int_\alpha^{n+1} k(y,\alpha)^{\beta p}u(y)\,dy\Big\}^{1/p} \times\Big\{\int_{n-1}^{\alpha} k(\alpha,y)^{\overline{p}'(1-\beta)}v(y)^{1-\overline{p}'}\,dy\Big\}^{1/\overline{p}'}$$

are finite, then there exists a $C > 0$ such that $\|Kf\|_{p,u,q} \leq C\|f\|_{\overline{p},v,\overline{q}}$ for all $f \in \ell^{\overline{q}}(L^{\overline{p}}_v)$.

Proof. We use the method of majorization quite similar to that used in the proof of Theorem 2.1. Due to the possible singularities of the kernel $k(x,y)$ on the diagonal $x = y$ a modification of the decomposition of Kf is required. Specifically, for $f \geq 0$ and for $n \leq x \leq n+1$,

$$(Kf)(x) = \int_{-\infty}^{n-1} k(x,y)f(y)\,dy + \int_{n-1}^{x} k(x,y)f(y)\,dy = T_1(x,n) + T_2(x,n).$$

If $U_n = \int_n^{n+1} u$ and $a_i = \int_{i-1}^{i} f$, then because of the monotonicity assumptions on $k(x,y)$,

$$\Big(\int_n^{n+1} T_1(x,n)^p u(x)\,dx\Big)^{1/p} \leq U_n^{1/p}T_1(n,n) = U_n^{1/p}\int_{-\infty}^{n-1} k(n,y)f(y)\,dy$$

$$= U_n^{1/p}\sum_{i=-\infty}^{n-1}\int_{i-1}^{i} k(n,y)f(y)\,dy \leq U_n^{1/p}\sum_{i=-\infty}^{n-1} k(n,i)a_i.$$

Therefore from [1, Theorem 4.1] it follows that

$$\|T_1(\cdot,n)\|_{p,u,q} \leq \Big(\sum_{n\in\mathbb{Z}} U_{n+1}^{q/p}\Big|\sum_{i=-\infty}^{n} k(n+1,i)a_i\Big|^q\Big)^{1/q} \leq AD_1\Big(\sum_{n\in\mathbb{Z}} V_n a_n^{\overline{q}}\Big)^{1/\overline{q}}$$

provided

$$\text{(3.3)}\qquad \sup_m\Big(\sum_{n=m}^{\infty} k(n+1,m)^{\gamma q}U_{n+1}^{q/p}\Big)^{1/q} \times\Big(\sum_{n=-\infty}^{m} k(m+1,n)^{(1-\gamma)\overline{q}'}V_n^{1-\overline{q}'}\Big)^{1/\overline{q}'} < \infty.$$

If we choose $V_n = (\int_{n-1}^{n} v^{1-\overline{p}})^{-\overline{q}/\overline{p}'}$, then (3.3) coincides with D_1 of (3.1). As in the proof of Theorem 2.1,

$$\Big(\sum_{n\in\mathbb{Z}} V_n a_n^{\overline{q}}\Big)^{1/\overline{q}} \leq \|f\|_{\overline{p},v,\overline{q}}.$$

Next observe that an application of [1, Theorem 2.1] yields

$$\Big(\int_n^{n+1} T_2(x,n)^p u(x)\,dx\Big)^{1/p} \leq \Big\{\int_{n-1}^{n+1} u(x)\Big(\int_{n-1}^{x} k(x,y)f(y)\,dy\Big)^p dx\Big\}^{1/p}$$

$$\leq AD_2\Big(\int_{n-1}^{n+1} vf^{\overline{p}}\Big)^{1/\overline{p}}$$

provided that D_2 given by (3.2) is finite. Therefore

$$\|T_2(\cdot,n)\|_{p,u,q} \le AD_2\Big\{\sum_{n\in\mathbb{Z}}\Big(\int_{n-1}^{n} vf^{\bar p} + \int_{n}^{n+1} vf^{\bar p}\Big)^{q/\bar p}\Big\}^{1/q}$$

$$\le 2AD_2\Big\{\sum_{n\in\mathbb{Z}}\Big(\int_{n}^{n+1} vf^{\bar p}\Big)^{q/\bar p}\Big\}^{1/q} \le 2AD_2\|f\|_{\bar p,v,\bar q},$$

where the last inequality follows from Proposition 1.1. Since $\|Kf\|_{p,u,q} \le \|T_1(\cdot,n)\|_{p,u,q} + \|T_2(\cdot,n)\|_{p,u,q}$ the result follows.

Remark 3.2. If $k(x,y)$ is defined on the larger set $\widetilde{\Delta} = \{(x,y)\in\mathbb{R}^2 : y\le x\}$ then the conditions (3.1) and (3.2) in Theorem 3.1 can be replaced by

$$\widetilde{D}_1 \equiv \sup_{m\in\mathbb{Z}}\Big\{\sum_{n=m}^{\infty} k(n,m)^{\gamma q}\Big(\int_{n}^{n+1} u\Big)^{q/p}\Big\}^{1/q} \times\Big\{\sum_{n=-\infty}^{m} k(m,n)^{(1-\gamma)\bar q'}\Big(\int_{n-1}^{n} v^{1-\bar p'}\Big)^{\bar q'/\bar p'}\Big\}^{1/\bar q'} < \infty$$

and

$$\widetilde{D}_2 \equiv \sup_{n\in\mathbb{Z}}\sup_{\alpha\in(n,n+1)}\Big(\int_{\alpha}^{n+1} k(y,\alpha)^{\beta p}u(y)\,dy\Big)^{1/p} \times\Big(\int_{n}^{\alpha} k(\alpha,y)^{\bar p'(1-\beta)}v(y)^{1-\bar p'}\,dy\Big)^{1/\bar p'} < \infty$$

to insure the boundedness of $K:\ell^{\bar q}(L_v^{\bar p}) \to \ell^q(L_u^p)$. The proof of this is quite similar to that of Theorem 3.1 and hence omitted.

Arguments similar to those used to prove Theorem 2.3 show that if D_1 and D_2 of Theorem 3.1 are finite, the dual operator K^* of K on $\ell^{q'}(L^{p'}_{u^{1-p'}})$ coincides with $J: g \to \int_t^\infty k(x,t)g(x)\,dx$. With an appropriate change of the names of weights and role of the indices we can state the dual result of Theorem 3.1 as follows:

Theorem 3.3. *Suppose u and v are weight functions ($u>0$) and $1<\bar p\le p<\infty$, $1<\bar q\le q<\infty$, $\beta,\gamma\in[0,1]$. If*

$$D_1^* \equiv \sup_{m\in\mathbb{Z}}\Big\{\sum_{n=-\infty}^{m} k(m+1,n)^{\gamma q}\Big(\int_{n-1}^{n} u\Big)^{q/p}\Big\}^{1/q} \times\Big\{\sum_{n=m+1}^{\infty} k(n,m)^{\bar q'(1-\gamma)}\Big(\int_{n}^{n+1} v^{1-\bar p'}\Big)^{\bar q'/\bar p'}\Big\}^{1/\bar q'}$$

and

$$D_2^* \equiv \sup_{n\in\mathbb{Z}}\sup_{\alpha\in(n-1,n+1)}\Big\{\int_{n-1}^{\alpha} k(\alpha,y)^{p\beta}u(y)\,dy\Big\}^{1/p} \times\Big\{\int_{\alpha}^{n+1} k(y,\alpha)^{(1-\beta)\bar p'}v(y)^{1-\bar p'}\,dy\Big\}^{1/\bar p'}$$

are finite, then there exists a $C>0$ such that $\|K^*f\|_{p,u,q} \le C\|f\|_{\bar p,v,\bar q}$ for all $f\in\ell^{\bar q}(L_v^{\bar p})$.

Note that this result corresponds to the statement regarding P^* in Theorem 2.3′.

The previous theorems and remarks apply in particular to operators of convolution type. For example if

$$(k*f)(x) = \int_{-\infty}^{x} k(x-y)f(y)\,dy$$

where $k(x)\ge 0$ is nonincreasing on $\mathbb{R}$, then we obtain from Theorem 3.1 at once

Corollary 3.4. *If p, $\bar p$, q, $\bar q$, β, γ are as in Theorem 3.1 and the weight functions satisfy*

$$\sup_{m\in\mathbb{Z}}\Big\{\sum_{n=m+1}^{\infty} k(n-m)^{\gamma q}\Big(\int_{n}^{n+1} u\Big)^{q/p}\Big\}^{1/q} \times\Big\{\sum_{n=-\infty}^{m} k(m+1-n)^{(1-\gamma)\bar q'}\Big(\int_{n-1}^{n} v^{1-\bar p'}\Big)^{\bar q'/\bar p'}\Big\}^{1/\bar q'} < \infty$$

and

$$\sup_{n\in\mathbb{Z}}\sup_{\alpha\in(n-1,n+1)}\Big(\int_{\alpha}^{n+1} k(y-\alpha)^{\beta p}u(y)\,dy\Big)^{1/p} \times\Big(\int_{n-1}^{\alpha} k(\alpha-y)^{\bar p'(1-\beta)}v(y)^{1-\bar p'}\,dy\Big)^{1/\bar p'} < \infty,$$

then $\|k*f\|_{p,u,q} \le C\|f\|_{\bar p,v,\bar q}$.

From this corollary and its dual one obtains easily mapping properties of various fractional integral operators in amalgams, thereby extending the weighted L^p results of [1] to weighted amalgams. Note in particular that if we take $k(x) = x^{\alpha-1}/\Gamma(\alpha)$, $0<\alpha<1$ and f supported in $(0,\infty)$ we obtain from Corollary 3.4 weighted amalgam inequalities for the Riemann–Liouville fractional integral operator. From this one also deduces estimates for the left handed fractional maximal functions M_α, $0<\alpha<1$, defined by

$$(M_\alpha f)(x)=\sup_{0<h<x} h^{\alpha-1}\int_{x-h}^{x+h}|f|$$

since $(M_\alpha f)(x)\le \Gamma(\alpha)(I_\alpha f)(x)$, where I_α is the Riemann–Liouville fractional integral operator (cf. [1]).

Corresponding results can also be obtained for the Weyl fractional integral operator by using the dual form of Corollary 3.4 based on Theorem 3.3.

4. The Hardy–Littlewood maximal operator in amalgams. In this section weighted amalgam norm inequalities are proved for the Hardy–Littlewood maximal and local maximal operators on $\mathbb{R}$ defined by

$$(Mf)(x)=\sup_{h>0}\frac{1}{2h}\int_{x-h}^{x+h}|f(y)|\,dy,\qquad (M_1f)(x)=\sup_{|I_x|\le 1}\frac{1}{|I_x|}\int_{I_x}|f(y)|\,dy$$

respectively, where $x\in\mathbb{R}$ and I_x is an interval centered at x with length $|I_x|$.

The weights considered here belong to certain A_p weight classes. Recall that $w\in A_p$, $1<p<\infty$, if for every $I\subset\mathbb{R}$, there is a constant $C>0$ such that

$$\left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-1/(p-1)}\,dx\right)^{p-1}\le C<\infty, \tag{4.1}$$

and $w\in A_1$ if

$$\frac{1}{|I|}\int_I w(x)\,dx\le C\operatorname*{ess\,inf}_{x\in I} w(x). \tag{4.2}$$

Note that if $w(x)=|x|^\alpha$, then $w\in A_p$, $p\ge 1$, if and only if $-1<\alpha<p-1$.

The definition of the two-weighted A_p class is similar. Thus $(u,v)\in A_p$, $1<p<\infty$, if for every $I\subset\mathbb{R}$,

$$\left(\frac{1}{|I|}\int_I u(x)\,dx\right)\left(\frac{1}{|I|}\int_I v(x)^{-1/(p-1)}\,dx\right)^{p-1}\le C<\infty.$$

These definitions are completely analogous in the discrete case. Thus, the discrete maximal function on $\mathbb{Z}$ is defined by

$$M^{\mathrm{d}}(a)(m)=\sup_{N\ge 0}\frac{1}{2N+1}\sum_{n=m-N}^{m+N}|a_n|,$$

where $a=\{a_n\}_{n\in\mathbb{Z}}$ is a sequence of real or complex numbers.

A positive sequence $w=\{w_n\}_{n\in\mathbb{Z}}$ is a weight in A_p^{d}, $1<p<\infty$, if there is a constant $C>0$ such that

$$\sup_{N\ge 0}\left(\frac{1}{2N+1}\sum_{n=N_0-N}^{N_0+N}w_n\right)\left(\frac{1}{2N+1}\sum_{n=N_0-N}^{N_0+N}w_n^{-1/(p-1)}\right)^{p-1}\le C<\infty, \tag{4.3}$$

where $N_0\in\mathbb{Z}$ is arbitrary.

The smallest constant C in (4.1), (4.2) and (4.3) is also denoted by A_p, A_1, A_p^{d}, respectively. Since integers with counting measure form a space of homogeneous type, a result of Calderón [5] (see also [10]) shows that M^{d} is bounded on the weighted sequence space ℓ_w^p, $1<p<\infty$, defined by

$$\ell_w^p=\Big\{a=\{a_n\}:\Big(\sum_{n\in\mathbb{Z}}|a_n|^p w_n\Big)^{1/p}<\infty\Big\},$$

whenever w satisfies (4.3).

We shall also need the notion of reverse Hölder inequality. A weight function w is said to satisfy the *reverse Hölder inequality* of order $r>1$, written $w\in \mathrm{RH}_r$, if for every $I\subset\mathbb{R}$, there is a constant $C>0$ such that

$$\left(\frac{1}{|I|}\int_I w^r\right)^{1/r}\le\frac{C}{|I|}\int_I w.$$

The results of this section are concerned with amalgams $\ell^q(L_w^p)$, $1<p,q<\infty$, $p\ne q$. For $q=p$ the amalgam reduces to L_w^p and a celebrated result of Muckenhoupt [12] then asserts that the condition $w\in A_p$ is necessary and sufficient for the boundedness of M.

Remark 4.1. It is well known that if $w\in A_p$, $1\le p<\infty$, then there exists an $r>1$ such that $w\in\mathrm{RH}_r$. In particular, if $w\in A_1$ then $w\in\mathrm{RH}_r$ for some $r>1$, and therefore $w^r\in A_1$ for some $r>1$. (See [8, Lemma IV.2.5, Theorem IV.2.7].) We shall use this fact in Theorem 4.5 below.

The first result is

THEOREM 4.2. *If* $1<p<q<\infty$ *and*

(i) $w\in A_p\cap A_{2-p/q}$,
(ii) $w\in\mathrm{RH}_{q/p}$,

then there exists $C>0$ *such that*

$$\|Mf\|_{p,w,q}\le C\|f\|_{p,w,q}. \tag{4.4}$$

Proof. Let $r = (q/p)'$. Then there exists $\{b_n\} = b \in \ell^r$, with $\|b\|_{\ell^r} = 1$, such that

$$\|Mf\|_{p,w,q}^p = \sum_n b_n \int_n^{n+1} (Mf)(x)^p w(x)\,dx = \int_{\mathbb{R}} (Mf)(x)^p u(x)\,dx\,,$$

where $u(x) = \sum_n b_n w(x)\chi_n(x)$. Since $w \in A_p$, there is a p_0, with $1 < p_0 < p$, such that $w \in A_{p_0}$ ([8]). Define $v(x) = \sum_{m\in\mathbb{Z}} \Lambda_m w(x)\chi_m(x)$, where for each $m \in \mathbb{Z}$, $\Lambda_m = \max(\alpha_m, \beta_m)$, with

$$\alpha_m = W_m^{-1} M^{\mathrm{d}}(\{b_k W_k\})(m), \quad W_n = \int_n^{n+1} w, \quad \beta_m = \sum_{k=m-2}^{m+2} b_k\,.$$

In what follows, we successively prove

(i) $(u, v) \in A_{p_0}$;

(ii) $\int |f|^p v \le C\|f\|_{p,w,q}^p$.

Since $(u, v) \in A_{p_0}$ implies $\int (Mf)^p u \le C \int |f|^p v$ (see [8, Corollary 1.3, p. 393]) the boundedness of M on $\ell^q(L_w^p)$ will follow.

Proof of (i). Let $I = [x_0 - h, x_0 + h]$, $x_0 \in [n_0, n_0 + 1]$. We consider two cases.

(a) Suppose $0 < h < 1$. Then

$$\lambda_1(I) = \frac{1}{|I|} \int_I u \le \Big(\sum_{n=n_0-1}^{n_0+1} b_n \Big) \Big(\frac{1}{|I|} \int_I w \Big)\,.$$

On the other hand, for $x \in I$,

$$v(x)^{1/(1-p_0)} = \sum_{m=n_0-1}^{n_0+1} \Lambda_m^{1/(1-p_0)} w(x)^{1/(1-p_0)} \chi_m(x)\,,$$

where for each m with $n_0-1 \le m \le n_0+1$, we have $\Lambda_m \ge \beta_m \ge \sum_{k=n_0-1}^{n_0+1} b_k$. As a consequence,

$$\lambda_2(I) = \Big(\frac{1}{|I|} \int_I v^{1/(1-p_0)} \Big)^{p_0-1}$$
$$\le \Big(\sum_{k=n_0-1}^{n_0+1} b_k \Big)^{-1} \Big(\frac{1}{|I|} \int_I w(x)^{1/(1-p_0)} \Big)^{p_0-1}\,,$$

which yields $\lambda_1(I)\lambda_2(I) \le A_{p_0}$. (Here we assumed without loss of generality that $\sum_{k=n_0-1}^{n_0+1} b_k > 0$; otherwise there is nothing to prove.)

(b) Suppose $h \ge 1$. Then

$$\frac{1}{|I|} \int_I u \le \frac{1}{2h} \sum_{k=n_0-[h]-2}^{n_0+[h]+2} b_k W_k \le C\gamma(n_0, [h])\,,$$

where $[\cdot]$ denotes the greatest integer function and

$$\gamma(n_0, [h]) = \min\{M^{\mathrm{d}}(\{b_k W_k\})(m) : m \in [n_0 - [h] - 2, n_0 + [h] + 2]\}\,.$$

Also, for $x \in I$,

$$v(x)^{1/(1-p_0)} = \sum_{m=n_0-[h]-2}^{n_0+[h]+2} \Lambda_m^{1/(1-p_0)} w(x)^{1/(1-p_0)} \chi_m(x)\,,$$

where $\Lambda_m^{-1} \le \alpha_m^{-1} \le W_m \gamma(n_0, [h])^{-1}$ for each $m \in [n_0 - [h] - 2, n_0 + [h] + 2]$. This implies

$$\Big(\frac{1}{|I|} \int_I u \Big) \Big(\frac{1}{|I|} \int_I v^{1/(1-p_0)} \Big)^{p_0-1}$$
$$\le C \Big\{ \frac{1}{2h} \sum_{m=n_0-[h]-2}^{n_0+[h]+2} W_m^{1/(p_0-1)} \Big(\int_m^{m+1} w^{1/(1-p_0)} \Big) \Big\}^{p_0-1} \le CA_{p_0}\,.$$

Assertion (i) now follows from (a) and (b).

Proof of (ii). By Hölder's inequality with exponents $r' = q/p$,

$$\int_{\mathbb{R}} |f|^p v = \sum_{m\in\mathbb{Z}} \Lambda_m \int_m^{m+1} |f|^p w \le \Big(\sum_m \Lambda_m^r \Big)^{1/r} \|f\|_{p,w,q}\,.$$

Now, $\Lambda_m = \max(\alpha_m, \beta_m) \le \alpha_m + \beta_m$, and by Minkowski's inequality this implies that

$$\Big(\sum_m \Lambda_m^r \Big)^{1/r} \le \Big(\sum_m \alpha_m^r \Big)^{1/r} + \Big(\sum_m \beta_m^r \Big)^{1/r}.$$

Clearly, $(\sum_m \beta_m^r)^{1/r} \le C\|b\|_{\ell^r} = C$, since $\|b\|_{\ell^r} = 1$.

On the other hand, from the boundedness of the discrete maximal function ([5], [10]),

$$\Big(\sum_m \alpha_m^r \Big)^{1/r} = \Big(\sum_m \frac{M^{\mathrm{d}}(\{b_k W_k\})(m)^r}{W_m^r} \Big)^{1/r}$$
$$\le C \Big(\sum_k b_k^r \Big)^{1/r} = C$$

provided $\{W_m^{-r}\} \in A_r^{\mathrm{d}}$, i.e.,

$$\sup_{N_0\in\mathbb{Z}}\sup_{N\geq 0}\Big(\frac{1}{2N+1}\sum_{n=N_0-N}^{N_0+N} W_n^{-r}\Big)\Big(\frac{1}{2N+1}\sum_{n=N_0-N}^{N_0+N} W_n^{r'}\Big)^{r-1}<\infty. \tag{4.5}$$

To show that this condition is satisfied, note first that Hölder's inequality and $w\in \mathrm{RH}_{r'}$ yields

$$1\leq\Big(\int_n^{n+1} w^{-r}\Big)\Big(\int_n^{n+1} w^{r'}\Big)^{r/r'}\leq C\Big(\int_n^{n+1} w^{-r}\Big)W_n^r,$$

so that $W_n^{-r}\leq C(\int_n^{n+1} w^{-r})$ and the first factor of (4.5) is majorized by

$$\frac{1}{2N+1}\int_{N_0-N}^{N_0+N+1} w^{-r}.$$

Similar arguments show that the second factor in (4.5) is dominated by

$$\begin{aligned}\Big(\frac{1}{2N+1}\sum_{n=N_0-N}^{N_0+N}\int_n^{n+1} w^{r'}\Big)^{r-1}&=\Big(\frac{1}{2N+1}\int_{N_0-N}^{N_0+N+1} w^{r'}\Big)^{r-1}\\&\leq C\Big(\frac{1}{2N+1}\int_{N_0-N}^{N_0+N+1} w\Big)^{r}.\end{aligned}$$

But $r=1/(1-p/q)$ and since $w\in A_{2-p/q}$ the product of these two terms is bounded and hence (4.5) holds.

This completes the proof of the theorem.

Remark 4.3. We now show that on the scale of A_1 weights the result is sharp. In fact we show that the condition $w\in \mathrm{RH}_{q/p}$ is optimal.

THEOREM 4.4. *Suppose $1<p<q<\infty$. Then for all $\varepsilon>0$ there is a weight w with $w^{q/p-\varepsilon}\in A_1$ and a function $f\in \ell^q(L_w^p)$ for which $Mf\equiv\infty$. Thus the estimate* (4.4) *fails.*

Proof. Recall that $|x|^{-\beta}\in A_1$ if and only if $0\leq\beta<1$. Let

$$w(x)=|x|^{-\beta}\quad\text{and}\quad f(x)=\sum_{n=1}^{\infty} n^{\alpha}\chi_n(x)$$

with $\alpha>0$ to be chosen. Then

$$\|f\|_{p,w,q}^q\sim\sum_{n=1}^{\infty} n^{-\beta q/p}n^{\alpha q}.$$

Clearly the sum is finite if

$$q(\beta/p-\alpha)>1. \tag{4.6}$$

It suffices therefore to show that for every β, $p/q<\beta<1$, there is an α such that (4.6) holds, but $Mf\equiv\infty$.

Let $x\in[n,n+1]$, $n\in\mathbb{N}$. Then for every $h>0$,

$$(Mf)(x)\geq\frac{1}{2h}\int_{x-h}^{x+h}\sum_{k=1}^{\infty} k^{\alpha}\chi_k(y)\,dy. \tag{4.7}$$

Now take $h=n^m$, where m is large. Since $x\sim n$, the right side of (4.7) is then larger than

$$\begin{aligned}n^{-m}\int_1^{n^m}\sum_{k=1}^{\infty} k^{\alpha}\chi_k(y)\,dy&\geq n^{-m}\sum_{k=1}^{n^m-1} k^{\alpha}\\&\geq n^{-m}\int_1^{n^m} y^{\alpha}\,dy\\&\sim n^{m(\alpha+1)-m}=n^{m\alpha}.\end{aligned}$$

Therefore, for every sufficiently large $m>0$ and $x\in[n,n+1]$, $(Mf)(x)\geq C_\alpha n^{m\alpha}$, that is, $Mf\equiv\infty$ on $[1,\infty)$ and so it is infinite everywhere. But now, if $q\beta/p>1$ then it is always possible to find $\alpha>0$ for which (4.6) holds. This proves the result.

Note that if $1<p<q<\infty$ and $w^{q/p}\in A_1$ then the conclusion of Theorem 4.2 holds. This follows at once from Theorem 4.2, since $w^{q/p}\in A_1$ implies $w\in \mathrm{RH}_{q/p}$ and also $w\in A_1$. But then $w\in A_s$ for $s>1$ and so $w\in A_p\cap A_{2-p/q}$.

THEOREM 4.5. *If $1<q<p<\infty$ and $w\in A_1$, then $\|Mf\|_{p,w,q}\leq C\|f\|_{p,w,q}$ for some $C>0$.*

Proof. The argument is based on that given in [8, Theorem VI.5.2, p. 555]. Let $\beta=q/(p-q)$ or $1/q=1/p+1/(p\beta)$. Then there is a positive sequence $\{u_n\}=u\in\ell^\beta$, with $\|u\|_{\ell^\beta}=1$, such that

$$\begin{aligned}\Big\{\sum_n\Big(\int_n^{n+1}|f|^pw\Big)^{q/p}\Big\}^{1/q}&=\Big\{\sum_n\Big[\Big(\int_n^{n+1}|f|^pw\Big)^{1/p}u_n^{-1/p}\Big]^p\Big\}^{1/p}\Big(\sum_n u_n^{\beta}\Big)^{1/\beta}\\&=\Big(\sum_n u_n^{-1}\int_n^{n+1}|f|^pw\Big)^{1/p}\end{aligned}$$

$$=\Big\{\int_{\mathbb{R}}|f(x)|^p\Big(\sum_n u_n^{-1}\chi_n(x)\Big)w(x)\,dx\Big\}^{1/p}.$$

Also, by Hölder's inequality with exponent p/q,

$$\|Mf\|_{p,w,q}\le\Big\{\sum_n v_n^{-1}\int_n^{n+1}|Mf|^p w\Big\}^{1/p}\Big(\sum_n v_n^{\beta}\Big)^{1/(\beta p)}$$

for any positive sequence $\{v_n\}$. Now since $p>q>1$, there is a $p_0<p$ such that $p_0-1>p-q$ and hence $(p_0-1)\beta=(p_0-1)q/(p-q)>q>1$. Therefore

$$\|[M^{\mathrm d}(\{u_n^{1/(p_0-1)}\})]^{p_0-1}\|_{\ell^\beta}\le C\|\{u_n\}\|_{\ell^\beta}=C$$

and choosing $v_m=[M^{\mathrm d}(\{u_n^{1/(p_0-1)}\})(m)]^{p_0-1}$ it follows that

$$\Big(\sum_n v_n^{\beta}\Big)^{1/\beta}\le C.$$

In order to prove (4.4) this argument shows that it suffices to prove

$$\int_{\mathbb{R}}|Mf(x)|^p\Big(\sum_n v_n^{-1}\chi_n(x)\Big)w(x)\,dx = C\int_{\mathbb{R}}|f(x)|^p\Big(\sum_n u_n^{-1}\chi_n(x)\Big)w(x)\,dx.\tag{4.8}$$

If we write

$$v(x)=w(x)\sum_n v_n^{-1}\chi_n(x)$$

and

$$u(x)=w(x)\sum_n u_n^{-1}\chi_n(x),$$

then (4.8) holds (cf. proof of Theorem 4.2) if the weight pair (v,u) satisfies A_{p_0} for some $p_0<p$. That is, we must show that

$$\sup_{I\subset\mathbb{R}}\Big(\frac{1}{|I|}\int_I v\Big)\Big(\frac{1}{|I|}\int_I u^{-1/(p_0-1)}\Big)^{p_0-1}\equiv A_{p_0}<\infty.\tag{4.9}$$

To establish (4.9), consider first $2h=|I|>2$. Let x be the center of I, and suppose $x\in[m,m+1]$, $m\in\mathbb{Z}$. Then in analogy with previous arguments, (4.9) is not larger than

$$\Big(\frac{1}{2h}\sum v_n^{-1}\int_n^{n+1}w\Big)\Big(\frac{1}{2h}\sum u_n^{1/(p_0-1)}\int_n^{n+1}w^{-1/(p_0-1)}\Big)^{p_0-1},\tag{4.10}$$

where summation is over all $n\in[m-[h]-1,m+[h]+1]$ (here $[\cdot]$ denotes the greatest integer function). Note that the second factor is obtained from

$$u(x)^{-1/(p_0-1)}=\sum_{n\in\mathbb{Z}}\chi_n(x)w(x)^{-1/(p_0-1)}u_n^{1/(p_0-1)}.$$

Now by Hölder's inequality with exponent $r>1$, the first factor of (4.10) is dominated by

$$\Big(\frac{1}{2h}\sum v_n^{-r'}\Big)^{1/r'}\Big\{\frac{1}{2h}\sum\Big(\int_n^{n+1}w\Big)^r\Big\}^{1/r}.$$

But since $w^r\in A_1$, for r sufficiently close to 1 (cf. Remark 4.1),

$$\begin{aligned}\Big\{\frac{1}{2h}\sum\Big(\int_n^{n+1}w\Big)^r\Big\}^{1/r}&\le\Big(\frac{1}{2h}\sum\int_n^{n+1}w^r\Big)^{1/r}\\&=\Big(\frac{1}{2h}\int_{m-[h]-1}^{m+[h]+1}w^r\Big)^{1/r}\\&\le C\operatorname*{ess\,inf}_{x\in[m-[h]-1,m+[h]+1]}w(x).\end{aligned}$$

Hence (4.10) is not larger than

$$\begin{aligned}&C\Big(\frac{1}{2h}\sum v_n^{-r'}\Big)^{1/r'}\\&\quad\times\Big\{\frac{1}{2h}\sum u_n^{1/(p_0-1)}\int_n^{n+1}\Big(\frac{\operatorname{ess\,inf}w(x)}{w(x)}\Big)^{1/(p_0-1)}dx\Big\}^{p_0-1}\\&\le C\Big(\frac{1}{2h}\sum v_n^{-r'}\Big)^{1/r'}\Big(\frac{1}{2h}\sum u_n^{1/(p_0-1)}\Big)^{p_0-1}\\&\le C\big(\max_{n\in[m-[h]-1,m+[h]+1]}v_n^{-1}\big)\\&\quad\times\big[\min_{k\in[m-[h]-1,m+[h]+1]}M^{\mathrm d}(\{u_n^{1/(p_0-1)}\})(k)\big]^{p_0-1}\le C.\end{aligned}$$

But the second factor is equal to $\min_{n\in[m-[h]-1,m+[h]+1]}v_n$. This proves the boundedness of (4.9) in the case $h>1$.

In case $h \le 1$, that is, $|I| \le 2$, the interval I can meet $[n, n+1]$ in at most three integers, say $n_0 - 1, n_0, n_0 + 1$. Therefore (4.9) is not larger than

$$\Big\{\sum_{n=n_0-1}^{n_0+1} v_n^{-1}\Big(\frac{1}{|I|}\int_I w\Big)\Big\}\Big\{\sum_{n=n_0-1}^{n_0+1} u_n^{1/(p_0-1)}\Big(\frac{1}{|I|}\int_I w^{-1/(p_0-1)}\Big)\Big\}^{p_0-1}$$

$$\le C\Big(\sum_{n=n_0-1}^{n_0+1} v_n^{-1}\Big)$$

$$\times \Big\{\sum_{n=n_0-1}^{n_0+1} u_n^{1/(p_0-1)}\Big(\frac{1}{|I|}\int_I \Big(\frac{\operatorname{ess\,inf} w(x)}{w(x)}\Big)^{1/(p_0-1)}\Big)\Big\}^{p_0-1}$$

$$\le C\Big(\sum_{n=n_0-1}^{n_0+1} v_n^{-1}\Big)\Big(\sum_{n=n_0-1}^{n_0+1} u_n^{1/(p_0-1)}\Big)^{p_0-1},$$

where we used the fact that $w \in A_1$. Since

$$C \min_{n\in[n_0-1,n_0+1]} v_n \ge \Big(\sum_{n=n_0-1}^{n_0+1} u_n^{1/(p_0-1)}\Big)^{p_0-1},$$

the last product is bounded independently of n_0. This proves the result.

As q in the amalgam norm $\|f\|_{p,w,q}$ becomes larger than p, the norm tends to emphasize more strongly the local behaviour of f. This is the reason why for the local maximal function M_1 (but not M) the weight class A_p, independent of q, is sufficient to assure boundedness between weighted amalgams. This is the content of Theorem 4.7 below.

The weight pair (u, v) is said to belong to $A_{s,\mathrm{loc}}$, the local weight class A_s, $s > 1$, if

$$\sup_{\substack{I\subset\mathbb{R}\\ |I|\le 1}} \Big(\frac{1}{|I|}\int_I u\Big)\Big(\frac{1}{|I|}\int_I v^{-1/(s-1)}\Big)^{s-1} < \infty. \tag{4.11}$$

The next lemma we require is a variant of a result of Muckenhoupt (cf. [8, Ch. IV, §1]).

LEMMA 4.6. *If $(u, v) \in A_{s,\mathrm{loc}}$, $s > 1$, then M_1 is of weak type (s, s) with respect to the weight pair (u, v). That is, for $\lambda > 0$,*

$$\int_{E_\lambda} u \le C\lambda^{-s} \int_{\mathbb{R}} |f|^s v,$$

where $E_\lambda = \{x : |(M_1 f)(x)| > \lambda\}$.

Proof. Without loss of generality assume f has compact support in $\mathbb{R}$, for once this case is proved, standard limiting arguments yield the general case.

Now if f has compact support, then E_λ is bounded and the proof follows along the lines of [8, pp. 391–392].

If $(u, v) \in A_{s,\mathrm{loc}}$ then for $|I| \le 1$,

$$\Big(\frac{1}{|I|}\int_I |f|\Big)^s\Big(\int_I u\Big) \le C\int_I |f|^s v. \tag{4.12}$$

To see this, observe that by Hölder's inequality,

$$\Big(\frac{1}{|I|}\int_I |f|\Big)^s \le \Big(\frac{1}{|I|}\int_I |f|^s v\Big)\Big(\frac{1}{|I|}\int_I v^{-1/(s-1)}\Big)^{s-1}$$

and multiplying by $\int_I u$ yields (4.12) with C the $A_{s,\mathrm{loc}}$ constant.

For each $x \in E_\lambda$ there is an interval I_x centered at x, with $|I_x| \le 1$, such that

$$\lambda < \frac{1}{|I_x|}\int_{I_x} |f|.$$

Clearly,

$$E_\lambda \subset \bigcup_{x\in E_\lambda} I_x$$

and by the Besicovitch covering lemma there is a countable subsequence $\{I_j\}$ whose union covers E_λ and $\sum_j \chi_{I_j}(x) \le C$. Therefore, by (4.12),

$$\int_{E_\lambda} u \le \sum_j \int_{I_j} u$$

$$\le C\sum_j \Big(\int_{I_j} |f|^s v\Big)\Big(\frac{1}{|I_j|}\int_{I_j} |f|\Big)^{-s}$$

$$\le C\lambda^{-s}\sum_j \int_{I_j} |f|^s v$$

$$\le C\lambda^{-s}\int_{\mathbb{R}} |f|^s v.$$

THEOREM 4.7. *If $1 < p < q < \infty$ and $w \in A_p$, then $\|M_1 f\|_{p,w,q} \le C\|f\|_{p,w,q}$.*

Proof. Arguing as in the proof of Theorem 4.2 it suffices to prove

$$\int_{\mathbb{R}} |(M_1 f)(x)|^p u(x)\,dx \le C\int_{\mathbb{R}} |f(x)|^p v(x)\,dx, \tag{4.13}$$

where

$$u(x) = \sum_{n\in\mathbb{Z}} b_n \chi_n(x) w(x),$$

$$v(x) = \sum_{n\in\mathbb{Z}} \chi_n(x) \Big(\sum_{m=n-2}^{n+2} b_m \Big) w(x),$$

with $\{b_n\} = b \in \ell^r$, $r = (q/p)'$ and $\|b\|_{\ell^r} = 1$. For if (4.13) holds, then as before the right side of (4.13) is dominated by $\|f\|_{p,w,q}^p$ and the result follows.

Again, the proof of Theorem 4.2 shows that the weight pair $(u, v) \in A_{p_0,\mathrm{loc}}$ for some $p_0 < p$. Hence by Lemma 4.6 with $s = p_0$, M_1 is of weak type (p_0, p_0) and since $A_{p_0,\mathrm{loc}} \subset A_{r,\mathrm{loc}}$ for $r > p_0$, M_1 is also of weak type (r, r) with respect to the same weight pair (u, v). Thus (4.13) follows from the Marcinkiewicz interpolation theorem.

Remark 4.8. The proof of this result, that is, $(u, v) \in A_{p_0,\mathrm{loc}}$, requires that $v(x) > 0$ a.e., which is the case if $b_n > 0$. However, the sequence $\{b_n\}$ was chosen so that the duality argument holds, which in fact is

$$b_n = C\Big(\int_n^{n+1} w|M_1 f|^p \Big)^{q/p-1}.$$

But this is clearly positive for nontrivial f on $[n, n+1]$. On the other hand, if $f \equiv 0$ a.e. on some interval $[n_0, n_0 + 1]$ then the right side of (4.13) is unaffected if we replace v on $[n_0, n_0 + 1]$ by

$$\max\Big(\varepsilon, \sum_{m=n_0-2}^{n_0+2} b_m\Big) w, \quad \varepsilon > 0.$$

In this case $(u, v) \in A_{p_0,\mathrm{loc}}$, where the $A_{p_0,\mathrm{loc}}$ constant is independent of ε and so (4.13) still holds.

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