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Received July 16, 1993
Revised version August 17, 1993 (3133)

STUDIA MATHEMATICA 109 (2) (1994)

Pointwise ergodic theorems for functions in Lorentz spaces L_{pq} with $p \neq \infty$

by

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Abstract. Let τ be a null preserving point transformation on a finite measure space. Assuming τ is invertible, P. Ortega Salvador has recently obtained sufficient conditions for the almost everywhere convergence of the ergodic averages in L_{pq} with $1 < p < \infty$, $1 < q < \infty$. In this paper we obtain necessary and sufficient conditions for the almost everywhere convergence, without assuming that τ is invertible and only assuming that $p \neq \infty$.

1. Introduction. If τ is an invertible null preserving transformation on a σ-finite measure space $(X, \mathcal{F}, \mu)$, then $A_{n,m}$ and M will denote the ergodic averages and the maximal operator, respectively, defined by

$$A_{n,m}f(x) = \frac{1}{n+m+1} \sum_{i=-n}^{m} f(\tau^i x)$$

and

$$Mf = \sup_{n,m \geq 0} A_{n,m}|f|\,.$$

In [6], Ortega studied the good weights W for M to be bounded in $L_{pq}(W d\mu)$ $(1 < p < \infty,\ 1 < q \leq \infty)$, under the additional assumption that τ is measure preserving. Among other things, he proved that $\|Mf\|_{pq;W d\mu} \leq C\|f\|_{pq;W d\mu}$ if and only if $\sup_{n,m\geq 0} \|A_{n,m}f\|_{p\infty;W d\mu} \leq C\|f\|_{pq;W d\mu}$, C being a positive constant, not necessarily the same at each occurrence. Applying this result he then considered a null preserving τ on a finite measure space and proved that if $\sup_{n,m\geq 0} \|A_{n,m}f\|_{p\infty} \leq C\|f\|_{pq}$, where $1 < p < \infty$ and $1 < q < \infty$, then for any f in $L_{pq}(\mu)$ the ergodic averages $A_{0,n}f$ converge almost everywhere as $n \to \infty$. It seems to the author that this condition for the validity of the pointwise ergodic theorem is too strong. In fact, as

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1991 *Mathematics Subject Classification*: Primary 47A35; Secondary 28D05.
Key words and phrases: pointwise ergodic theorems, L_{pq} spaces, null preserving transformations, measure preserving transformations, positive contractions on L_1 spaces.

is seen in [6], this condition implies $\|Mf\|_{pq} \le C\|f\|_{pq}$, and thus $A_{0,n}f$ converges strongly in $L_{pq}(\mu)$ as $n \to \infty$. However, in general, the pointwise ergodic theorem does not imply the mean ergodic theorem. (To see this, a counterexample due to Ryll-Nardzewski [7] is sufficient.) So the author thinks that it would be of interest to find necessary and sufficient conditions for this almost everywhere convergence. This is the starting point for the study in this paper. We will find necessary and sufficient conditions, without assuming that τ is invertible and only assuming that $p \neq \infty$.

For this purpose, we first consider a positive linear contraction T on $L_1(\mu)$ and prove a ratio ergodic theorem. As a corollary, we obtain the desired result.

2. Preliminaries and results. Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space and $M^+(\mu)$ denote the space of all nonnegative extended real-valued measurable functions on X. As usual, two functions f and g are not distinguished provided that $f = g$ a.e. on X. Let $V \in M^+(\mu)$ be such that $V > 0$ a.e. on X. The space $L_{pq}(V d\mu)$, where $0 < p, q < \infty$ or $0 < p \le \infty$ and $q = \infty$, is the collection of all measurable functions f on X such that $\|f\|_{pq;Vd\mu} < \infty$, where

$$\|f\|_{pq;Vd\mu} = \begin{cases} [q\int_0^\infty (\int_{\{|f|>t\}} V\,d\mu)^{q/p} t^{q-1}\,dt]^{1/q} & (q \neq \infty), \\ \sup_{t>0} t(\int_{\{|f|>t\}} V\,d\mu)^{1/p} & (q = \infty). \end{cases}$$

Note that $\|1_E\|_{pq;Vd\mu} = (\int_E V\,d\mu)^{1/p}$, 1_E being the indicator function of $E \in \mathcal{F}$. Also, $\|f\|_{pp;Vd\mu} = (\int |f|^p V\,d\mu)^{1/p} = \|f\|_{p;Vd\mu}$ and hence $L_{pp}(Vd\mu) = L_p(Vd\mu)$. The basic properties of Lorentz spaces $L_{pq}(Vd\mu)$ are explained in Hunt [4]. When $V = 1$ a.e. on X, we write $\|f\|_{pq}$ and $L_{pq}(\mu)$ instead of $\|f\|_{pq;Vd\mu}$ and $L_{pq}(Vd\mu)$, respectively.

Let T be a positive linear contraction on $L_1(\mu)$; thus $\|Tf\|_1 \le \|f\|_1$ for all $f \in L_1(\mu)$ and $TL_1^+(\mu) \subset L_1^+(\mu)$, where $L_1^+(\mu) = L_1(\mu) \cap M^+(\mu)$. In order to extend the domain of T to $M^+(\mu)$, fix any $f \in M^+(\mu)$ and take $f_n \in L_1^+(\mu)$, $n = 1, 2, \ldots$, such that $f_n \uparrow f$ a.e. on X. Then define

$$Tf = \lim_n Tf_n \quad \text{a.e. on } X.$$

It is easily seen that by this process T can be uniquely extended to an operator on $M^+(\mu)$ satisfying $T(f+g) = Tf + Tg$ and $T(\alpha f) = \alpha Tf$ for all $f, g \in M^+(\mu)$ and constants α, $0 \le \alpha < \infty$. For simplicity, the following notation will be used:

$$R_0^n(f,e) = \Big(\sum_{i=0}^n T^i f\Big) \Big/ \Big(\sum_{i=0}^n T^i e\Big) \quad \text{and} \quad M(f,e) = \sup_{n\ge0} R_0^n(f,e)$$

for $f \in M^+(\mu)$ and $e \in L_1^+(\mu)$, with $e > 0$ a.e. on X.

We also consider a null preserving transformation $\tau : X \to X$. By definition τ is *null preserving* if it is measurable and satisfies $\mu(\tau^{-1}E) = 0$ for all $E \in \mathcal{F}$ with $\mu(E) = 0$. We call τ *ergodic* if $E \in \mathcal{F}$ and $\tau^{-1}E = E$ imply $\mu(E) = 0$ or $\mu(X \setminus E) = 0$, and *conservative* if there exists no $E \in \mathcal{F}$ such that $\tau^{-1}E \subset E$ and $\mu(E \setminus \tau^{-1}E) > 0$. As is well known, if an operator $T : L_1(\mu) \to L_1(\mu)$ is defined by the relation

$$\int_E Tf\,d\mu = \int_{\tau^{-1}E} f\,d\mu \quad (f \in L_1(\mu),\ E \in \mathcal{F}),$$

then T becomes a positive linear contraction on $L_1(\mu)$. T will be referred to as the Frobenius–Perron operator associated with τ.

We are now in a position to state the first result.

THEOREM 1. *Let T be a positive linear contraction on $L_1(\mu)$. Let $V \in M^+(\mu)$ with $V > 0$ a.e. on X and $e \in L_1^+(\mu)$ with $e > 0$ a.e. on X. Let $0 < p < \infty$, $0 < q \le \infty$, and suppose r satisfies $0 < r \le 1$, $r < p$ and $r \le q$. Then the following are equivalent:*

(a) *For any $f \in L_{pq}^+(Vd\mu)$, $\lim_n R_0^n(f,e)$ exists and is finite a.e. on X.*
(b) *$M(f,e) < \infty$ a.e. on X for all $f \in L_{pq}^+(Vd\mu)$.*
(c) *There exists a $U \in M^+(\mu)$, with $U > 0$ a.e. on X, such that*

$$\int_{\{M(f,e)>\lambda\}} U\,d\mu \le \Big(\frac{1}{\lambda}\|f\|_{pq;Vd\mu}\Big)^r \quad (\lambda > 0,\ f \in L_{pq}^+(Vd\mu)).$$

(d) *There exists a $U \in M^+(\mu)$, with $U > 0$ a.e. on X, such that*

$$\liminf_n \int_{\{R_0^n(f,e)>\lambda\}} U\,d\mu \le \Big(\frac{1}{\lambda}\|f\|_{pq;Vd\mu}\Big)^r \quad (\lambda > 0,\ f \in L_{pq}^+(Vd\mu)).$$

Proof. By virtue of the Radon–Nikodym theorem, we may and do assume, without loss of generality, that $\mu(X) = 1$.

(a)⇒(b). Obvious.

(b)⇒(c). We recall (cf. [4], §2) that $L_{pq}(Vd\mu)$ is a linear space and has a complete invariant metric ϱ such that

$$\varrho(0,f) = \varrho(0,|f|) \quad \text{and} \quad \varrho(0,cf) = |c|^r \varrho(0,f) \tag{1}$$

for all $f \in L_{pq}(Vd\mu)$ and constants c, and further such that

$$\|f\|_{pq;Vd\mu} \le [\varrho(0,f)]^{1/r} \le (p/(p-r))^{1/r}\|f\|_{pq;Vd\mu}. \tag{2}$$

It follows that $L_{pq}(Vd\mu)$ is an F-space, with the topology induced by $\|\cdot\|_{pq;Vd\mu}$. Next we notice that the mapping $f \to R_0^n(f,e)$ from $L_{pq}^+(Vd\mu)$ to the space $L_0(\mu)$ of all real-valued measurable functions on X is continuous in measure. In fact, if this is not the case, then there exists a sequence $\{f_k\}$ in $L_{pq}^+(Vd\mu)$ with $\varrho(0,f_k) < 2^{-k}$ and $\mu(\{R_0^n(f_k,e) > \varepsilon\}) > \delta$ $(k \ge 1)$ for some positive reals ε and δ. Then the function $f = \sum_{k=1}^{\infty} f_k$ is in $L_{pq}^+(V d\mu)$, and writing $E_k = \{R_0^n(f_k, e) > \varepsilon\}$, we have

$$R_0^n(f,e)(x) = \sum_{k=1}^{\infty} R_0^n(f_k,e)(x) \geq \varepsilon \sum_{k=1}^{\infty} 1_{E_k}(x).$$

But, since $\delta < \mu(E_k) \leq \mu(X) = 1$ for all $k \geq 1$, the function $h(x) = \sum_{k=1}^{\infty} 1_{E_k}(x)$ satisfies $h = \infty$ on a set of positive measure, which contradicts (b). Consequently, the mapping $f \to R_0^n(f,e)$ can be uniquely extended to a mapping from the full space $L_{pq}(V d\mu)$ to $L_0(\mu)$ and it is also continuous in measure. By this and the completeness of the invariant metric ϱ, (b) implies that there exists a positive decreasing function $C(\lambda)$ defined for $\lambda > 0$ and tending to zero as $\lambda \to \infty$ such that for all $f \in L_{pq}^+(V d\mu)$ with $\varrho(0,f) \leq 1$ we have

$$\mu(\{M(f,e) > \lambda\}) \leq C(\lambda) \quad (\lambda > 0). \tag{3}$$

(Cf. [3], pp. 2–3, for a proof.) This yields that if $\{f_k\}$ is a sequence in $L_{pq}(V d\mu)$ such that $\sum_{k=1}^{\infty} \varrho(0, f_k) \leq 1$, then the function $g = \sum_{k=1}^{\infty} |f_k|$ satisfies

$$\mu(\{M(g,e) > \lambda\}) \leq C(\lambda) \quad (\lambda > 0), \tag{4}$$

because $\varrho(0,g) \leq \sum_{k=1}^{\infty} \varrho(0, |f_k|) = \sum_{k=1}^{\infty} \varrho(0, f_k) \leq 1$.

For a moment, fix a real number $K > 0$ and define $\Sigma(K)$ to be the collection of all sets $E \in \mathcal{F}$, with $\mu(E) > 0$, such that

$$\mu(E)(M(f,e)(x))^r > K^r \quad \text{a.e. on } E \tag{5}$$

for some $f \in L_{pq}^+(V d\mu)$ with $\varrho(0,f) = 1$. Since $\mu(X) < \infty$, if $\Sigma(K)$ is not empty, there exists a disjoint sequence $\{E_i\}$ of sets in $\Sigma(K)$ such that

$$E \in \Sigma(K) \quad \text{implies} \quad \sup_i \mu(E \cap E_i) > 0. \tag{6}$$

Writing $E(K) = \bigcup_i E_i$ and $c_i = \mu(E_i)^{1/r}$, and choosing f_i in $L_{pq}^+(V d\mu)$ with $\varrho(0, f_i) = 1$, so that (5) holds with f_i and E_i instead of f and E, we then see that

$$\sup_i M(c_i f_i, e)(x) > K \quad \text{a.e. on } E(K), \tag{7}$$

and

$$\sum_i \varrho(0, c_i f_i) = \sum_i |c_i|^r = \sum_i \mu(E_i) \leq \mu(X) = 1. \tag{8}$$

It follows from (4), (7) and (8) that $\mu(E(K)) \leq C(K) \to 0$ as $K \to \infty$. Thus for each $n \geq 1$, we can choose a sufficiently large K_n so that

$$\mu(X \setminus B_n) < \frac{1}{n}, \quad \text{where } B_n = X \setminus E(K_n). \tag{9}$$

Since the inequality

$$\mu(\{x \in B_n : M(f,e)(x) > \lambda\})(M(f,e)(y))^r \geq \mu(\{x \in B_n : M(f,e)(x) > \lambda\})\lambda^r \quad (\lambda > 0)$$

holds for each y in the set $\{x \in B_n : M(f,e)(x) > \lambda\}$, (6) gives

$$\mu(\{x \in B_n : M(f,e)(x) > \lambda\}) \leq (K_n/\lambda)^r \quad (\lambda > 0,\ f \in L_{pq}^+(V d\mu) \text{ with } \varrho(0,f) = 1). \tag{10}$$

If $\Sigma(K)$ is empty for some $K > 0$, then the above argument implies that (10) holds with $B_n = X$ and $K_n = K$. Hence, in any case, (9) and (10) hold.

In order to finish the proof, we take a sequence $\{a_n\}$ of positive reals such that

$$\sum_{n=1}^{\infty} a_n K_n^r \left(\frac{p}{p-r}\right) = 1,$$

and define

$$U(x) = \sum_{n=1}^{\infty} a_n 1_{B_n}(x).$$

By (9), $U > 0$ a.e. on X; and by (2) if $f \in L_{pq}^+(V d\mu)$ and $\|f\|_{pq;V d\mu} = (p/(p-r))^{-1/r}$ then $\varrho(0,f) \leq 1$. Thus by (10),

$$\begin{aligned} \int_{\{M(f,e)>\lambda\}} U\, d\mu &\leq \sum_{n=1}^{\infty} a_n \mu(B_n \cap \{M(f,e) > \lambda\}) \\ &\leq \sum_{n=1}^{\infty} a_n \left(\frac{K_n}{\lambda}\right)^r = \sum_{n=1}^{\infty} a_n \left(\frac{K_n}{\lambda}\right)^r \frac{p}{p-r} \|f\|_{pq;V d\mu}^r \\ &= \left(\frac{1}{\lambda}\|f\|_{pq;V d\mu}\right)^r \quad (\lambda > 0). \end{aligned}$$

This proves (c).

(c)$\Rightarrow$(d). Obvious.

(d)$\Rightarrow$(a). Let $f \in L_{pq}^+(V d\mu)$. We apply the Neveu–Chacon identification theorem for the limit of the Chacon–Ornstein ratio ergodic theorem (see, e.g. [5], Chapter 3) to infer that the ratio ergodic averages $R_0^n(f,e)(x)$ converge a.e. on X, but the limit may be ∞. Thus for the proof it suffices to show that $\mu(\{M(f,e) = \infty\}) = 0$. To do this, let $\lambda > 0$. Then

$$\{M(f,e) = \infty\} \subset \liminf_n \{R_0^n(f,e) > \lambda\}.$$

Thus, putting $E = \{M(f,e) = \infty\}$, we obtain, by (d) and Fatou's lemma,

$$\int_E U\, d\mu \leq \liminf_n \int_{\{R_0^n(f,e)>\lambda\}} U\, d\mu \leq \lambda^{-r} \|f\|_{pq;V d\mu}^r.$$

Letting $\lambda \to \infty$, we get $\int_E U\,d\mu = 0$ and $\mu(E) = 0$.

The proof is complete.

We now apply Theorem 1 to a null preserving transformation $\tau : X \to X$ on a finite measure space. For simplicity we set

$$A_n f(x) = \frac{1}{n}\sum_{i=0}^{n-1} f(\tau^i x), \qquad MAf(x) = \sup_{n\ge 1} |A_n f(x)|.$$

THEOREM 2. *Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $\tau : X \to X$ be a null preserving transformation. Let $0 < p < \infty$, $0 < q \le \infty$ and suppose r satisfies $0 < r \le 1$, $r < p$ and $r \le q$. Then the following are equivalent:*

(a) *For any $f \in L_{pq}(\mu)$, $\lim_n A_n f$ exists and is finite a.e. on X.*

(b) *$MAf < \infty$ a.e. on X for all $f \in L_{pq}(\mu)$.*

(c) *There exists a $U \in M^+(\mu)$, with $U > 0$ a.e. on X, such that*

$$\int_{\{MAf>\lambda\}} U\,d\mu \le \left(\frac{1}{\lambda}\|f\|_{pq}\right)^r \quad (\lambda > 0,\ f \in L_{pq}(\mu)).$$

(d) *There exists a $U \in M^+(\mu)$, with $U > \infty$ a.e. on X, such that*

$$\sup_{n\ge 1} \int_{\{|A_n f|>\lambda\}} U\,d\mu \le \left(\frac{1}{\lambda}\|f\|_{pq}\right)^r \quad (\lambda > 0,\ f \in L_{pq}(\mu)).$$

(e) *For any $E \in \mathcal{F}$, $\lim_n n^{-1}\sum_{i=0}^{n-1}\mu(\tau^{-i}E)$ exists, and further there exists a $U \in M^+(\mu)$, with $U > 0$ a.e. on X, such that*

$$\liminf_n \int_{\{|A_n f|>\lambda\}} U\,d\mu \le \left(\frac{1}{\lambda}\|f\|_{pq}\right)^r \quad (\lambda > 0,\ f \in L_{pq}(\mu)).$$

Proof. The implications (a)$\Rightarrow$(b) and (c)$\Rightarrow$(d) are obvious; (b)$\Rightarrow$(c) follows as in the proof of (b)$\Rightarrow$(c) in Theorem 1.

(d)$\Rightarrow$(e). We may suppose that $0 < U \le 1$ on X. Given an $\varepsilon > 0$, choose $\delta > 0$ so that $\mu(\{U < \delta\}) < \varepsilon$. Then, since $0 \le A_n 1_E \le 1$ on X for $E \in \mathcal{F}$, (d) implies that

$$\begin{aligned}\int (A_n 1_E)\,d\mu &\le \delta^{-1}\int (A_n 1_E) U\,d\mu + \mu(\{U<\delta\})\\ &\le \delta^{-1}\Big(\int_{\{A_n 1_E>\lambda\}} U\,d\mu + \lambda\int U\,d\mu\Big) + \varepsilon\\ &\le \delta^{-1}\left(\frac{1}{\lambda}\|1_E\|_{pq}\right)^r + \delta^{-1}\lambda\mu(X) + \varepsilon \quad (\lambda > 0).\end{aligned}$$

Letting $\lambda > 0$ sufficiently small to majorize the middle term by ε and then letting $\|1_E\|_{pq} = \mu(E)^{1/p} \downarrow 0$, we see that

$$\lim_{\mu(E)\to 0}\ \sup_{n\ge 1} n^{-1}\sum_{i=0}^{n-1}\mu(\tau^{-i}E) = 0.$$

It follows that the set $\{n^{-1}\sum_{i=0}^{n-1} T^i 1 : n \ge 1\}$ is weakly sequentially compact in $L_1(\mu)$, where T denotes the Frobenius–Perron operator on $L_1(\mu)$ which is associated with τ. By the mean ergodic theorem (cf. e.g. [2], p. 661), the averages $n^{-1}\sum_{i=0}^{n-1} T^i 1$ converge in the norm topology of $L_1(\mu)$. This proves the first part of (e); the second part is trivial.

(e)$\Rightarrow$(a). The Vitali–Hahn–Saks theorem implies that the set function $\nu(E) = \lim_n n^{-1}\sum_{i=0}^{n-1}\mu(\tau^{-i}E)$ $(E \in \mathcal{F})$ is a (countably additive) finite measure, absolutely continuous with respect to μ, and clearly it is invariant under τ. Thus if we write

$$Y = \{x : (d\nu/d\mu)(x) > 0\} \quad\text{and}\quad Z = X \setminus Y,$$

then $\nu(\tau^{-1}Z) = \nu(Z) = 0$, and so neglecting ν-null sets we may suppose that $Y \subset \tau^{-1}Y$. Therefore τ can be considered to be a measure preserving transformation on the measure space (Y, ν). We now consider $V = d\mu/d\nu$ (which is defined on Y), $e = 1_Y$, and the positive linear contraction T_Y on $L_1(Y, \nu)$ defined by

$$T_Y f(x) = f(\tau x) \quad (x \in Y,\ f \in L_1(Y, \nu)),$$

and apply Theorem 1(d) to see that for every $f \in L_{pq}(Y, \mu)$, $\lim_n A_n f(x)$ exists and is finite a.e. on Y.

To finish the proof, it suffices to show that $\mu(X \setminus \lim_n \tau^{-n} Y) = 0$. But this is immediate, since $X \setminus \lim_n \tau^{-n} Y$ is an invariant set under τ and contained in Z.

The proof is complete.

Remarks. (a) In Theorem 2 the hypothesis $(p, q) \ne (\infty, \infty)$ is essential. In fact, by using Chacon's example [1] we see that there exists a conservative, ergodic and invertible null preserving transformation τ on a finite measure space $(X, \mathcal{F}, \mu)$ such that

$$\liminf_n A_n f = 0 \quad \text{a.e.} \quad \text{and} \quad \limsup_n A_n f = 1 \quad \text{a.e. on } X$$

for some $f \in L_\infty^+(\mu)$. But clearly (b) in Theorem 2 holds when $(p, q) = (\infty, \infty)$.

(b) Let $(X, \mathcal{F}, \mu)$ be a *nonatomic* finite measure space and τ an ergodic null preserving transformation on $(X, \mathcal{F}, \mu)$. Then to each q, with $1 < q \le \infty$, there corresponds an $f \in L_{1q}(\mu)$ such that for almost all $x \in X$ the averages $A_n f(x)$ fail to converge to a finite limit. In fact, by [4], there exists an $f \in L_{1q}(\mu)$ which does not belong to $L_1(\mu)$; using this fact and the argument in Proposition of [8], the desired result follows. We also note that the same result holds even if $L^+_{1q}(\mu)$ is replaced by $L^+_{p'q'}(\mu)$ with $p' < 1$, because $\mu(X) < \infty$ implies $L_{1q}(\mu) \subset L_{p'q'}(\mu)$ with $p' < 1$.

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Received September 9, 1993
Revised version November 9, 1993
(3162)

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