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Two-weight norm inequalities for the fractional maximal operator on spaces of homogeneous type

by

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Abstract. We give a characterization of the pairs of weights (v,w), with w in the class A_∞ of Muckenhoupt, for which the fractional maximal function is a bounded operator from $L^p(X, v\,d\mu)$ to $L^q(X, w\,d\mu)$ when $1 < p \le q < \infty$ and X is a space of homogeneous type.

In 1990, C. Perez ([P]) gave a characterization of the pairs of weights (v,w) for which the fractional maximal function over cubes in $\mathbb{R}^n$ is bounded as an operator from $L^p(\mathbb{R}^n, v\,dx)$ to $L^q(\mathbb{R}^n, w\,dx)$, $1 < p \le q < \infty$, when $v^{-1/(p-1)}$ belongs to Muckenhoupt's class A_∞. The main purpose of this work is to extend the result of C. Perez to spaces of homogeneous type. One of most important technical points of the proof is, as in [P], a suitable modification of the Calderón–Zygmund method. In particular, we extend the method H. Aimar and R. Macías ([AM]) use to obtain Muckenhoupt's theorem on weighted L^p boundedness of the Hardy–Littlewood maximal function in spaces of homogeneous type.

The definitions and the statement of the main result are in Section 1. Section 2 is dedicated to the proof of that result.

1. By a *quasi-distance* on a set X we mean a nonnegative function $d(\cdot,\cdot)$ defined on $X \times X$ such that

(1.1) for every x and y in X, $d(x,y) = 0$ iff $x = y$,
(1.2) for every x and y in X, $d(x,y) = d(y,x)$,
(1.3) there exists a finite constant $K > 0$ such that for every x, y and z in X,
$$d(x,y) \le K(d(x,z) + d(z,y)).$$

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A quasi-distance $d(\cdot,\cdot)$ defines a uniform structure on X. The balls $B(x,r) = \{y : d(x,y) < r\}$, $r > 0$, form a basis of neighborhoods of x for the topology induced by the uniformity on X.

Let X be a set endowed with a quasi-distance $d(\cdot,\cdot)$ and assume that a positive measure μ, defined on a σ-algebra of subsets of X which contains the balls $B(x,r)$, is given and has the property that there exist two finite constants, $a > 1$ and A, such that

$$(1.4)\qquad 0 < \mu(B(x,ar)) \le A\mu(B(x,r)) < \infty$$

for every $x \in X$ and $r > 0$. Such a set X with d and μ will be called a *space of homogeneous type* and denoted by (X,d,μ).

In 1979, R. Macías and C. Segovia ([MS1]) proved that, if (X,d,μ) is given, one can always find a continuous quasi-distance $d'(\cdot,\cdot)$ equivalent to $d(\cdot,\cdot)$ (i.e. there exist c_1 and c_2 such that $c_1 d'(x,y) \le d(x,y) \le c_2 d'(x,y)$, $\forall x,y \in X$). In the following, we always assume that the quasi-distance d is continuous.

Let (X,d,μ) be a space of homogeneous type. For $0 \le \gamma < 1$, we denote by M_γ the fractional maximal operator defined by

$$M_\gamma f(x) = \sup_B \frac{1}{\mu(B)^{1-\gamma}} \int_B |f(y)|\,d\mu(y), \qquad f \in L^1_{\mathrm{loc}}(X,d\mu),$$

where the sup is taken over all balls B such that $x \in B$. Note that $M_\gamma f$ is a measurable function.

A *weight* w is a nonnegative function in $L^1_{\mathrm{loc}}(X,d\mu)$ and we shall use $w(A)$ to denote $\int_A w\,d\mu$. We say that a weight w belongs to A_∞ if there exist finite positive constants C and δ such that

$$(1.5)\qquad \frac{\mu(E)}{\mu(B)} \le C\left(\frac{w(E)}{w(B)}\right)^\delta,$$

for every ball B and every measurable set $E \subset B$. It can be proved that this inequality is equivalent to a similar one where μ appears instead of w and conversely (see [CF] and [MS2]).

The main result of this work is the following theorem.

(1.6) THEOREM. *Suppose $0 \le \gamma < 1$ and $1 < p \le q < \infty$. Let (w,v) be a pair of weights with $\sigma = v^{-1/(p-1)} \in A_\infty$. Then*

$$(1.7)\qquad \|M_\gamma f\|_{L^q(X,wd\mu)} \le C\|f\|_{L^p(X,vd\mu)} \quad \textit{for all } f \in L^p(X,vd\mu)$$

if and only if

$$(1.8)\qquad \frac{w(B)^{p/q}\sigma(B)^{p-1}}{\mu(B)^{(1-\gamma)p}} \le C < \infty \quad \textit{for every ball } B \subset X.$$

(1.9) Remark. The equivalence between (1.7) and (1.8) for the case $p < q$ was obtained by R. Wheeden ([W]) simultaneously with the authors but under the weaker hypothesis that σ satisfies a condition like (1.4). His techniques are very different from ours and are based on some previous results for the fractional integral operator obtained in [SW].

(1.10) Remark. By using Theorem (1.6) it is possible to obtain a similar result for the fractional operator defined, for each $\gamma \in (0,1)$, by

$$I_\gamma f(x) = \int_X \frac{f(y)}{\mu(B(y,d(x,y)))^{1-\gamma}}\,d\mu(y), \qquad f \in L^1(X,d\mu).$$

In fact, by following the reasoning for $\mathbb{R}^n$ in [MW] and using results of [MT], we can prove $\|I_\gamma f\|_{L^q(X,wd\mu)} \approx \|M_\gamma f\|_{L^p(X,wd\mu)}$, for each $\gamma \in (0,1)$ and each $q \in (0,\infty)$, whenever $w \in A_\infty$. From this and (1.6), it follows that (1.8) is equivalent to (1.7) with M_γ replaced by I_γ, whenever w and σ belong to A_∞. This last result is an extension to the case of spaces of homogeneous type of one of C. Perez ([P]) and it was previously obtained by E. Sawyer and R. Wheeden in [SW] using different techniques.

2. In this section, we prove Theorem (1.6). For this purpose, we need two covering lemmas.

(2.1) LEMMA. *Let E be a bounded subset of X and assume that for each $x \in E$ there exist $y(x) \in X$ and $r(x) > 0$ such that $x \in B(y(x), r(x))$. Then there exists a sequence $\{B(y(x_i), r(x_i))\}$ of disjoint balls such that $E \subset \bigcup_{i=1}^\infty B(y(x_i), 5K^2 r(x_i))$, where K is the constant of* (1.3).

Proof. See [CW], p. 69. ∎

Now, we prove a covering lemma that extends a result of H. Aimar and R. Macías ([AM]) to the case $\mu(X) < \infty$. First, we introduce some notation. If $B = B(x,r)$, we write $\widetilde{B}$ for $B(x, 5K^2 r)$ and $\widehat{B}$ for $B(x, 15K^5 r)$. Let D be such that $\mu(\widehat{B}) \le D\mu(B)$ for every B. Fix $\gamma \in [0,1)$. If f is a nonnegative integrable function and E a measurable set, we denote $\mu(E)^{\gamma-1}\int_E f\,d\mu$ by $m_E f$. Let $b \ge 2D^2 + 1$ and, for each $k \in \mathbb{Z}$, let $\Omega_k = \{y \in X : b^{k+1} \ge M_\gamma f(y) > b^k\}$. Note that, if $\mu(X) < \infty$, then $m_X f \le M_\gamma f(x)$ for all $x \in X$. In this case, for each f, we denote by k_0 the integer such that $b^{k_0+1} \ge m_X f > b^{k_0}$. Then, clearly, $\Omega_k = \emptyset$ for $k < k_0$.

(2.2) LEMMA. *For any nonnegative integrable function f with bounded support, and any $k \in \mathbb{Z}$ such that $\Omega_k \ne \emptyset$, there exists a sequence $\{B_i^k\}_{i\in\mathbb{N}}$ of balls satisfying:*

$$(2.3)\qquad \Omega_k \subset \bigcup_{i=1}^\infty \widetilde{B}_i^k.$$

$$(2.4)\qquad B_i^k \cap B_j^k = \emptyset \ \textit{if } i \ne j.$$

(2.5) *If $\mu(X)=\infty$, then for every B_i^k there exists $x_i^k \in B_i^k$ such that if r_i^k is the radius of B_i^k, $r \geq 5K^2 r_i^k$ and $x_i^k \in B = B(y,r)$, then $b^{k+1} \geq M_\gamma f(x_i^k) \geq m_{B_i^k} f > b^k \geq m_B f$.*

(2.6) *If $\mu(X) < \infty$, then (2.5) still holds for $k > k_0$, but if $k = k_0$ we only have one ball $B_1^{k_0}$ such that $\Omega_{k_0} \subset B_1^{k_0} = X$ and $b^{k_0+1} \geq M_\gamma f(x_1^{k_0}) \geq m_{B_1^{k_0}} f > b^{k_0}$, for some $x_1^{k_0} \in B_1^{k_0}$.*

(2.7) *If $x \notin \bigcup_{j=k}^\infty \bigcup_{i=1}^\infty \widetilde{B}_i^j$ and $M_\gamma f(x) < \infty$, then $M_\gamma f(x) \leq b^k$.*

(2.8) *Let $I_j^k = \{(l,n) \in \mathbb{Z} \times \mathbb{N} : l \geq k+2,\ \widetilde{B}_n^l \cap \widetilde{B}_j^k \neq \emptyset\}$ and let $A_j^k = \bigcup_{(l,n)\in I_j^k} \widetilde{B}_n^l$. Then $2\mu(A_j^k) \leq \mu(B_j^k)$.*

(2.9) *Let $E_j^k = \widetilde{B}_j^k - A_j^k$. Then $2\mu(E_j^k) \geq \mu(\widetilde{B}_j^k)$ and $\mu(X - \bigcup_{k,j} E_j^k) = 0$. If $x \in E_j^k$ and $M_\gamma f(x) < \infty$, then $M_\gamma f(x) \leq b^{k+2}$.*

(2.10) *Let $F_j^k = B_j^k - A_j^k$. Then $\mu(F_j^k) \geq \mu(\widetilde{B}_j^k)/(2A)$ and*

$$\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \chi_{F_j^k}(x) \leq 3,$$

where χ_E denotes the characteristic function of the set E.

Proof. In order to obtain (2.3)–(2.6) we first assume $\mu(X) = \infty$. If $x \in \Omega_k$, the integrability of f implies that the set $R_k(x) = \{r > 0 : m_B f > b^k,\ x \in B(y,r)\}$ is bounded. We can choose $r(x) \in R_k(x)$ in such a way that if $r \geq 5K^2 r(x)$, then $r \notin R_k(x)$. Thus, there is a point $y(x) \in X$ such that

$$b^{k+1} \geq M_\gamma f(x) \geq m_{B(y(x),r(x))} f > b^k \geq m_{B(y,r)} f \tag{2.11}$$

whenever $r \geq 5K^2 r(x)$ and $x \in B(y,r)$. The boundedness of the support of f implies that of Ω_k, therefore Lemma (2.1) can be applied to obtain a sequence $\{B_i^k\}$ satisfying (2.3)–(2.5). If $\mu(X) < \infty$ and $k > k_0$, it is easy to see that we can still find $r(x) \in R_k(x)$ and $y(x) \in X$ such that (2.11) holds. Then, by applying (2.1) again, we obtain (2.3), (2.4) and the first part of (2.6). If $k = k_0$, we can choose $x \in \Omega_{k_0}$ and $r > 0$ such that $B(x,r) = X$. Then, with $x_1^{k_0} = x$ and $r_1^{k_0} = r$ we have the last part of (2.6). Now, (2.7) follows easily from (2.3)–(2.5).

In order to get (2.8), let us first show that if $l \geq k+2$, $n \in \mathbb{N}$ and $\widetilde{B}_n^l \cap \widetilde{B}_j^k \neq \emptyset$, then

$$\widetilde{B}_n^l \subset \widehat{B}_j^k, \tag{2.12}$$

even more: $r_n^l \leq r_j^k$. Indeed, suppose that $r_n^l > r_j^k$. Then $\widetilde{B}_j^k \subset \widetilde{B}_n^l$. Thus

from (2.5) and (2.6) we get

$$\begin{aligned} b^{k+1} &\geq m_{\widehat{B}_n^l} f \geq (\mu(B_n^l)\mu(\widehat{B}_n^l)^{-1})^{1-\gamma} m_{B_n^l} f \\ &\geq D^{-1} m_{B_n^l} f > D^{-1} b^l \geq D^{-1} b^{k+2}, \end{aligned}$$

which is a contradiction. Now, (2.5), (2.6), (2.4) and (2.12) yield (2.8) in the following way:

$$\begin{aligned} \mu(A_j^k) &\leq \sum_{(l,n)\in I_j^k} \mu(\widetilde{B}_n^l) \leq D \sum_{(l,n)\in I_j^k} \Big(b^{-l} \int_{B_n^l} f\,d\mu\Big)^{1/(1-\gamma)} \\ &\leq D\Big(\sum_{l=k+2}^{\infty} b^{-l/(1-\gamma)}\Big)\Big(\int_{\widehat{B}_j^k} f\,d\mu\Big)^{1/(1-\gamma)} \\ &\leq D^2 b^{-(k+1)/(1-\gamma)} (b^{1/(1-\gamma)} - 1)^{-1} \mu(B_j^k)(m_{\widehat{B}_j^k} f)^{1/(1-\gamma)} \leq \mu(B_j^k)/2. \end{aligned}$$

In order to prove (2.9), let x be a point such that $M_\gamma f(x) < \infty$; then $x \in \Omega_k$ for some $k \in \mathbb{Z}$. By (2.3), $x \in \widetilde{B}_j^k$ for some $j \in \mathbb{N}$. Assume that $x \in A_j^k$; then there exists $(l,n) \in I_j^k$ such that $x \in \widetilde{B}_j^l$, and from (2.5) and (2.6) we obtain

$$M_\gamma f(x) \geq m_{\widetilde{B}_n^l} f \geq D^{-1} m_{B_n^l} f > D^{-1} b^{k+2} > b^{k+1},$$

which is a contradiction. Thus the sequence $\{E_j^k\}$ is a covering of $\{x : M_\gamma f(x) < \infty\}$. On the other hand, on account of the weak type $(1, 1/(1-\gamma))$ of M_γ (the proof uses the same technique as in the case $\gamma = 0$; see, for example, [C]), we have $\mu(\{x : M_\gamma f(x) = \infty\}) = 0$. Thus (2.9) is proved.

From (2.4) we see that

$$\sum_{j=1}^{\infty} \chi_{F_j^k}(x) \leq \chi_{\bigcup_{j=1}^\infty F_j^k}(x) \leq \chi_{\bigcup_{j=1}^\infty E_j^k}(x),$$

for any $k \in \mathbb{Z}$. By definition of E_j^k it follows that no point of X belongs to more than three of the sets $\bigcup_{j=1}^\infty E_j^k$. Thus, we get

$$\sum_{k=-\infty}^{\infty} \sum_{j=1}^{\infty} \chi_{F_j^k}(x) \leq \sum_{k=-\infty}^{\infty} \chi_{\bigcup_{j=1}^\infty E_j^k}(x) \leq 3,$$

which is (2.10). This finishes the proof of the lemma.

With this lemma, we have the tools to prove our main result.

Proof of Theorem (1.6). The inequality (1.8) follows easily from (1.7) by taking $f = v^{-1/(p-1)}\chi_B$ for each ball B. Now, assume that (1.8) holds. Let f be a nonnegative function in $L^p(v\,d\mu)$. First, in order to prove

(1.7), we suppose that f has bounded support. Then, from the above lemma, with the same notation, we have

$$(M_\gamma f(x))^q = \sum_k (M_\gamma f(x))^q \chi_{\Omega_k}(x) \le b^q \sum_k b^{kq} \chi_{\Omega_k}(x)$$
$$\le b^q \sum_{k,i} b^{kq} \chi_{\widetilde{B}_i^k}(x) \le b^q \sum_{k,i} (m_{b_i^k} f)^q \chi_{\widetilde{B}_i^k}(x)\,.$$

This inequality, (1.8), (2.10) and the fact that $\sigma \in A_\infty$ allow us to obtain

$$\begin{aligned}\|M_\gamma f\|^p_{L^q(w d\mu)} &\le C \sum_{k,i} (m_{B_i^k} f)^p w(\widetilde{B}_i^k)^{p/q} \\ &\le C \sum_{k,i} \frac{w(\widetilde{B}_i^k)^{p/q} \sigma(B_i^k)^{p-1}}{\mu(B_i^k)^{(1-\gamma)p}} \bigg(\frac{1}{\sigma(B_i^k)} \int_{B_i^k} f\sigma^{-1} \sigma\, d\mu \bigg)^p \sigma(B_i^k) \\ &\le C \sum_{k,i} \bigg(\frac{1}{\sigma(B_i^k)} \int_{B_i^k} (f\sigma^{-1}) \sigma\, d\mu \bigg)^p \sigma(F_i^k) \\ &\le C \|M^\sigma (f\sigma^{-1})\|^p_{L^p(\sigma d\mu)}\,,\end{aligned}$$

where M^σ is the Hardy–Littlewood maximal operator on the space of homogeneous type $(X, d, \sigma d\mu)$. Then, from the boundedness of this operator, we get

$$\|M_\gamma f\|_{L^q(w d\mu)} \le C \|f\sigma^{-1}\|_{L^p(\sigma d\mu)} = C\|f\|_{L^p(v d\mu)}\,,$$

which is (1.8). When f does not have bounded support, the result follows by using an obvious density argument.

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