Thus, (3.26) is bounded by a constant times

$$\sum_{j,k} a_{jk}^{q/p}\Big(\frac{1}{a_{jk}}\int_{Q_j^k} f\sigma\,d\mu\Big)^q .$$

Since $\sigma d\mu$ is a doubling measure, we may apply Lemma (3.15) of [SW1] directly to see that the last sum is at most $c\|f\|^q_{L^p_{\sigma d\mu}}$, which proves Theorem 4, provided that we verify the condition

$$\sum_{Q_j^k\subset Q_s^t} a_{jk}^{q/p}\le c a_{st}^{q/p}.$$

This condition is proved exactly as (3.20) in [SW1], using $q>p$ and the fact that $\sigma d\mu$ is a doubling measure, and in fact does not require the maximality of the Q_j^k.

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Pseudotopologies with applications to one-parameter groups, von Neumann algebras, and Lie algebra representations

by

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Abstract. For any pair E, F of pseudotopological vector spaces, we endow the space $L(E,F)$ of all continuous linear operators from E into F with a pseudotopology such that, if G is a pseudotopological space, then the mapping $L(E,F)\times L(F,G)\ni(f,g)\to gf\in L(E,G)$ is continuous. We use this pseudotopology to establish a result about differentiability of certain operator-valued functions related with strongly continuous one-parameter semigroups in Banach spaces, to characterize von Neumann algebras, and to establish a result about integration of Lie algebra representations.

0. Introduction. If E is a Banach space and $L(E)$ is the space of all continuous linear operators in E, then, if $L(E)$ is endowed with the standard norm topology, then the composition of operators in $L(E)$ is continuous. When $L(E)$ is equipped with either the strong operator topology or weak operator topology, the composition of operators in $L(E)$ fails to be continuous unless E is finite-dimensional. If F is a Fréchet space with a topology that cannot be determined by a single norm, then, as proved by Bastiani [B] and Keller [Ke], there is no reasonable topology on $L(F)$ under which the composition of operators in $L(F)$ is continuous. In this paper, for any pair E, F of pseudotopological vector spaces, we endow the space $L(E,F)$ of all continuous linear operators from E into F with a pseudotopology such that, if G is a pseudotopological space, then the mapping $L(E,F)\times L(F,G)\ni(f,g)\to gf\in L(E,G)$ is continuous. We use this pseudotopology to establish a result about differentiability of certain operator-valued functions related to strongly continuous one-parameter semigroups in Banach spaces, to characterize von Neumann

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algebras, and to establish a result about integration of Lie algebra representations.

1. Pseudotopological spaces. We begin by recalling a number of elementary facts from the theory of pseudotopological spaces. A more detailed account can be found in [F-B].

If E is a set, then 2^E denotes the set of all subsets of E.

DEFINITION 1.1. A *filter* on a set E is a nonempty subset $\mathfrak{X}$ of 2^E such that

(i) $\emptyset \notin \mathfrak{X}$,
(ii) if $X_1 \in \mathfrak{X}$ and $E \supset X_2 \supset X_1$, then $X_2 \in \mathfrak{X}$,
(iii) if $X_1, X_2 \in \mathfrak{X}$, then $X_1 \cap X_2 \in \mathfrak{X}$.

DEFINITION 1.2. A *filter basis* on a set E is a nonempty subset $\mathfrak{B}$ of 2^E such that

(i) $\emptyset \notin \mathfrak{B}$,
(ii) if $X_1, X_2 \in \mathfrak{B}$, then there exists $X_3 \in \mathfrak{B}$ such that $X_3 \subset X_1 \cap X_2$.

If E is a subset of a set F and $\mathfrak{B}$ is a filter basis on E (in particular, if $\mathfrak{B}$ is a filter on E), then, obviously, $\mathfrak{B}$ is a filter basis on F.

Given a filter basis $\mathfrak{B}$ on a set E, we denote by $[\mathfrak{B}]$ the filter on E such that $A \in [\mathfrak{B}]$ if and only if there exists $B \in \mathfrak{B}$ such that $B \subset A$. $[\mathfrak{B}]$ is called the *filter generated by* $\mathfrak{B}$. For $B \subset E$, we abbreviate $[\{B\}]$ as $[B]$, and, for $x \in E$, we contract $[\{x\}]$ to $[x]$.

The set of all filters on a given set E is endowed with a partial order $\prec$ defined by $\mathfrak{X}_1 \prec \mathfrak{X}_2$ if and only if $\mathfrak{X}_1 \supset \mathfrak{X}_2$.

For a family $\{\mathfrak{X}_i : i \in I\}$ of filters on a set E, the filter $\sup_{i\in I} \mathfrak{X}_i$ in the sense of the order $\prec$ always exists—it is the filter generated by the filter consisting of all sets of the form $\bigcup_{i\in I} X_i$, where $X_i \in \mathfrak{X}_i$.

For $i = 1, \ldots, n$, let E_i be a set and $\mathfrak{X}_i$ be a filter on E_i. By $\mathfrak{X}_1 \times \ldots \times \mathfrak{X}_n$ we denote the filter on $E_1 \times \ldots \times E_n$ generated by the filter basis $\{A_1 \times \ldots \times A_n\colon A_i \in \mathfrak{X}_i, i = 1, \ldots, n\}$. If F is a set and f is a mapping from $E_1 \times \ldots \times E_n$ into F, then by $f(\mathfrak{X}_1, \ldots, \mathfrak{X}_n)$ we denote the filter $[f(\mathfrak{X}_1 \times \ldots \times \mathfrak{X}_n)]$ on F. In particular, if E is a vector space over $\mathbb{R}$ or $\mathbb{C}$ and $\mathfrak{X}$ and $\mathfrak{Y}$ are filters on E and $\mathcal{V}$ is a filter on $\mathbb{R}$ or $\mathbb{C}$, in this way we can define the filters $\mathfrak{X} + \mathfrak{Y}$ and $\mathcal{V}\mathfrak{X}$ on E.

DEFINITION 1.3. A *pseudotopology* on a set E is a map assigning to every $x \in E$ a set of filters on E, each called a filter convergent to x, in such a way that, if $\mathfrak{X} \downarrow_x E$ denotes that the filter $\mathfrak{X}$ converges to x, then the following conditions are satisfied:

(i) if $\mathfrak{X} \downarrow_x E$ and $\mathfrak{X}_1 \prec \mathfrak{X}$, then $\mathfrak{X}_1 \downarrow_x E$,
(ii) if $\mathfrak{X}_1 \downarrow_x E$ and $\mathfrak{X}_2 \downarrow_x E$, then $\sup(\mathfrak{X}_1, \mathfrak{X}_2) \downarrow_x E$,
(iii) $[x] \downarrow_x E$.

DEFINITION 1.4. By a *pseudotopological space* we mean a set endowed with a pseudotopology.

DEFINITION 1.5. Let E be a pseudotopological space. A subset A of E is said to be *closed* if, for any filter $\mathfrak{B}$ on A,

$$[\mathfrak{B}] \downarrow_x E \Rightarrow x \in A\,.$$

DEFINITION 1.6. Let E and F be pseudotopological spaces. A mapping f from E into F is said to be *continuous* at $x \in E$ if

$$\mathfrak{X} \downarrow_x E \Rightarrow f(\mathfrak{X}) \downarrow_{f(x)} F\,.$$

2. Pseudotopological vector spaces. Let $\mathbb{R}$ be the field of real numbers and let $\mathbb{K}$ be either the field of real numbers or the field of complex numbers. A vector space E over $\mathbb{K}$ is called a *pseudotopological vector space*, PVS for short, if E is a pseudotopological space such that the mappings $E \times E \ni (x, y) \to x + y \in E$ and $\mathbb{K} \times E \ni (t, x) \to tx \in E$ are continuous.

The pseudotopology of a PVS is determined by the family of all filters convergent to 0. We write $\mathfrak{X} \downarrow E$ in place of $\mathfrak{X} \downarrow_0 E$.

Let $\mathcal{V}$ be the filter generated by the family of all open neighbourhoods of 0 on $\mathbb{K}$.

It is clear that the pseudotopology on a PVS satisfies the following conditions:

(i) if $\mathfrak{X}_1 \downarrow E$ and $\mathfrak{X}_2 \downarrow E$, then $\mathfrak{X}_1 + \mathfrak{X}_2 \downarrow E$,
(ii) if $\mathfrak{X} \downarrow E$ and $t \in \mathbb{K}$, then $t\mathfrak{X} \downarrow E$,
(iii) if $\mathfrak{X} \downarrow E$, then $\mathcal{V}\mathfrak{X} \downarrow E$,
(iv) if $x \in E$, then $\mathcal{V}[x] \downarrow E$.

THEOREM 2.1 ([F-B, 2.4.4]). *A PVS E is a topological vector space if and only if*

$$(\sup_{\mathfrak{X}\downarrow E} \mathfrak{X}) \downarrow E\,. \tag{2.1}$$

If (2.1) *is satisfied (so that E is a topological vector space), then $\sup_{\mathfrak{X}\downarrow E} \mathfrak{X}$ coincides with the filter generated by the family of all open neighbourhoods of 0 in E.*

THEOREM 2.2 ([F-B, 2.8.7]). *If E and F are PVS's, then a linear mapping from E into F is continuous if and only if it is continuous at 0 in E.*

For a pair E, F of PVS's, we denote by $L(E, F)$ the space of all continuous linear functions from E to F. We abbreviate $L(E, E)$ as $L(E)$.

DEFINITION 2.1. A filter $\mathfrak{X}$ on a PVS E is called *quasi-bounded* if $\mathcal{V}\mathfrak{X} \downarrow E$.

DEFINITION 2.2. A subset A of a PVS E is called *quasi-bounded* if the filter $[A]$ is quasi-bounded.

It is easy to establish the following result.

THEOREM 2.3. *Let E be a locally convex vector space with a topology determined by a family of pseudonorms $\{p_i : i \in I\}$ and $\mathfrak{X}$ be a filter on E. Then $\mathfrak{X}$ is quasi-bounded if and only if, for each $i \in I$, there exist $B_i \in \mathfrak{X}$ such that*

$$\sup_{x \in B_i} p_i(x) < \infty .$$

DEFINITION 2.3. A pseudotopological space E is called *Hausdorff* if

$$\mathfrak{X} \downarrow_x E \ \& \ \mathfrak{X} \downarrow_y E \Rightarrow x = y .$$

Hereafter all PVS's will be assumed to be Hausdorff.

3. Differentiability. Given a pair E, F of PVS's and a mapping r from E into F, let Θr be the mapping from $\mathbb{R} \times E$ into F given by

$$\Theta r(t,x) = \begin{cases} r(tx)/t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

DEFINITION 3.1. Let E and F be PVS's. A mapping r from E into F is called a *remainder* if $r(0) = 0$, and, for each quasi-bounded filter $\mathfrak{X}$ on E, we have $\Theta r(\mathcal{V}, \mathfrak{X}) \downarrow F$.

LEMMA 3.1 ([F-B, 3.2.1]). *If E and F are PVS's, f is a mapping from E into F, and x is an element of E, then there exists at most one linear mapping l from E into F such that the mapping*

$$r(h) = f(x+h) - f(x) - l(h) \tag{3.1}$$

is a remainder.

DEFINITION 3.2. Let E and F be PVS's. A mapping f from E into F is said to be *differentiable* at $x \in E$ if there exists $l \in L(E,F)$ such that r defined by (3.1) is a remainder. If f is differentiable at x, then l is called the *derivative* of f at x and is denoted by $Df(x)$ or by $f'(x)$.

DEFINITION 3.3. A net $(x_\alpha)_{\alpha \in A}$ in a PVS E indexed by the directed set $(A, \leq)$ is said to *converge* to $x \in E$ if the filter generated by the filter basis $\{\{x_\beta : \alpha \leq \beta\} : \alpha \in A\}$ is convergent to x in E.

LEMMA 3.2 ([F-B, 4.3.1]). *Let E and F be PVS's and f be a mapping from E into F differentiable at $x \in E$. Then, for any $v \in E$,*

$$\lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} = f'(x)v$$

exists in the sense of Definition 3.3.

The following two theorems will be very important in the sequel.

THEOREM 3.1 ([F-B, 3.3.1]). *If E, F, and G are PVS's, f is a mapping from E into F differentiable at $x \in E$, and g is a mapping from F into G differentiable at $f(x) \in F$, then $g \circ f$ is differentiable at x and*

$$D(g \circ f)(x) = Dg(f(x))Df(x) .$$

THEOREM 3.2 ([F-B, 4.2.1]). *If E, F, and G are PVS's, and f is a linear and continuous mapping from E into F, and g is a bilinear continuous mapping from $E \times F$ into G, then f and g are differentiable and*

$$Df(x)(h) = f(h) \quad (x, h \in E),$$

$$Dg(x_1, x_2)(h_1, h_2) = g(x_1, h_2) + g(h_1, x_2) \quad (x_1, h_1 \in E,\ x_2, h_2 \in F) .$$

We now introduce a class of pseudotopological vector spaces that is not specified in [F-B] but will be of direct relevance in the subsequent considerations.

For any $r > 0$, let $I_r = \{t \in \mathbb{K} : |t| < r\}$.

DEFINITION 3.4. A PVS E is called a *C-pseudotopological vector space*, CPVS for short, if

$$\mathfrak{X} \downarrow E \Rightarrow I_r \mathfrak{X} \downarrow E \text{ for each } r > 0 .$$

Obviously, every locally convex topological vector space is a CPVS.

THEOREM 3.3. *Let E be a CPVS, f be a mapping from $\mathbb{R}^n$ into E, and x be an element of $\mathbb{R}^n$. Then f is differentiable at x if and only if there exists $l \in L(\mathbb{R}^n, E)$ such that*

$$\lim_{\|h\| \to 0} \frac{f(x+h) - f(x) - l(h)}{\|h\|} = 0 .$$

Proof. Suppose that f is differentiable at x, and let $r(h) = f(x+h) - f(x) - f'(x)h$ for each $h \in \mathbb{R}^n$. For each $s > 0$, let $B_s = \{h \in \mathbb{R}^n : \|h\| < s\}$. Since r is a remainder, $\Theta r(\mathcal{V}, \mathfrak{B}) \downarrow E$ for the quasi-bounded filter $\mathfrak{B}$ on $\mathbb{R}^n$ generated by the family $\{B_s : s > 0\}$. Thus, if $C \in \Theta r(\mathcal{V}, \mathfrak{B})$, then there exist $b > 0$ and $s > 0$ such that

$$C \supset \Theta r(I_b, B_s) = \left\{ \frac{f(x+th) - f(x) - f'(x)th}{t} : 0 < |t| < b, \|h\| < s \right\} \cup \{0\} .$$

Let $B = q(B_{bs})$, where

$$q(y) = \frac{f(x+y) - f(x) - f'(x)y}{\|y\|} \quad \text{for } y \in E \setminus \{0\}$$

and $q(0) = 0$. Clearly, $sB \subset C$, and so $C \in sq(\mathcal{V}^n)$, showing that $sq(\mathcal{V}^n) \prec \Theta r(\mathcal{V}, \mathfrak{B})$ and, hence, $sq(\mathcal{V}^n) \downarrow E$ and finally $q(\mathcal{V}^n) \downarrow E$. (We did not use here the assumption that E is a CPVS.)

Now, let $\mathfrak{B}$ be a quasi-bounded filter on $\mathbb{R}^n$ and suppose that $q(\mathcal{V}^n) \downarrow E$. Then $B_b \in \mathfrak{B}$ for some $b > 0$ and, as E is a CPVS, we have $I_b q(\mathcal{V}^n) \downarrow E$.

To complete the proof, it suffices to show that $\Theta r(\mathcal{V}, \mathfrak{B}) \prec I_b q(\mathcal{V}^n)$. Given $D \supset I_b q(B_a)$ with $a > 0$, let $C = \Theta r(I_{a/b}, B_b)$. Clearly, $C \in \Theta r(\mathcal{V}, \mathfrak{B})$ and since

$$\Theta r(t,h) = \begin{cases} \|h\| \dfrac{|t|}{t} \dfrac{f(x+th) - f(x) - f'(x)th}{\|th\|} & \text{if } t \neq 0 \text{ and } h \in E - \{0\}, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that $\Theta r(t,h) \in I_b q(B_a)$ for $t \in I_{a/b}$ and $h \in B_b$. Hence $C \subset I_b q(B_a) \subset D$, showing that $D \in \Theta r(\mathcal{V}, \mathfrak{B})$. ∎

4. Pseudotopologies in $L(E,F)$. Following [F-B], for every pair E, F of PVS's, we introduce a certain pseudotopology on the space $L(E,F)$ of all linear continuous mappings from E into F. The space $L(E,F)$ equipped with this pseudotopology will be denoted by $B(E,F)$.

DEFINITION 4.1. Let E and F be PVS's. A filter $\mathfrak{X}$ on $L(E,F)$ is said to be *convergent to* 0 in $B(E,F)$ if, for any quasi-bounded filter $\mathfrak{B}$ on E, $\mathfrak{X}(\mathfrak{B})$ converges to 0 in F, that is,

$$\mathfrak{X} \downarrow B(E,F) \Leftrightarrow (\mathcal{V}\mathfrak{B} \downarrow E \Rightarrow \mathfrak{X}(\mathfrak{B}) \downarrow F). \tag{4.1}$$

THEOREM 4.1 ([F-B, 6.1.10]). *If E and F are normed spaces, then the pseudotopology of $B(E,F)$ coincides with the norm topology of $L(E,F)$.*

THEOREM 4.2 ([F-B, 6.3.1]). *If E, F, and G are PVS's, then the mapping $B(E,F) \times B(F,G) \ni (f,g) \to gf \in B(E,G)$ is continuous.*

THEOREM 4.3. *If E is a PVS and F is a CPVS, then $B(E,F)$ is a CPVS.*

Proof. If $\mathfrak{X} \downarrow B(E,F)$ and $\mathcal{V}\mathfrak{B} \downarrow E$, then, since $\mathfrak{X}$ consists of linear mappings, we have $(I_a\mathfrak{X})(\mathfrak{B}) = I_a(\mathfrak{X}(\mathfrak{B}))$ for each $a > 0$, and so, for each $a > 0$, $(I_a\mathfrak{X})(\mathfrak{B})$ converges to 0 in F whenever $\mathfrak{X}(\mathfrak{B}) \downarrow F$, showing that $B(E,F)$ is CPVS. ∎

The next three propositions concern an important case where E and F are locally convex topological spaces but in $L(E,F)$ a pseudotopology is used. We omit the proofs which are easy and schematic.

PROPOSITION 4.1. *Let E and F be locally convex topological vector spaces with topologies determined by families of pseudonorms $\{p_i : i \in I\}$ and $\{q_j : j \in J\}$, respectively, and let $\mathfrak{X}$ be a filter on $B(E,F)$. Then $\mathfrak{X}$ is quasi-bounded if and only if, for each $j \in J$, there exist a finite subset $H(j)$ of I, $X_j \in \mathfrak{X}$, and a positive number m_j such that, for each $f \in X_j$ and each $x \in E$,*

$$q_j(fx) \le m_j \sum_{i \in H(j)} p_i(x). \tag{4.2}$$

PROPOSITION 4.2. *Let E and F be locally convex topological vector spaces with topologies determined by families of pseudonorms $\{p_i : i \in I\}$ and $\{q_j : j \in J\}$, respectively. If f is a mapping from $\mathbb{R}^n$ into $B(E,F)$ such that, for each $j \in J$, there exist a finite subset $H(j)$ of I and a mapping k from $\mathbb{R}^n$ into $\mathbb{R}^+$ with $\lim_{t\to t_0} k(t) = 0$ such that*

$$q_j(f(t)x - f(t_0)x) \le k(t) \sum_{i \in H(j)} p_i(x),$$

then f is continuous at t_0.

Using Theorem 3.3, we can also prove the following.

PROPOSITION 4.3. *Let E and F be locally convex topological vector spaces with topologies determined by families of pseudonorms $\{p_i : i \in I\}$ and $\{q_j : j \in J\}$, respectively. If f is a mapping from $\mathbb{R}^n$ into $B(E,F)$ such that, for each $j \in J$, there exist a finite subset $H(j)$ of I and a mapping k from $\mathbb{R}^n$ into $\mathbb{R}^+$ with $\lim_{t\to 0} k(t) = 0$ and $l \in L(\mathbb{R}^n, B(E,F))$ such that*

$$q_j\left(\frac{f(s+h)x - f(s)x - l(h)x}{\|h\|}\right) \le k(h) \sum_{i \in H(j)} p_i(x),$$

then f is differentiable at s and $f'(s) = l$.

We now introduce a new "weaker" pseudotopology on $L(E,F)$. The space $L(E,F)$ equipped with this pseudotopology will be denoted by $Q(E,F)$.

DEFINITION 4.2. Let E and F be PVS's. A filter $\mathfrak{X}$ on $L(E,F)$ is said to be *convergent to* 0 in $Q(E,F)$ if

$$\text{for each } x \in E \text{ there exists a filter } \mathfrak{B}_x \text{ on } F \text{ such that } \mathfrak{X}(x) \prec \mathcal{V}\mathfrak{B}_x \text{ and } \mathcal{V}\mathfrak{B}_x \downarrow_0 F \tag{4.3}$$

and

$$\mathfrak{X} \text{ is quasi-bounded in } B(E,F). \tag{4.4}$$

It is easy to see that $Q(E,F)$ is well defined for every pair E, F of PVS's. We have the following obvious

FACT 4.1. *If E is a PVS and F is a locally convex topological space, then (4.3) is equivalent to "strong convergence" to 0, that is,*

$$\mathfrak{X}(x) \downarrow F \quad \textit{for each } x \in E. \tag{4.3'}$$

Note that if E and F are normed vector spaces, then, in view of Theorem 2.2, $\mathfrak{X} \downarrow Q(E,F)$ whenever $\mathfrak{X}$ strongly converges to 0 and $\mathfrak{X}$ contains a set bounded in the norm topology in $L(E,F)$. Note, moreover, that, unless E is finite-dimensional, $Q(E,F)$ is not a topological space for the filter $\mathfrak{U} = \sup_{\mathfrak{X} \downarrow Q(E,F)} \mathfrak{X}$ does not contain a bounded set.

In view of Propositions 4.1–4.3, we have the following three propositions:

PROPOSITION 4.4. *Let E and F be locally convex topological vector spaces and $\mathfrak{X}$ be a filter on $L(E,F)$. Then $\mathfrak{X} \downarrow Q(E,F)$ if and only if $\mathfrak{X}(x) \downarrow F$ for each $x \in E$ and for each $j \in J$, there exist a finite subset $H(j)$ of J and $X_j \in \mathfrak{X}$ such that (4.2) holds for each $f \in X_j$ and each $x \in E$.*

PROPOSITION 4.5. *Let E and F be locally convex topological vector spaces, f be a mapping from $\mathbb{R}^n$ into $Q(E,F)$, and s be an element of $\mathbb{R}^n$. If, for each $x \in E$, $\lim_{t\to s} f(t)x$ exists in the sense of Definition 3.3 and, for some $\varepsilon > 0$, the set $\{f(t) : |t-s| < \varepsilon\}$ is quasi-bounded in $B(E,F)$, then f is continuous at s.*

PROPOSITION 4.6. *Let E and F be locally convex topological vector spaces, f be a mapping from $\mathbb{R}^n$ into $Q(E,F)$, and t be an element of $\mathbb{R}^n$. If, for some $l \in L(\mathbb{R}^n, Q(E,F))$ and all $x \in E$,*

$$\lim_{h\to 0} \frac{f(t+h)x - f(t)x - l(h)x}{\|h\|} = 0$$

and, for some $\varepsilon > 0$, the set

$$\left\{\frac{f(t+h) - f(t) - l(h)}{\|h\|} : \|h\| < \varepsilon\right\}$$

is quasi-bounded in $B(E,F)$, then f is differentiable at t and $f'(t) = l$.

The following is a simple but important result.

THEOREM 4.4. *For any pair E, F of PVS's, $Q(E,F)$ is a CPVS.*

Proof. If $\mathfrak{X} \downarrow Q(E,F)$, then $\mathfrak{X}$ is quasi-bounded in $B(E,F)$ and, for any $x \in E$, there exists a filter $\mathfrak{B}_x$ on F such that $\mathfrak{X}(x) \prec \mathcal{V}\mathfrak{B}_x$ and $\mathcal{V}\mathfrak{B}_x \downarrow F$. To show that $I_a\mathfrak{X} \downarrow Q(E,F)$ for each $a > 0$, note that, for each $a > 0$, $I_a\mathfrak{X}(x) \prec I_a\mathcal{V}\mathfrak{B}_x = \mathcal{V}\mathfrak{B}_x$ so $I_a\mathfrak{X}$ satisfies (4.3). The quasi-boundedness of $I_a\mathfrak{X}$ follows from the quasi-boundedness of $\mathfrak{X}$ and the equality $\mathcal{V}I_a\mathfrak{X} = \mathcal{V}\mathfrak{X}$. ∎

We now state the main result of the present section.

THEOREM 4.5. *If E, F, and G are PVS's, then the mapping*

$$Q(E,F) \times Q(F,G) \ni (f,g) \to gf \in Q(E,G)$$

is continuous.

Proof. Suppose that $\mathfrak{X} \downarrow_f Q(E,F)$ and $\mathfrak{Y} \downarrow_g Q(F,G)$ and let $(\mathfrak{B}_x)_{x\in E}$ and $(\mathfrak{U}_y)_{y\in F}$ be the corresponding families of filters satisfying (4.3) such that:

(i) $(\mathfrak{X} - f)(x) \prec \mathcal{V}\mathfrak{B}_x$ and $\mathcal{V}\mathfrak{B}_x \downarrow F$ for each $x \in E$,
(i′) $\mathfrak{X} - f$ is quasi-bounded in $B(E,F)$, i.e., $\mathcal{V}\mathfrak{B} \downarrow E \Rightarrow \mathcal{V}(\mathfrak{X} - f)\mathfrak{B} \downarrow F$,
(ii) $(\mathfrak{Y} - g)(y) \prec \mathcal{V}\mathfrak{U}_y$ and $\mathcal{V}\mathfrak{U}_y \downarrow G$ for each $y \in F$,
(ii′) $\mathfrak{Y} - g$ is quasi-bounded in $B(F,G)$, i.e., $\mathcal{V}\mathfrak{U} \downarrow F \Rightarrow \mathcal{V}(\mathfrak{Y} - g)\mathfrak{U} \downarrow G$.

Note that

$$\mathfrak{Y}(\mathfrak{X}) - g(f) \prec (\mathfrak{Y} - g)(\mathfrak{X} - f) + (\mathfrak{Y} - g)(f) + g(\mathfrak{X} - f).$$

Hence, to show that $\mathfrak{Y}(\mathfrak{X}) - g(f)$ satisfies (4.3) and (4.4), it suffices to show that (4.3) and (4.4) hold for all three summands of the right hand side of the above relation.

To verify that the three summands satisfy (4.3) note that, if $x \in E$, then:

1° By (i), $(\mathfrak{Y} - g)(\mathfrak{X} - f)(x) \prec (\mathfrak{Y} - g)(\mathcal{V}\mathfrak{B}_x) = \mathcal{V}(\mathfrak{Y} - g)\mathfrak{B}_x$ and, by (i), (i′), (ii′), and (4.3), $\mathcal{V}(\mathfrak{Y} - g)\mathfrak{B}_x$ tends to 0 in G.
2° $(\mathfrak{Y} - g)(f(x)) \prec \mathcal{V}\mathfrak{U}_{f(x)}$ and, by (ii), $\mathcal{V}\mathfrak{U}_{f(x)} \downarrow G$.
3° $g(\mathfrak{X} - f)(x) \prec g(\mathcal{V}\mathfrak{B}_x)$ and, by the continuity of g and (i), $g(\mathcal{V}\mathfrak{B}_x) \downarrow G$.

To verify that the three summands satisfy (4.4), suppose that $\mathcal{V}\mathfrak{B} \downarrow E$. Then:

1° $\mathcal{V}(\mathfrak{Y} - g)(\mathfrak{X} - f)(\mathfrak{B}) = \mathcal{V}(\mathfrak{Y} - g)(\mathcal{V}(\mathfrak{X} - f))(\mathfrak{B})$. By (i′), $\mathcal{V}(\mathfrak{X} - f)(\mathfrak{B}) \downarrow F$, and so, by (ii′), $\mathcal{V}(\mathfrak{Y} - g)(\mathcal{V}(\mathfrak{X} - f))(\mathfrak{B}) \downarrow G$.
2° $\mathcal{V}(\mathfrak{Y} - g)(f(\mathfrak{B})) \downarrow G$ for $f(\mathfrak{B})$ is quasi-bounded.
3° $\mathcal{V}g(\mathfrak{X} - f)(\mathfrak{B}) \downarrow G$ on account of (i′), the continuity of g, and the identity $\mathcal{V}g(\mathfrak{X} - f)(\mathfrak{B}) = g(\mathcal{V}(\mathfrak{X} - f))(\mathfrak{B})$. ∎

5. Applications and examples

5.1. *Strongly continuous semigroups.* Let E be a Banach space, $(S(t))_{t\geq 0}$ be a strongly continuous one-parameter semigroup in E, and A be the infinitesimal generator of $(S(t))_{t\geq 0}$ with domain $D(A)$. As is well known, the space $D = \bigcap_{n=1}^{\infty} D(A^n)$ is dense in E and D is a core for A, that is, the closure of the restriction of A to D coincides with A ([D, Theorem 1.43]). It is easy to see that $A(D) \subset D$ and $S(t)(D) \subset D$ for each $t \geq 0$. Moreover, D is a Fréchet space under the topology determined by the family of pseudonorms $\{p_k : k \in \mathbb{N}\}$ defined inductively by

$$p_0(x) = \|x\|, \quad \ldots, \quad p_{k+1}(x) = p_k(x) + p_k(Ax) \quad (k = 0, 1, \ldots).$$

As a consequence of Theorems 3.3 and 4.5, and the fact that, for each $t \geq 0$, A and $S(t)$ commute on D, we obtain the following

THEOREM 5.1. *If E is a Banach space and $(S(t))_{t\geq 0}$ is a strongly continuous semigroup in E, then the mapping $\mathbb{R} \ni t \to S(t) \in Q(D,D)$ is C^∞ and, for each $n \in \mathbb{N}$,*

$$\frac{d^n S(t)}{dt^n} = A^n S(t) = S(t)A^n.$$

Note that formally the above formula is the same as formula 1.24 in [D].

5.2. *Von Neumann algebras.* We recall that a von Neumann algebra on a Hilbert space H is a *-subalgebra of $B(H)$ closed in one (and then in all) of the following six topologies: weak operator topology, σ-weak operator topology, strong operator topology, σ-strong operator topology, strong* operator topology, σ-strong* operator topology (see [B-R]). In quantum mechanics, statistical mechanics, and quantum field theory, strongly continuous unitary one-parameter groups with values in a von Neumann algebra play a vital role. If $\mathcal{M}$ is a von Neumann algebra contained in $B(H)$, then, unless H is finite-dimensional, the composition of elements of $\mathcal{M}$ is continuous under none of the above topologies on $\mathcal{M}$. Moreover, unless H is finite-dimensional, there exists no reasonable topology on $\mathcal{M}$ under which the composition of elements of $\mathcal{M}$ is continuous. Thus, in general, one cannot regard one-parameter groups with values in $\mathcal{M}$ as continuous homomorphisms of topological algebras. It turns out, however, that any strongly continuous unitary one-parameter group with values in $\mathcal{M}$ is continuous and the composition of elements of $\mathcal{M}$ is continuous under the pseudotopology $Q(H,H)$ in $\mathcal{M}$. The first statement follows from Proposition 4.5, and the second from Theorem 4.5.

Moreover, we have the following characterization of von Neumann algebras among *-subalgebras of $B(H)$, where H is a Hilbert space, in terms of $Q(H,H)$.

THEOREM 5.2. *Let H be a Hilbert space and $\mathcal{M}$ be a *-subalgebra of $B(H)$. Then $\mathcal{M}$ is a von Neumann algebra if and only if $\mathcal{M}$ is closed in $Q(H,H)$.*

Proof. Let $\mathcal{M}$ be a von Neumann algebra. Then $\mathcal{M}$ is closed in the strong operator topology and consequently it is closed in the pseudotopology $Q(H,H)$.

Now suppose that $\mathcal{M}$ is closed in $Q(H,H)$ and let $\mathcal{N}$ be the closure of $\mathcal{M}$ in the strong operator topology. To prove that $\mathcal{N}=\mathcal{M}$, it suffices to show that, for each $A\in\mathcal{N}$, there exists a filter $\mathfrak{X}$ on $\mathcal{M}$ such that $\mathfrak{X}$ is convergent to A in $Q(H,H)$. By the Kaplansky theorem (cf. [B-R, 2.4.16]), for each $A\in\mathcal{N}$ and each neighbourhood U of A in the strong operator topology, the set $\mathrm{A}_U=\{B\in\mathcal{M}\cap U:\|B\|\le\|A\|\}$ is not empty. Thus the family

$$\mathfrak{B}=\{\mathrm{A}_U : U \text{ a neighbourhood of } A \text{ in the strong operator topology}\}$$

is a filter basis on $\mathcal{M}$, the filter $[\mathfrak{B}]$ converges strongly to A and, by Proposition 4.1, is quasi-bounded in $B(H)$. ■

5.3. *Representations of Lie groups and Lie algebras.* Let G be a finite-dimensional real Lie group and $\mathfrak{g}$ be its Lie algebra. Given $x\in G$, let T_xG be the tangent space of G at x. We identify $\mathfrak{g}$ with T_eG, where e is the neutral element of G. For each $\xi\in\mathfrak{g}$, let $\mathbb{R}\ni t\to e^{t\xi}\in G$ be the homomorphism of $\mathbb{R}$ in G whose differential at 0 evaluated at 1 is ξ. Given $y\in G$ and $\xi\in T_xG$ ($x\in G$), denote by $y\xi$ the differential of the mapping $G\ni x\to yx$ at x evaluated at ξ, and by ξy the differential of the mapping $G\ni x\to xy$ at x evaluated at ξ. For each $x\in G$ and each $\xi\in\mathfrak{g}$, the element $(x\xi)x^{-1}$ of $\mathfrak{g}$ coincides with $x(\xi x^{-1})$ and is customarily denoted as $x\xi x^{-1}$. Note that $x\xi x^{-1}$ coincides with the differential of the mapping $G\ni y\to xyx^{-1}\in G$ at e evaluated at ξ and is then also denoted by $\mathrm{Ad}\,x\xi$. Given $\xi,\eta\in\mathfrak{g}$, let $[\xi,\eta]$ denote the Lie bracket of ξ and η, that is, the differential of the mapping $G\ni x\to\mathrm{Ad}\,x\eta\in\mathfrak{g}$ at e evaluated at ξ.

Let M be a pseudotopological algebra, that is, an algebra whose underlying vector space is a PVS such that the multiplication is continuous in the corresponding pseudotopology. Let ϱ be a twice differentiable representation of G in $\mathcal{M}$.

Differentiating both sides of the equality

$$\varrho(xyx^{-1})=\varrho(x)\varrho(y)\varrho(x^{-1})\qquad (x,y\in G) \tag{5.1}$$

with respect to y at e gives, for each $\xi\in\mathfrak{g}$,

$$\varrho(x)d\varrho(e;\xi)\varrho(x^{-1})=d\varrho(e;x\xi x^{-1}). \tag{5.2}$$

Write $\Gamma(\xi)$ for $d\varrho(e;\xi)$. Differentiating both sides of (5.2) with respect to x at e yields, for each $\xi\in\mathfrak{g}$ and each $\eta\in\mathfrak{g}$,

$$\Gamma(\xi)\Gamma(\eta)-\Gamma(\eta)\Gamma(\xi)=\Gamma([\xi,\eta]). \tag{5.3}$$

An important problem is to determine conditions under which a given linear mapping $\Gamma:\mathfrak{g}\to\mathcal{M}$ satisfying (5.3) can be identified with the mapping $\xi\to d\varrho(e;\xi)$ for some twice differentiable representation ϱ of G in $\mathcal{M}$. In contributing to solution of this problem, we establish a result similar to a result of [T-W]. The latter was proved for the so-called b-structures, and hence was not directly applicable to strongly continuous representations in Banach spaces or Hilbert spaces.

We start with the following

DEFINITION 5.1. Let $\mathcal{M}$ be a pseudotopological algebra, $\mathcal{M}_0$ be the set of all invertible elements in $\mathcal{M}$, and Γ be a linear mapping from $\mathfrak{g}$ into a pseudotopological algebra $\mathcal{M}$ with identity, satisfying (5.3). We say that a manifold $M\subset G$ containing e is *integrable* if there exists a differentiable mapping $\varrho:M\to\mathcal{M}_0$ such that

$$\varrho(e)=\mathrm{Id}, \tag{5.4}$$

$$\varrho(x)\Gamma(\xi)\varrho(x)^{-1}=\Gamma(\mathrm{Ad}\,x\xi)\quad\text{for each } x\in M \text{ and each } \xi\in\mathfrak{g}, \tag{5.5}$$

$$d\varrho(x;\xi)=\Gamma(\xi x^{-1})\varrho(x)\quad\text{for each } x\in M \text{ and each } \xi\in\mathfrak{g}. \tag{5.6}$$

PROPOSITION 5.1. *Let $\boldsymbol{\Gamma}$ be a linear mapping from $\mathfrak{g}$ into a pseudotopological algebra $\mathcal{M}$ with identity, satisfying* (5.3), *and M be a integrable manifold of maximal dimension. Then T_eM is a subalgebra of $\mathfrak{g}$ and if N is integrable, then $T_eN \subset T_eM$.*

Sketch of the proof. The proof is divided into a number of steps.

a) Let M and N be two integrable submanifolds with corresponding mappings ϱ_1 and ϱ_2 satisfying (5.4)–(5.6), respectively, such that the mapping $M \times N \ni (x,y) \to xy \in MN$ is regular at (e,e), where, clearly, $MN = \{z \in \boldsymbol{G} : z = xy,\ x \in M,\ y \in N\}$. Then MN is a locally integrable submanifold of $\boldsymbol{G}$. Setting $\varrho(xy) = \varrho_1(x)\varrho_2(y)$ for $x \in M$ and $y \in N$ we infer that MN is integrable.

b) Let M be an integrable manifold with a corresponding mapping ϱ satisfying (5.4)–(5.6). Then, for each $x \in M$, xMx^{-1} is an integrable manifold. Indeed, if we set $\varrho_1(xyx^{-1}) = \varrho(x)\varrho(y)\varrho(x)^{-1}$ for $y \in M$, then ϱ_1 satisfies (5.4)–(5.6) with respect to xMx^{-1}.

c) Let M be an integrable manifold of maximal dimension and let N be an integrable manifold. Then $T_eN \subset T_eM$, for otherwise there is a one-dimensional submanifold P of N such that $e \in P$ and $T_eP \cap T_eM = \{0\}$, and so, by a), PM is an integrable manifold of dimension greater than the dimension of M, a contradiction.

d) Let M be an integrable manifold of maximal dimension. By b) and c), for each $x \in M$, $T_e(xMx^{-1}) = x(T_eM)x^{-1}$ is contained in T_eM, and so, for each $\eta \in T_eM$, $\operatorname{Ad} x\eta$ is in T_eM. Thus, for any $\xi, \eta \in T_eM$, the differential $[\xi,\eta]$ of the mapping $\boldsymbol{G} \ni x \to \operatorname{Ad} x\eta \in \mathfrak{g}$ at e evaluated at ξ is also in T_eM. Hence T_eM is a subalgebra of $\mathfrak{g}$. ∎

As a simple consequence of Proposition 5.1, we obtain

PROPOSITION 5.2. *Let $\boldsymbol{\Gamma}$ be a linear mapping from $\mathfrak{g}$ into a pseudotopological algebra $\mathcal{M}$ with identity, satisfying* (5.4) *and S be a subset of $\mathfrak{g}$ Lie algebraically generating $\mathfrak{g}$ such that, for each $\eta \in S$, there exists a differentiable group $(U(t,\eta))_{t\in\mathbb{R}}$ in $\mathcal{M}_0$ such that*

$$\frac{dU(t,\eta)}{dt} = \boldsymbol{\Gamma}(\eta)U(t,\eta) \quad \textit{for each } \eta \in S, \tag{5.7}$$

$$U(t,\eta)\boldsymbol{\Gamma}(\xi)U(-t,\eta) = \boldsymbol{\Gamma}((\operatorname{Ad} e^{t\eta})\xi) \quad \textit{for each } \xi, \eta \in S. \tag{5.8}$$

Then there exists a differentiable local representation ϱ of $\boldsymbol{G}$ such that

$$d\varrho(e;\xi) = \boldsymbol{\Gamma}(\xi) \quad \textit{for each } \xi \in \mathfrak{g}. \tag{5.9}$$

Proof. Given $t \in \mathbb{R}$ and $\eta \in S$, let $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ be the Lie algebra representations of $\mathfrak{g}$ in $\mathcal{M}$ given by

$$\boldsymbol{\Gamma}_1(\xi) = U(t,\eta)\boldsymbol{\Gamma}(\xi)U(-t,\eta) \quad (\xi \in \mathfrak{g}),$$
$$\boldsymbol{\Gamma}_2(\xi) = \boldsymbol{\Gamma}(\operatorname{Ad} e^{t\eta}\xi) \quad (\xi \in \mathfrak{g}).$$

Since $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ coincide on S and S Lie algebraically generates $\mathfrak{g}$, it follows that $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ coincide everywhere. Hence (5.8) holds for each $\xi \in \mathfrak{g}$ and all the one-dimensional submanifolds $\{e^{t\eta} : t \in \mathbb{R}\}$ $(\eta \in S)$ are integrable. Let M be an integrable manifold of maximal dimension with a corresponding mapping ϱ satisfying (5.4)–(5.6). By Proposition 5.1, T_eM contains S and is a subalgebra of $\mathfrak{g}$, whence $T_eM = \mathfrak{g}$. Hence, in particular, M contains an open neighbourhood of e in $\boldsymbol{G}$. Since the mapping $M \times M \ni (x,y) \to xy \in MM$ is regular at (e,e), we infer, reasoning as in step a) of the proof of Proposition 5.1, that $\varrho(xy) = \varrho(x)\varrho(y)$ for any $x, y \in M$, showing that ϱ is a local representation of $\boldsymbol{G}$. ∎

As a consequence of Proposition 5.2, we obtain the following theorem which in one form or another appears in many papers on integration of Lie algebra representations (cf. [K], [J-M], and [R]).

THEOREM 5.3. *Let $\boldsymbol{\Gamma}$ be a Lie algebra representation of $\mathfrak{g}$ in a locally convex space E, D be a dense linear subset of E such that $\boldsymbol{\Gamma}(\xi)(D) \subset D$ for each $\xi \in \mathfrak{g}$, and S be a subset of $\mathfrak{g}$ Lie algebraically generating $\mathfrak{g}$ such that, for each $\eta \in S$, the restriction of $\boldsymbol{\Gamma}(\eta)$ to D is a pregenerator of a strongly continuous group $(U(t,\eta))_{t\in\mathbb{R}}$ of continuous operators satisfying the following conditions*:

$$U(t,\eta)(D) \subset D \quad \textit{for each } t \in \mathbb{R} \textit{ and each } \eta \in S, \tag{5.10}$$

$$U(t,\eta)\boldsymbol{\Gamma}(\xi)U(-t,\eta)f = \boldsymbol{\Gamma}((\operatorname{Ad} e^{t\eta})\xi)f \quad \textit{for each } t \in \mathbb{R},\ \textit{each } \xi,\eta \in S, \textit{ and each } f \in D. \tag{5.11}$$

Then there exists a strongly continuous local representation ϱ of $\boldsymbol{G}$ of continuous operators such that $d\varrho(e;\xi)f = \boldsymbol{\Gamma}(\xi)f$ for each $f \in D$.

Proof. Arguing as in the proof of Proposition 5.2, we first prove that (5.11) holds for each $\xi \in \mathfrak{g}$. Let $\xi_1, \ldots, \xi_N$ be a basis of $\mathfrak{g}$, $\{p_i : i \in I\}$ be a family of pseudonorms determining the topology of E, and $\{p_{i,n} : i \in I,\ n = 0,1,\ldots\}$ be the family of pseudonorms defined inductively by

$$p_{i,0} = p_i, \quad \ldots, \quad p_{i,n+1}(f) = p_{i,n}(f) + \sum_{j=1}^{N} p_{i,n}(\boldsymbol{\Gamma}(\xi_j)f) \quad (i \in I,\ n = 0,1,\ldots).$$

Equip D with the topology determined by the latter family. Using Theorem 5.1 and (5.11) for ξ running over the whole of $\mathfrak{g}$, it is easy to show that the mapping $\mathbb{R} \ni t \to U(t,\eta) \in Q(D,D)$ is C^∞ and that $\boldsymbol{\Gamma}$ as mapping from $\mathfrak{g}$

into $Q(D,D)$ satisfies (5.3), (5.7) and (5.8). Now the theorem follows upon applying Proposition 5.2.

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On the representation of uncountable symmetric basic sets and its applications

by

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Abstract. It is shown that every uncountable symmetric basic set in an F-space with a symmetric basis is equivalent to a basic set generated by one vector. We apply this result to investigate the structure of uncountable symmetric basic sets in Orlicz and Lorentz spaces.

I. Introduction. There are three results about Banach spaces with an uncountable symmetric basis having some relevance to the subject of this paper. Firstly, it was shown by renorming arguments in [T$_1$] that if X is a Banach space with a symmetric basis $\{e_\alpha\}_{\alpha\in A}$ which contains a subspace isomorphic to $c_0(\Gamma)$ (resp. to $\ell^1(\Gamma)$) for an uncountable set Γ then X itself is isomorphic to $c_0(A)$ (resp. to $\ell^1(A)$). Later, using this result and combinatorial considerations, Drewnowski [D$_1$] proved that for nonseparable Banach spaces with a symmetric basis, all uncountable symmetric bases are equivalent. Recently, in the special context of Orlicz spaces, Rodriguez-Salinas [R] has given necessary and sufficient conditions for isomorphic embeddings of Orlicz spaces $h_N(\Gamma)$ into a space $h_M(A)$ for uncountable sets $\Gamma\subseteq A$.

Our aim in the present paper is to analyze the above results in a general framework, that is, to generalize them to F-spaces, i.e. complete metric linear spaces, to determine the context of validity of possible connected extensions, and finally, to give some new applications.

For example, we prove that for the class of F-spaces the above mentioned result of [T$_1$] on spaces containing an isomorphic copy of $c_0(\Gamma)$ can be

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