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Uniqueness of complete norms for quotients of Banach function algebras

by

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Abstract. We prove that every quotient algebra of a unital Banach function algebra A has a unique complete norm if A is a Ditkin algebra. The theorem applies, for example, to the algebra $A(\Gamma)$ of Fourier transforms of the group algebra $L^1(G)$ of a locally compact abelian group (with identity adjoined if Γ is not compact). In such algebras non-semisimple quotients $A(\Gamma)/\overline{J(E)}$ arise from closed subsets E of Γ which are sets of non-synthesis. Examples are given to show that the condition of Ditkin cannot be relaxed. We construct a variety of mutually non-equivalent norms for quotients of the Mirkil algebra M, which fails Ditkin's condition at only one point of Φ_M.

1. Let $(\mathfrak{A}, \|\cdot\|)$ be a Banach algebra. Then $\mathfrak{A}$ has a *unique complete norm* if any algebra norm with respect to which $\mathfrak{A}$ is a Banach algebra is equivalent to the given norm $\|\cdot\|$. It is well-known that each semisimple Banach algebra has a unique complete norm: this is Johnson's uniqueness of norm theorem ([5], [2], [7]). In this note we wish to investigate when a quotient of a Banach function algebra has a unique complete norm.

Let A be an algebra. Then the set of characters, or non-zero multiplicative linear functionals, on A is denoted by Φ_A. In the case where A is a unital Banach algebra, Φ_A is a compact space with respect to the weak $*$-topology. Now let A be a semisimple, commutative, unital Banach algebra. Then we regard A as a Banach function algebra on Φ_A. We first recall some standard definitions. Let $f \in A$. The *zero set* of f is $\mathbf{Z}(f) = \{\varphi \in \Phi_A : f(\varphi) = 0\}$, and the *hull* of an ideal I in A is $h(I) = \bigcap\{\mathbf{Z}(f) : f \in I\}$. For a closed set $E \subset \Phi_A$, set

$$J(E) = \{f \in A : \mathbf{Z}(f) \text{ is a neighbourhood of } E\},$$

$$I(E) = \{f \in A : E \subset \mathbf{Z}(f)\}.$$

The set E is a *set of synthesis* for A if $I(E)$ is the only closed ideal in A whose hull is E. The algebra A is *regular* if, for each closed set E in Φ_A and each $\varphi \in \Phi_A \setminus E$, there exists $f \in I(E)$ with $f(\varphi) = 1$.

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Let A be a regular Banach function algebra on Φ_A, and let I be a closed ideal in A with hull E, say. Then $\overline{J(E)} \subset I \subset I(E)$, and E is a set of synthesis if and only if $\overline{J(E)} = I(E)$. Set $\mathfrak{A} = A/I$. Then $\mathfrak{A}$ is a commutative Banach algebra (with respect to the quotient norm), and the radical of $\mathfrak{A}$ is $I(E)/I$. Thus $\mathfrak{A}$ is semisimple in the case where $I = I(E)$, but non-semisimple quotients arise when E is not a set of synthesis and $I \neq I(E)$. We shall enquire when these quotients have a unique complete norm.

Let A be a unital Banach function algebra on Φ_A, and take $\varphi \in \Phi_A$. We write J_φ for $J(\{\varphi\})$ and M_φ for the maximal ideal $I(\{\varphi\}) = \ker\varphi$. The algebra A is *strongly regular* if $\bar{J}_\varphi = M_\varphi \quad (\varphi \in \Phi_A)$, i.e., if each singleton is a set of synthesis. A strongly regular algebra is necessarily regular.

In §3, we shall first give an easy example which shows that, for a regular Banach function algebra A which is not strongly regular, it may be that a quotient $M_\varphi/\bar{J}_\varphi$ is both infinite-dimensional and has zero multiplication; for such an algebra, each Banach space norm on $M_\varphi/\bar{J}_\varphi$ is a Banach algebra norm, and so there are complete algebra norms on $A/\bar{J}_\varphi$ which are not equivalent to the quotient norm. Thus we shall concentrate on the question of the uniqueness of norm for quotients A/I in the case where A is strongly regular.

In fact, a condition a little stronger than strong regularity is required to obtain a positive result. Let A be a unital Banach function algebra on Φ_A. Then A *satisfies Ditkin's condition* at $\varphi \in \Phi_A$ if, for each $f \in M_\varphi$, there is a sequence (f_k) in J_φ such that $ff_k \to f$ in M_φ, and A is a *Ditkin algebra* if it satisfies Ditkin's condition at each $\varphi \in \Phi_A$. We shall show in §2 that, for a Ditkin algebra A, each quotient A/I does have a unique complete norm; an example in §3 will show that this is not necessarily the case if A only satisfies the weaker condition of being strongly regular. For this algebra we shall construct a variety of complete algebra norms not equivalent to the quotient norm.

A unital Banach function algebra A is a *strong Ditkin algebra* if each maximal ideal M_φ of A has a bounded approximate identity in J_φ. For example, let Γ be a locally compact abelian group, and let $A(\Gamma)$ be the algebra of Fourier transforms of the group algebra $L^1(G)$, where G is the dual group of Γ. Then $A(\Gamma)$ (with identity adjoined in the case where Γ is not compact) is a strong Ditkin algebra; the algebras $A(\Gamma)$ have non-semisimple quotients whenever Γ is not discrete, and the theorem of §2 will apply to these examples.

The class of Ditkin algebras is strictly larger than the class of strong Ditkin algebras. For example, take α with $0 < \alpha < 1$. Then $L^1_\alpha(\mathbb{R}^n)$ consists of the measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_\alpha = \int_{\mathbb{R}^n} |f(\mathbf{t})|(1+|\mathbf{t}|)^\alpha \, d\mathbf{t} < \infty,$$

where $|\mathbf{t}| = (t_1^2 + \ldots + t_n^2)^{1/2}$ for $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n$. It is standard that $(L^1_\alpha(\mathbb{R}^n), \|\cdot\|_\alpha)$ is a commutative Banach algebra with respect to convolution multiplication ([4], [8]). Denote by $A_\alpha(\mathbb{R}^n)$ the algebra of Fourier transforms of elements of $L^1_\alpha(\mathbb{R}^n)$. Then $A_\alpha(\mathbb{R}^n)$ is a Banach function algebra on $\Phi_{A_\alpha} = \mathbb{R}^n$, and the algebra $A_\alpha(\mathbb{R}^n)^\#$ formed by adjoining an identity to $A_\alpha(\mathbb{R}^n)$ is a Ditkin algebra ([8, VI.3.3]), but it is not a strong Ditkin algebra. It is proved in [8, II.7.3] that the sphere S_{n-1} in $\mathbb{R}^n$ is a set of non-synthesis for $A_\alpha(\mathbb{R}^n)$ whenever $n \geq 3$, and that the circle S_1 in $\mathbb{R}^2$ is a set of non-synthesis for $A_\alpha(\mathbb{R}^2)$ if and only if $\alpha \geq 1/2$. Thus the algebras $A_\alpha(\mathbb{R}^n)$ may have non-semisimple quotients. Each of these algebras has discontinuous point derivations at each character, but nevertheless the theorem of §2 implies that quotients of these algebras always have a unique complete norm.

2. Let A be a Ditkin algebra. We shall prove in this section that each quotient A/I has a unique complete norm.

We start from a standard result about regular algebras. Let A be a unital Banach function algebra on Φ_A, and let I be a closed ideal in A. A function f on Φ_A *belongs locally to I at $\varphi \in \Phi_A$* if there exists $g \in I$ such that $f - g \in J_\varphi$, and f *belongs locally to I on Φ_A* if f belongs locally to I at each point $\varphi \in \Phi_A$. In the case where A is a unital, regular Banach function algebra, each function which belongs locally to a closed ideal I already belongs to I ([8, 2.1.3]); this is the *localization lemma*.

Let I_1 and I_2 be closed ideals in a unital Banach function algebra A. Then I_2 *belongs locally to I_1 at $\varphi \in \Phi_A$* if there is a neighbourhood U_φ of φ such that, for each $f \in I_2$, there exists $g \in I_1$ with $\mathbf{Z}(f-g) \supset U_\varphi$. The following lemma is proved in [8, 2.6.4].

2.1. LEMMA. *Let A be a Ditkin algebra on Φ_A, and let I_1 and I_2 be closed ideals in A with $I_1 \subsetneq I_2$ and such that $h(I_1) = h(I_2)$. Set*

$$P(I_1, I_2) = \{\varphi \in \Phi_A : I_2 \text{ does not belong locally to } I_1 \text{ at } \varphi\}.$$

Then $P(I_1, I_2)$ is a non-empty, perfect subset of the boundary of $h(I_1)$. ∎

The theorem will also use the following standard *stability lemma* of automatic continuity theory; we state the result for epimorphisms between Banach algebras, but the result applies more generally (e.g., [9, Lemma 1.6]). Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let $\theta : \mathfrak{B} \to \mathfrak{A}$ be a homomorphism. Then the *separating space* $\mathfrak{S}(\theta)$ of θ is defined by

$$\mathfrak{S}(\theta) = \{a \in \mathfrak{A} : \text{there exists } (b_n) \text{ in } \mathfrak{B} \text{ such that } b_n \to 0 \text{ and } \theta(b_n) \to a\}.$$

2.2. LEMMA. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras, and let $\theta : \mathfrak{B} \to \mathfrak{A}$ be an epimorphism. Then $\mathfrak{S}(\theta)$ is a closed ideal in $\mathfrak{A}$ and, for each sequence (a_n) in $\mathfrak{A}$, there exists $n_0 \in \mathbb{N}$ such that*

$$\overline{a_1 \ldots a_n \mathfrak{S}(\theta)} = \overline{a_1 \ldots a_{n_0} \mathfrak{S}(\theta)} \quad (n \geq n_0). \ ∎$$

2.3. THEOREM. *Let A be a unital Ditkin algebra on Φ_A, and let I be a closed ideal in A. Then each epimorphism from a Banach algebra onto A/I is automatically continuous.*

Proof. Set $\mathfrak{A} = A/I$, and let $\theta : \mathfrak{B} \to \mathfrak{A}$ be an epimorphism from a Banach algebra $\mathfrak{B}$ onto $\mathfrak{A}$. Set $E = h(I)$, so that $\overline{J(E)} \subset I \subset I(E)$, and the radical of $\mathfrak{A}$ is $\mathfrak{R} = I(E)/I$. Certainly $\mathfrak{S}(\theta) \subset \mathfrak{R}$.

We first *claim* that, for each closed ideal K in A with $h(K) = E$ and $I \subsetneq K$, there exists $f \in A$ with

$$I \subsetneq \overline{fK + I} \subsetneq K. \tag{1}$$

By Lemma 2.1, the set $P(I, K)$ is a non-empty, perfect subset of ∂E. Choose $\varphi, \psi \in P(I, K)$ with $\varphi \neq \psi$, and choose $f \in A$ such that $f|U_\varphi = 1$ and $f|U_\psi = 0$ for neighbourhoods U_φ and U_ψ of φ and ψ, respectively; this is possible because A is regular, and hence normal. We have $\overline{fK} \neq K$ because each function of $\overline{fK}$ is zero on U_ψ, and this is not true of each function in K because $h(K) = E$ and $U_\psi \not\subset E$.

To obtain a contradiction, assume that $I = \overline{fK + I}$. Then $I = fK + I$, and, for each $g \in K$, there exists $h \in I$ with $\mathbf{Z}(g - h) \supset U_\varphi$. Thus K belongs locally to I at φ, a contradiction of the fact that $\varphi \in P(I, K)$, and so $I \neq \overline{fK + I}$.

Choose $h \in A$ such that $h|V_\psi = 1$ for a neighbourhood V_ψ of ψ and such that $h|(\Phi_A \backslash U_\psi) = 0$. Now, to obtain a contradiction, assume that $\overline{fK + I} = K$, and take $g \in K$. Then there exist sequences (f_n) in K and (h_n) in I such that $ff_n + h_n \to g$ in A. We have $ff_nh + h_nh \to gh$ in A. But $fh = 0$ and $\{h_n : n \in \mathbb{N}\} \subset I$ and so $gh \in I$. Since $\mathbf{Z}(g - gh) \supset V_\psi$, we have shown that K belongs locally to I at ψ, a contradiction of the fact that $\psi \in P(I, K)$. Thus $\overline{fK + I} \neq K$.

Let $\pi : A \to \mathfrak{A}$ be the quotient map, and set $K = \pi^{-1}(\mathfrak{S}(\theta))$, so that K is a closed ideal in A with $I \subset K \subset I(E)$, and $h(K) = E$. Assume that $I \neq K$. By induction from (1), we obtain a sequence (f_n) in A such that

$$\overline{f_1 \ldots f_{n+1}K + I} \subsetneq \overline{f_1 \ldots f_nK + I} \quad (n \in \mathbb{N}).$$

Since $f_1 \ldots f_nK + I \supset I$, we have

$$\pi(\overline{f_1 \ldots f_nK + I}) = \overline{\pi(f_1 \ldots f_nK + I)} \quad (n \in \mathbb{N}),$$

and so

$$\overline{a_1 \ldots a_{n+1}\mathfrak{S}(\theta)} \subsetneq \overline{a_1 \ldots a_n\mathfrak{S}(\theta)} \quad (n \in \mathbb{N}),$$

where $a_j = \pi(f_j)$ $(j \in \mathbb{N})$. But this contradicts the stability lemma, Lemma 2.2.

We conclude that $I = K$, that $\mathfrak{S}(\theta) = \{0\}$, and that θ is continuous, as required. ■

2.4. COROLLARY. *Let A be a unital Ditkin algebra, and let I be a closed ideal in A. Then A/I has a unique complete norm.* ■

3. In this section, we first give an example of a regular, unital Banach function algebra A such that $A/\bar{J}_\varphi$ fails to have a unique complete norm for some $\varphi \in \Phi_A$, and second an example of a strongly regular, unital Banach function algebra A with a closed ideal I such that A/I does not have a unique complete norm.

3.1. THEOREM. *There is a regular, unital Banach function algebra A on Φ_A with $\varphi \in \Phi_A$ such that $A/\bar{J}_\varphi$ does not have a unique complete norm.*

Proof. For $\alpha = (\alpha_k) \in \mathbb{C}^{\mathbb{N}}$ and $n \in \mathbb{N}$, set

$$p_n(\alpha) = \frac{1}{n}\sum_{k=1}^{n} k|\alpha_{k+1} - \alpha_k|,$$

and set

$$M = \{\alpha \in c_0 : \sup p_n(\alpha) < \infty\}.$$

For $\alpha \in M$, set $p(\alpha) = \sup p_n(\alpha)$ and $\|\alpha\| = \sup|\alpha_n| + p(\alpha)$. Then it is easily checked that $(M, \|\cdot\|)$ is a self-adjoint Banach function algebra on $\mathbb{N}$ (with respect to the pointwise product). Set $A = M^{\#}$, so that we may regard A as a unital, self-adjoint Banach function algebra on $\mathbb{N}_\infty$, the one-point compactification of $\mathbb{N}$.

Take $f \in A$ with $f(x) \neq 0$ $(x \in \mathbb{N}_\infty)$, say $\delta = \inf|f(x)|$. Then $p(1/f) \leq p(f)/\delta^2$, and so f is invertible in A. It follows that the character space of A is $\mathbb{N}_\infty$.

Clearly M contains c_{00}, the space of sequences which are eventually zero, and so A is regular.

For $k, n \in \mathbb{N}$, we write $\delta_k = (\delta_{j,k} : j \in \mathbb{N})$ and $e_n = \sum_{k=1}^{n} \delta_k$.

Take $\alpha = (\alpha_n)$ and $\beta = (\beta_n)$ in M, and take $\varepsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $|\alpha_n| + |\beta_n| < \varepsilon$ $(n \geq n_0)$. For $n \geq n_0$, we have

$$\|\alpha\beta - \alpha\beta e_n\| \leq \varepsilon(\varepsilon + p(\alpha) + p(\beta)),$$

and so $\alpha\beta e_n \to \alpha\beta$ in M as $n \to \infty$. Thus $M^2 \subset \bar{J}_\infty = \bar{c}_{00}$, and the algebra $M/\bar{J}_\infty$ has zero multiplication.

We now show that $M/\bar{J}_\infty$ is infinite-dimensional. In fact, we shall show that M is non-separable, which implies the result. For each $S \subset \mathbb{N}$, set

$$L(S) = \bigcup\{[2^n, 2^{n+1}) \cap \mathbb{N} : n \in S\},$$

and set

$$\alpha_k^{(S)} = \frac{(-1)^k}{k} \quad (k \in L(S)), \quad \alpha_k^{(S)} = 0 \quad (k \in \mathbb{N}\backslash L(S)).$$

Clearly each $\alpha^{(S)} = (\alpha_k^{(S)} : k \in \mathbb{N})$ belongs to c_0, and

$$k|\alpha_{k+1}^{(S)} - \alpha_k^{(S)}| \le 2 \quad (k \in \mathbb{N}),$$

and so $\alpha^{(S)}$ belongs to M with $\|\alpha^{(S)}\| \le 3$. Now let S and T be distinct subsets of $\mathbb{N}$, and set $\beta = \alpha^{(S)} - \alpha^{(T)}$. Choose $m \in (S\backslash T) \cup (T\backslash S)$. For $k \in \{2^m, \ldots, 2^{m+1} - 2\}$, we have

$$k|\beta_{k+1} - \beta_k| = \frac{2k+1}{k+1} \ge 1,$$

and so

$$p_{2^{m+1}}(\beta) \ge \frac{1}{2^{m+1}}(2^m - 1) \ge \frac{1}{4}.$$

Thus $\|\alpha^{(S)} - \alpha^{(T)}\| \ge 1/4$ whenever S and T are distinct subsets of $\mathbb{N}$. Since the power set of $\mathbb{N}$ is uncountable, M is non-separable.

The result follows. ∎

Before giving our second example, we prove two general results.

Let $\mathfrak{A}$ be an algebra, and take $\varphi \in \Phi_A$. A *point derivation at* φ is a linear functional d on $\mathfrak{A}$ such that

$$d(ab) = d(a)\varphi(b) + d(b)\varphi(a) \quad (a, b \in \mathfrak{A}).$$

Clearly, a linear functional d on $\mathfrak{A}$ is a point derivation at φ if and only if $d|(\ker\varphi)^2 = 0$ and $d(e) = 0$, where e is the identity of $\mathfrak{A}$, and so there are discontinuous point derivations at φ in the case where $(\ker\varphi)^2$ has infinite codimension in $\ker\varphi$.

3.2. PROPOSITION. *Let $(\mathfrak{A}, \|\cdot\|)$ be a commutative, unital Banach algebra with radical $\mathfrak{R}$. Suppose that there exists $\varphi \in \Phi_{\mathfrak{A}}$ and $a_0 \in \mathfrak{R}$ such that $\mathfrak{R} = \mathbb{C}a_0$, and there exists a discontinuous point derivation d at φ with $d(a_0) = 0$. Then the formula*

$$|||a||| = \|a + d(a)a_0\| \quad (a \in \mathfrak{A})$$

defines a complete algebra norm $|||\cdot|||$ on $\mathfrak{A}$ which is not equivalent to the given norm.

Proof. We have $aa_0 = \varphi(a)a_0$ $(a \in \mathfrak{A})$. The map $\theta : a \mapsto a + d(a)a_0$, $\mathfrak{A} \to \mathfrak{A}$, is an endomorphism on $\mathfrak{A}$. Suppose that $\theta(a) = 0$. Then $a \in \mathbb{C}a_0$, and hence $d(a) = 0$ and $a = 0$. Now, for $a \in \mathfrak{A}$, set $b = a - d(a)a_0$. Then $\theta(b) = a$, again because $d(a_0) = 0$. Thus θ is an automorphism of $\mathfrak{A}$. It follows that $|||\cdot|||$ is a complete algebra norm on $\mathfrak{A}$; it is not equivalent to $\|\cdot\|$ because d is discontinuous. ∎

A Banach algebra $\mathfrak{A}$ has a *Wedderburn decomposition* if there exists a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$ (as a semi-direct product), where $\mathfrak{R}$ is the radical of $\mathfrak{A}$; the decomposition is a *strong Wedderburn decomposition* if $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$.

3.3. PROPOSITION. *Let $(\mathfrak{A}, \|\cdot\|)$ be a unital Banach algebra with radical $\mathfrak{R}$. Suppose that $\mathfrak{R}^2 = \{0\}$ and that $\mathfrak{A}$ has a Wedderburn decomposition $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$ which is not a strong Wedderburn decomposition. Then the formula*

$$|||a||| = \|b + \mathfrak{R}\| + \|r\| \quad (a = b + r \in \mathfrak{B} \oplus \mathfrak{R})$$

defines a complete algebra norm on $\mathfrak{A}$ which is not equivalent to the given norm.

Proof. We have $\mathfrak{B} \cong \mathfrak{A}/\mathfrak{R}$, and so $a \mapsto \|a + \mathfrak{R}\| = \|b + \mathfrak{R}\|$ is a complete algebra norm on $\mathfrak{B}$. Thus $|||\cdot|||$ is a Banach space norm on the linear space $\mathfrak{B} \oplus \mathfrak{R}$.

Take $a_1 = b_1 + r_1$ and $a_2 = b_2 + r_2$ in $\mathfrak{B} \oplus \mathfrak{R}$. For each $s_1, s_2 \in \mathfrak{R}$, we have

$$\begin{aligned} |||a_1a_2||| &\le \|b_1 + \mathfrak{R}\|\,\|b_2 + \mathfrak{R}\| + \|b_1r_2\| + \|r_1b_2\| \\ &= \|b_1 + \mathfrak{R}\|\,\|b_2 + \mathfrak{R}\| + \|(b_1 + s_1)r_2\| + \|r_1(b_2 + s_2)\| \end{aligned}$$

because $\mathfrak{R}^2 = \{0\}$, and so

$$\begin{aligned} |||a_1a_2||| &\le \|b_1 + \mathfrak{R}\|\,\|b_2 + \mathfrak{R}\| + \|b_1 + \mathfrak{R}\|\,\|r_2\| + \|b_2 + \mathfrak{R}\|\,\|r_1\| \\ &\le |||a_1|||\,|||a_2|||. \end{aligned}$$

Thus $|||\cdot|||$ is an algebra norm on $\mathfrak{A}$.

The norm $|||\cdot|||$ is not equivalent to $\|\cdot\|$ on $\mathfrak{A}$ because $\mathfrak{B}$ is closed in $(\mathfrak{A}, |||\cdot|||)$, but $\mathfrak{B}$ is not closed in $(\mathfrak{A}, \|\cdot\|)$ by hypothesis. ∎

We now present an example of a Banach function algebra which originates with Mirkil ([6]) and which was considered further by Atzmon ([1]) and in [4, Example 3.6].

We start with the Banach space $(L^2(\mathbb{T}), \|\cdot\|_2)$, identifying $\mathbb{T}$ with the interval $[-\pi, \pi]$, and we set $S = [-\pi/2, \pi/2]$.

Define

$$M = \{f \in L^2(\mathbb{T}) : f|S \in C(S)\},$$

and define

$$\|f\| = \|f\|_2 + |f|_S = \frac{1}{\sqrt{2\pi}}\Big(\int_{-\pi}^{\pi} |f(\theta)|^2\,d\theta\Big)^{1/2} + |f|_S \quad (f \in M).$$

Then $(M, \|\cdot\|)$ is a commutative Banach algebra with respect to convolution multiplication on $\mathbb{T}$, and the trigonometric polynomials are dense in M ([6]).

We now identify M with its algebra of Fourier transforms on $\mathbb{Z}$; following [6], we note that

$$M^2 \subset l^1(\mathbb{Z}). \tag{2}$$

The algebra $A = M^{\#}$ formed by adjoining an identity to M is a Banach function algebra on $\mathbb{Z}_\infty$, the one-point compactification of $\mathbb{Z}$. The ideal J_∞

corresponds to the set of trigonometric polynomials in M, and so $\bar{J}_\infty = M$ and A is a strongly regular Banach function algebra with $\Phi_A = \mathbb{Z}_\infty$.

The map

$$f \mapsto (f, f|S), \quad M \to L^2(\mathbb{T}) \oplus C(S),$$

is an isometric linear isomorphism when the Banach space $L^2(\mathbb{T}) \oplus C(S)$ has the norm $\|(f,g)\| = \|f\|_2 + |g|_S$, and so each element of the dual space M' can be represented by a measure on $\mathbb{T}$ of the form

$$\nu = gd\theta + \mu, \tag{3}$$

where $g \in L^2(\mathbb{T})$, $\mu \in M(S)$, the space of measures on S, and $\|\nu\| = \max\{\|g\|_2, \|\mu\|\}$. The action of ν is given by

$$\langle f, \nu\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\overline{g(\theta)}\,d\theta + \int_S f(\theta)\,d\bar{\mu}(\theta) \quad (f \in M,\ \nu \in M').$$

Define $2\mathbb{Z} = \{2n : n \in \mathbb{Z}\}$ and $E = 2\mathbb{Z} \cup \{\infty\}$, a closed subset of $\mathbb{Z}_\infty$. It was proved by Atzmon ([1]) that E is a set of non-synthesis for A; the following result extends this observation.

3.4. PROPOSITION. *The closed ideal $\overline{J(E)}$ has codimension one in $I(E)$.*

Proof. Define g_0 on $\mathbb{T}$ by setting

$$g_0(\theta) = 1 \quad (|\theta| \le \pi/2), \quad g_0(\theta) = -1 \quad (\pi/2 < |\theta| \le \pi).$$

Then $g_0 \in M$, and the Fourier transform $\widehat{g}_0$ of g_0 is given by

$$\widehat{g}_0(k) = \frac{2}{\pi}\int_0^{\pi} g_0(\theta)\cos k\theta\,d\theta = \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right) \quad (k \in \mathbb{Z}\backslash\{0\})$$

with $\widehat{g}_0(0) = 0$. Thus $\widehat{g}_0|E = 0$, and so $g_0 \in I(E)$.

Set $\mu_0 = \delta_{\pi/2} + \delta_{-\pi/2}$ (where δ_x is the point mass at x), so that $\mu_0 \in M(S)$. Let Z be the function $\theta \mapsto e^{i\theta}$ on $\mathbb{T}$. As was proved in [1],

$$\langle Z^k, \mu_0\rangle = e^{ik\pi/2} + e^{-ik\pi/2} = 2\cos\left(\frac{k\pi}{2}\right) \quad (k \in \mathbb{Z}),$$

and so $\langle Z^k, \mu_0\rangle = 0$ for $k \in \mathbb{Z}\backslash 2\mathbb{Z}$. Thus $\mu_0|\overline{J(E)} = 0$. However,

$$\langle g_0, \mu_0\rangle = g_0(\pi/2) + g_0(-\pi/2) = 2,$$

and so $g_0 \notin \overline{J(E)}$.

Now take $\nu \in M'$ with $\nu|\overline{J(E)} = 0$. For $k \in 2\mathbb{Z}$, we have $e^{ik(\theta-\pi)} = e^{ik\theta}$ $(|\theta| \le \pi)$, and, if $k \in \mathbb{Z}\backslash 2\mathbb{Z}$, we have $\langle Z^k, \nu\rangle = 0$ because $Z^k \in J(E)$, and so

$$\langle S_\pi f, \nu\rangle = \langle f, \nu\rangle \quad (f \in J_\infty),$$

where $(S_\pi f)(\theta) = f(\theta - \pi)$. It follows that $\langle S_\pi f, \nu\rangle = \langle f, \nu\rangle$ for all $f \in M$, and so, regarding ν as a measure on $\mathbb{T}$, we have

$$\nu(T + \pi) = \nu(T) \tag{4}$$

for every Borel set $T \subset \mathbb{T}$. Let $\nu = gd\theta + \mu$, as in (3). Then $\mu|(-\pi/2, \pi/2)$ has the form $hd\theta$ for some $h \in L^2(S)$, and so, in fact, ν has the form

$$\nu = gd\theta + \alpha\delta_{\pi/2} + \beta\delta_{-\pi/2}$$

for some $g \in L^2(\mathbb{T})$ and some $\alpha, \beta \in \mathbb{C}$. Clearly, by (4) again, we have $\beta = \alpha$, and so $\nu = gd\theta + \alpha\mu_0$ for some $g \in L^2(\mathbb{T})$ and $\alpha \in \mathbb{C}$.

We have

$$\langle f, \nu\rangle = \sum_{k=-\infty}^{\infty} \widehat{f}(k)\overline{\widehat{g}(k)} + \alpha\langle f, \mu_0\rangle \quad (f \in M).$$

For $k \in \mathbb{Z}\backslash 2\mathbb{Z}$, necessarily $\widehat{g}(k) = 0$ because $\langle Z^k, \nu\rangle = \langle Z^k, \mu_0\rangle = 0$, and so $\langle f, \nu\rangle = \alpha\langle f, \mu_0\rangle$ $(f \in I(E))$. This shows that $\nu|I(E) = \alpha\mu_0|I(E)$, and so $\dim(I(E)/\overline{J(E)}) = 1$. Thus $\overline{J(E)}$ has codimension one in $I(E)$. ■

3.5. PROPOSITION. *The ideal $M^2 + I(E)$ has infinite codimension in M.*

Proof. Write $L = M^2 + I(E)$.

For $n \in \mathbb{N}$, set $\theta_n = (1 + 2^{-n})\pi/2$, and define g_n on $\mathbb{T}$ by

$$g_n(\theta) = 1 \quad (|\theta| \le \theta_n), \quad g_n(\theta) = 0 \quad (\theta_n < |\theta| \le \pi/2).$$

Then $g_n \in M$ and

$$\widehat{g}_n(k) = \frac{1}{k\pi}\sin(k\theta_n) \quad (k \in \mathbb{Z}\backslash\{0\}),$$

with $\widehat{g}_n(0) = 0$.

We *claim* that $\{g_n + L : n \in \mathbb{N}\}$ is linearly independent in M/L. Indeed, suppose that $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, that $f \in M^2$, and that $h \in I(E)$ with

$$\alpha_1 g_1 + \ldots + \alpha_m g_m = f + h.$$

Set $k_r = (2r+1)2^m$ $(r \in \mathbb{N})$. It follows from (2) that, for each $\varepsilon > 0$, there exists $r \in \mathbb{N}$ such that $|\widehat{f}(k_r)| < \varepsilon/k_r$. We have $\widehat{h}(k_r) = 0$ because $k_r \in 2\mathbb{Z}$. Also $k_r(1 + 2^{-m}) \in 2\mathbb{Z} + 1$, and so $|\widehat{g}_m(k_r)| = 1/k_r\pi$, whereas, for $n = 1, \ldots, m-1$, $k_r(1 + 2^{-n}) \in 2\mathbb{Z}$, and so $\widehat{g}_n(k_r) = 0$. It follows that $|\alpha_m| < \pi\varepsilon$. Hence $\alpha_m = 0$, and, successively, $\alpha_{m-1} = \ldots = \alpha_1 = 0$, establishing the claim.

The result follows. ■

Set $\mathfrak{A} = A/\overline{J(E)}$, $\mathfrak{M} = M/\overline{J(E)}$, and $\mathfrak{R} = \operatorname{rad}\mathfrak{A} = I(E)/\overline{J(E)}$ in the above notation.

The following proposition is contained in [4, Example 3.6], and corrects an unfortunate misprint in the statement of that result.

3.6. PROPOSITION. *The algebra $\mathfrak{A}$ has a Wedderburn decomposition $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$ which is such that $\mathfrak{M}^2 \subset \mathfrak{B}$. Further, $\mathfrak{A}$ has no strong Wedderburn decomposition.*

Proof. As in [4, Example 3.6], $\mathfrak{M}^2 \cap \mathfrak{R} = \{0\}$. Let $\mathfrak{X}$ be a linear subspace of $\mathfrak{M}$ such that $\mathfrak{M} = \mathfrak{M}^2 \oplus \mathfrak{R} \oplus \mathfrak{X}$, and set $\mathfrak{B} = (\mathfrak{M}^2 \oplus \mathfrak{X})^{\#}$. Then $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$ is a Wedderburn decomposition with $\mathfrak{M}^2 \subset \mathfrak{B}$. It is noted in [4, Example 3.6] that $\mathfrak{A}$ does not have any strong Wedderburn decomposition. ■

We can now establish the following theorem.

3.7. THEOREM. *There is a strongly regular Banach function algebra A on Φ_A and a closed subset E of Φ_A which is of non-synthesis for A such that the quotient algebra $A/\overline{J(E)}$ has a complete algebra norm which is not equivalent to the quotient norm. Further, the radical of $A/\overline{J(E)}$ has dimension one.*

The Banach function algebra A is the algebra $M^{\#}$ described above, and E is the subset $2\mathbb{Z} \cup \{\infty\}$ of $\Phi_A = \mathbb{Z} \cup \{\infty\}$. Certainly, E is of non-synthesis and the radical of $\mathfrak{A} = A/\overline{J(E)}$ has dimension one. We shall complete the proof of the theorem by describing three families of complete algebra norms on $\mathfrak{A}$ such that each member of each family is not equivalent to the quotient norm. Further, any two members of two distinct families are mutually inequivalent. Set $\mathfrak{R} = I(E)/\overline{J(E)}$, as before, and define $a_0 = g_0 + \overline{J(E)} \in \mathfrak{R}\backslash\{0\}$, so that $\mathfrak{R} = \mathbb{C}a_0$.

First, let φ be the character on $\mathfrak{A}$ whose kernel is $\mathfrak{M} = M/\overline{J(E)}$. Certainly $aa_0 = 0 \quad (a \in \mathfrak{M})$ because $MJ(E) \subset \overline{J(E)}$. It follows from Proposition 3.5 that $\mathfrak{M}^2$ has infinite codimension in $\mathfrak{M}$, and so there are discontinuous point derivations d on $\mathfrak{A}$ at φ; we may suppose that $d(a_0) = 0$. By Proposition 3.2, the formula

$$|||a|||_1 = \|a + d(a)a_0\| \quad (a \in \mathfrak{A})$$

defines a complete algebra norm on $\mathfrak{A}$ which is not equivalent to the quotient norm.

Second, we note that $\mathfrak{R}^2 = \{0\}$ and that, by Proposition 3.6, $\mathfrak{A}$ has a Wedderburn decomposition $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$ with $\mathfrak{M}^2 \subset \mathfrak{B}$ which is not a strong Wedderburn decomposition. By Proposition 3.3, the formula

$$|||a|||_2 = \|b + \mathfrak{R}\| + \|r\| \quad (a = b + r \in \mathfrak{B} \oplus \mathfrak{R})$$

defines a complete algebra norm on $\mathfrak{A}$ which is not equivalent to the quotient norm. Note that $|||\cdot|||_2$ depends on the choice of the linear subspace $\mathfrak{X}$ such that $\mathfrak{M} = \mathfrak{M}^2 \oplus \mathfrak{R} \oplus \mathfrak{X}$.

To see that any norm of the form $|||\cdot|||_1$ is not equivalent to any norm of the form $|||\cdot|||_2$, we argue as follows. For each point derivation d on $\mathfrak{A}$ at φ, we have $d|\mathfrak{M}^2 = 0$, and so every norm of the form $|||\cdot|||_1$ satisfies the condition

$$|||a|||_1 = \|a\| \quad (a \in \mathfrak{M}^2).$$

However, since $\mathfrak{M}^2 \subset \mathfrak{B}$, every norm of the form $|||\cdot|||_2$ satisfies the condition

$$|||a|||_2 = \|a + \mathfrak{R}\| \quad (a \in \mathfrak{M}^2).$$

Since $\mathfrak{M}^2$ is dense in $(\mathfrak{M}, \|\cdot\|)$, there is a sequence (a_n) in $\mathfrak{M}^2$ with $a_n \to a_0$, and then $|||a_n|||_1 \to \|a_0\| \neq 0$ and $|||a_n|||_2 \to \|a_0 + \mathfrak{R}\| = 0$. Hence $|||\cdot|||_1$ and $|||\cdot|||_2$ are not equivalent. Notice that we do have the relation

$$|||a|||_2 \leq |||a|||_1 \quad (a \in \mathfrak{M}^2).$$

We now show how to construct a third family of complete algebra norms $|||\cdot|||_3$ on $\mathfrak{A}$, each of which is not equivalent to the quotient norm $\|\cdot\|$. It is convenient to construct the norms $|||\cdot|||_3$ on the maximal ideal $\mathfrak{M}$ of $\mathfrak{A}$; the norms can be extended to $\mathfrak{A}$ by setting $|||a + \alpha e|||_3 = |||a|||_3 + |\alpha|$ in the usual way.

Let $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$ be a Wedderburn decomposition of $\mathfrak{A}$, as before, and set $\mathfrak{B}_0 = \mathfrak{B} \cap \mathfrak{M}$; each $a \in \mathfrak{M}$ has a unique expression $a = b + \alpha a_0$, where $b \in \mathfrak{B}_0$ and $\alpha \in \mathbb{C}$. The Gelfand map on $\mathfrak{A}$ is denoted by $\mathcal{G}$, and we have an isomorphism

$$\mathcal{G}|\mathfrak{B}_0 : \mathfrak{B}_0 \to \mathfrak{M}/\mathfrak{R}.$$

Since $\mathfrak{M} = M/\overline{J(E)}$ and $\mathfrak{R} = I(E)/\overline{J(E)}$, there is an isometric isomorphism

$$\tau : \mathfrak{M}/\mathfrak{R} \to M/I(E) = M(E).$$

Thus, for each $a \in \mathfrak{M}$, there is a unique element $h_a \in M(E)$ such that $h_a = \tau(a + \mathfrak{R})$. For $b \in \mathfrak{B}_0$, the correspondence $b \mapsto h_b$, $\mathfrak{B}_0 \to M(E)$, is an isomorphism. Now take $f \in M$ (so that $\widehat{f}$ is regarded as a function on $\mathbb{Z}_\infty$). Set $a = f + \overline{J(E)}$, say $a = b + \alpha a_0$, where $b \in \mathfrak{B}_0$ and $\alpha \in \mathbb{C}$. Now h_b is a function on $\mathbb{Z}_\infty$, and we see that $\widehat{f}|E = h_b|E$.

We now construct norms $|||\cdot|||_3$ on $\mathfrak{M}$.

Define a linear functional λ on $\mathfrak{M}$ as follows. For $a \in \mathfrak{M}^2 \subset \mathfrak{B}_0$, set

$$\lambda(a) = \sum\{h_a(k) : k \in E\}.$$

Then extend λ to be a linear functional on $\mathfrak{M}^2$ with $\lambda(a_0) = 0$.

We *claim* that

$$|\lambda(ab)| \leq \|a + \mathfrak{R}\| \, \|b + \mathfrak{R}\| \quad (a, b \in \mathfrak{M}). \tag{5}$$

First, for functions f on $\mathbb{Z}$ (respectively, on E), define

$$\|f\|_2 = \left(\sum\{|f(k)|^2 : k \in \mathbb{Z}\}\right)^{1/2}, \quad \|f\|_{2,E} = \left(\sum\{|f(k)|^2 : k \in E\}\right)^{1/2}.$$

Then

$$|\lambda(ab)| \leq \|h_a\|_{2,E}\|h_b\|_{2,E} \quad (a, b \in \mathfrak{M}).$$

Also, for $a \in \mathfrak{M}$,

$$\|h_a\|_{M(E)} = \inf\{\|\widehat{g}\| : g \in M,\ g|E = h_a\} \geq \inf\{\|\widehat{g}\|_2 : g \in M,\ g|E = h_a\} \geq \|h_a\|_{2,E},$$

and so, for $a, b \in \mathfrak{M}$,

$$|\lambda(ab)| \leq \|h_a\|_{M(E)}\|h_b\|_{M(E)} = \|a + \mathfrak{R}\|\,\|b + \mathfrak{R}\|,$$

giving (5).

Define

$$|||a|||_3 = \max\{\|a + \mathfrak{R}\|,\ |\lambda(a) - \alpha|\} \quad (a = b + \alpha a_0 \in \mathfrak{M}).$$

Clearly $|||\cdot|||_3$ is a Banach space norm on $\mathfrak{M}$. For $a_1 = b_1 + \alpha_1 a_0$ and $a_2 = b_2 + \alpha_2 a_0$ in $\mathfrak{M}$, we have $a_1a_2 = b_1b_2$ because $\mathfrak{M}\mathfrak{R} = \{0\}$, and so

$$\begin{aligned} |||a_1a_2|||_3 &= \max\{\|a_1a_2 + \mathfrak{R}\|,\ |\lambda(a_1a_2)|\} \\ &\leq \|a_1 + \mathfrak{R}\|\,\|a_2 + \mathfrak{R}\| \quad \text{using (5)} \\ &\leq |||a_1|||_3|||a_2|||_3 . \end{aligned}$$

Thus $|||\cdot|||_3$ is a complete algebra norm on $\mathfrak{M}$.

To see that $|||\cdot|||_3$ is not equivalent to the quotient norm $\|\cdot\|$ or to any norm of the form $|||\cdot|||_1$ or $|||\cdot|||_2$, consider the functions f_n on $\mathbb{T}$ defined by

$$f_n(z) = \frac{1+z^n}{1+z} = 1 - z + z^2 - \ldots + (-1)^{n-1}z^{n-1} \quad (z \in \mathbb{T})$$

for $n \in \mathbb{N}$. Since f_n is a trigonometric polynomial, $\widehat{f_n}$ belongs to M^2. Set $a_n = f_n + \overline{J(E)}$ $(n \in \mathbb{N})$, so that $a_n \in \mathfrak{B}_0$ $(n \in \mathbb{N})$. Then

$$\|a_n\| \leq \|f_n\| = \sqrt{2} + n^{1/2} \quad (n \in \mathbb{N}).$$

However,

$$|||a_n|||_3 \geq |\lambda(a_n)| = \left|\sum\{\widehat{f}_n(k) : k \in 2\mathbb{Z}\}\right| \geq \frac{1}{2}(n-1).$$

Thus $|||a_n|||_2 \leq |||a_n|||_1 = \|a_n\| = O(n^{1/2})$, whereas $|||a_n|||_3 \geq (n-1)/2$ for $n \in \mathbb{N}$, and so $|||\cdot|||_3$ is not equivalent to any of the norms $\|\cdot\|$, $|||\cdot|||_1$, or $|||\cdot|||_2$.

In summary, we have

$$|||a|||_2 \leq |||a|||_1 = \|a\|, \quad |||a|||_2 \leq |||a|||_3 \quad (a \in \mathfrak{M}^2).$$

However, there is no constant C such that $\|a\| \leq C|||a|||_3$ $(a \in \mathfrak{M}^2)$.

We finally remark that there is an upper bound to the values $|||a|||$ for $a \in \mathfrak{M}^2$ for each complete algebra norm $|||\cdot|||$ on $\mathfrak{M}$.

The *projective norm* $\|\cdot\|_\pi$ is defined on $\mathfrak{M}^2$ by

$$\|a\|_\pi = \inf\left\{\sum_{j=1}^n \|b_j\|\,\|c_j\| : a = \sum_{j=1}^n b_jc_j \text{ for } b_1,\ldots,b_n,c_1,\ldots,c_n \in \mathfrak{M}\right\}.$$

3.8. Proposition. *Let $|||\cdot|||$ be a complete algebra norm on $\mathfrak{A}$. Then there is a constant C such that*

$$|||a||| \leq C\|a\|_\pi \quad (a \in \mathfrak{M}^2).$$

Proof. Let $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{R}$ be the Wedderburn decomposition of $\mathfrak{A}$ described before, and set

$$\nu = \mathcal{G}|\mathfrak{B} : \mathfrak{B} \to \mathfrak{A}/\mathfrak{R} = A(E).$$

Then ν is an isomorphism, with inverse

$$\theta : A(E) \to \mathfrak{B} \subset \mathfrak{A},$$

say. We regard θ as a map from $A(E)$ into $(\mathfrak{A}, |||\cdot|||)$, and set $\mathfrak{S} = \mathfrak{S}(\theta)$, the separating space of θ. Certainly $\mathfrak{S} \subset \mathfrak{R}$.

Suppose that $\mathfrak{S} = \{0\}$. Then θ is continuous, and

$$|||a||| \leq \|\theta\|\,\|a + \mathfrak{R}\| \leq \|\theta\|\,\|a\| \leq \|\theta\|\,\|a\|_\pi \quad (a \in \mathfrak{M}^2).$$

In the alternative case, we have $\mathfrak{S} = \mathfrak{R}$. Set

$$\mathcal{I}(\theta) = \{F \in A(E) : \theta(F)\mathfrak{S} = \{0\}\},$$

so that $\mathcal{I}(\theta)$ is the continuity ideal of θ. Clearly

$$\mathcal{I}(\theta) = \{F \in A(E) : \theta(F) \in \mathfrak{M}\} = \{F \in A(E) : F \in \nu(\mathfrak{M})\} = M(E),$$

and so the hull of $\mathcal{I}(\theta)$ is $\{\infty\}$.

Since $A(E)$ is a regular Banach function algebra on E, it follows from [3, Proposition 1.3] that there exists a constant C with

$$|||\theta(FG)||| \leq C\|F\|\,\|G\| \quad (F, G \in M(E)).$$

Now take $b, c \in \mathfrak{M}$, and set $F = b + \mathfrak{R}$, $G = c + \mathfrak{R}$, regarding F and G as elements of $M(E)$, as before. Since $\mathfrak{M}^2 \subset \mathfrak{B}$, $\nu(bc)$ is defined in $A(E)$, and indeed

$$\nu(bc) = bc + \mathfrak{R} = FG,$$

whence $\theta(FG) = bc$. Thus

$$|||bc||| = |||\theta(FG)||| \leq C\|b\|\,\|c\|.$$

Finally, take $a = \sum_{j=1}^n b_jc_j \in \mathfrak{M}^2$. Then

$$|||a||| \leq \sum_{j=1}^n |||b_jc_j||| \leq C\sum_{j=1}^n \|b_j\|\,\|c_j\|,$$

and so $|||a||| \leq C\|a\|_\pi$, as required. ∎

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