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Spectral radius formula for commuting Hilbert space operators

by

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Abstract. A formula is given for the (joint) spectral radius of an n-tuple of mutually commuting Hilbert space operators analogous to that for one operator. This gives a positive answer to a conjecture raised by J. W. Bunce in [1].

Let H be a Hilbert space. Denote by $B(H)$ the algebra of all bounded linear operators on H. Let $T = (T_1, \ldots, T_n) \in B(H)^n$ be an n-tuple of pairwise commuting operators on H. The symbol $\sigma(T)$ will stand for the *Harte spectrum* of T, i.e. $(\lambda_1, \ldots, \lambda_n) \notin \sigma(T)$ if there exist operators $U_1, \ldots, U_n$ and $V_1, \ldots, V_n$ in $B(H)$ such that $\sum_{j=1}^n U_j(T_j-\lambda_j) = I$ and $\sum_{j=1}^n (T_j-\lambda_j)V_j = I$ (here we write for simplicity $T_j - \lambda_j$ instead of $T_j - \lambda_j I$). We shall also need the *approximate point spectrum* of T, i.e. the set

$$\sigma_\pi(T) = \Big\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \inf\Big\{\sum_{j=1}^n \|(T_j - \lambda_j)x\| : x \in H,\ \|x\| = 1\Big\} = 0\Big\}.$$

The *spectral radius* of T is defined to be the number

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$$

where

$$|\lambda| = |(\lambda_1, \ldots, \lambda_n)| = \Big(\sum_{j=1}^n |\lambda_j|^2\Big)^{1/2}.$$

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As was shown in [2] (cf. also [6]), $r(T)$ does not depend upon the choice of a joint spectrum of T, in particular one can replace $\sigma(T)$ by $\sigma_\pi(T)$ in the above formula without changing the value of $r(T)$. Further, define the operator $M_T : B(H) \to B(H)$ by the formula

$$M_T(X) = \sum_{j=1}^n T_j^* X T_j.$$

This operator plays an important role in the study of commuting Hilbert space operators (see [5] and [7]). It is easy to see that

$$M_T^s(X) = \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha|=s} \frac{s!}{\alpha!} T^{*\alpha} X T^{\alpha} \quad (s = 1, 2, \ldots)$$

where $\mathbb{Z}_+^n$ is the set of all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \geq 0$ $(j = 1, \ldots, n)$, and, as usual, $|\alpha| = \sum_{j=1}^n \alpha_j$, $\alpha! = \alpha_1! \ldots \alpha_n!$, $T^\alpha = T_1^{\alpha_1} \ldots T_n^{\alpha_n}$ and $T^* = (T_1^*, \ldots, T_n^*)$.

THEOREM 1. *Let $T = (T_1, \ldots, T_n)$ be a mutually commuting n-tuple of Hilbert space operators. Then*

$$r(T) = r(M_T)^{1/2} = \inf_s \Big\| \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha|=s} \frac{s!}{\alpha!} T^{*\alpha} T^\alpha \Big\|^{1/(2s)}.$$

Proof. By [4], Theorem 3.4 (cf. also [3], Theorem 1), we have

$$\sigma(M_T) \subset \Big\{ \sum_{j=1}^n \lambda_j \mu_j : (\lambda_1, \ldots, \lambda_n) \in \sigma(T^*) \text{ and } (\mu_1, \ldots, \mu_n) \in \sigma(T) \Big\}.$$

Thus for every $\nu \in \sigma(M_T)$ we get

$$|\nu| = \Big| \sum_{j=1}^n \lambda_j \mu_j \Big| \leq \Big(\sum_{j=1}^n |\lambda_j|^2 \Big)^{1/2} \Big(\sum_{j=1}^n |\mu_j|^2 \Big)^{1/2} \leq r(T)^2,$$

since obviously $r(T^*) = r(T)$. Hence we obtain

$$r(T) \geq r(M_T)^{1/2} = \inf_s \|M_T^s\|^{1/(2s)}$$
$$\geq \inf_s \|M_T^s(I)\|^{1/(2s)} = \inf_s \Big\| \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha|=s} \frac{s!}{\alpha!} T^{*\alpha} T^\alpha \Big\|^{1/(2s)}.$$

On the other hand, if $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma_\pi(T)$, there exists a sequence (x_k) in H such that $\|x_k\| = 1$ for all k and $(T_j - \lambda_j)x_k \to 0$ as $k \to \infty$ $(j = 1, \ldots, n)$. Thus we have

$$\Big\| \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha|=s} \frac{s!}{\alpha!} T^{*\alpha} T^\alpha \Big\| \geq \Big\langle \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha|=s} \frac{s!}{\alpha!} T^{*\alpha} T^\alpha x_k, x_k \Big\rangle$$
$$= \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha|=s} \frac{s!}{\alpha!} \|T^\alpha x_k\|^2$$
$$\to \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha|=s} \frac{s!}{\alpha!} |\lambda_1|^{2\alpha_1} \ldots |\lambda_n|^{2\alpha_n}$$
$$= \Big(\sum_{j=1}^n |\lambda_j|^2 \Big)^s = |\lambda|^{2s}.$$

This implies

$$\inf_s \Big\| \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha|=s} \frac{s!}{\alpha!} T^{*\alpha} T^\alpha \Big\|^{1/(2s)} \geq |\lambda|$$

for every $\lambda \in \sigma_\pi(T)$ and so, as $\max\{|\lambda| : \lambda \in \sigma_\pi(T)\} = r(T)$ by [2], the left hand side in the above formula is no smaller than $r(T)$. This concludes the proof.

Remarks. 1. For one operator the formula given in Theorem 1 coincides with the usual formula for the spectral radius of an operator.

2. It is easy to see that $M_T^s(I) = \sum_{f \in F(s,n)} T_f^* T_f$ where $F(s,n)$ is the set of all functions from $\{1, \ldots, s\}$ to $\{1, \ldots, n\}$ and $T_f = T_{f(1)} \ldots T_{f(s)}$ for $f \in F(s,n)$. Thus, for a commuting n-tuple $T = (T_1, \ldots, T_n)$, we have

$$r(T) = \inf_s \Big\| \sum_{f \in F(s,n)} T_f^* T_f \Big\|^{1/(2s)},$$

which gives a positive answer to a conjecture of J. W. Bunce (see [1], p. 30).

In order to have a complete analogy to a single operator case let us introduce the following notation:

For $T = (T_1, \ldots, T_n) \in B(H)^n$ define the norm

$$\|T\| = \sup\Big\{ \Big(\sum_{j=1}^n \|T_j x\|^2 \Big)^{1/2} : x \in H,\ \|x\| = 1 \Big\},$$

i.e. $\|T\|$ is the norm of the operator $\widetilde{T} : H \to \bigoplus_{j=1}^n H$ defined by $\widetilde{T}x = (T_1 x, \ldots, T_n x)$. Further, for $T = (T_1, \ldots, T_n) \in B(H)^n$ and $S = (S_1, \ldots, S_m) \in B(H)^m$ set

$$TS = (T_1 S_1, \ldots, T_1 S_m, T_2 S_1, \ldots, T_2 S_m, \ldots, T_n S_m) \in B(H)^{nm}$$

and define inductively $T^{s+1} = T \cdot T^s$ $(s = 1, 2, \ldots)$.

Then the previous spectral radius formula can be expressed in the familiar way:

THEOREM 2. *Let $T=(T_1,\dots,T_n)$ be an n-tuple of pairwise commuting operators on a Hilbert space H. Then*

$$r(T)=\inf_s \|T^s\|^{1/s}=\lim_{s\to\infty}\|T^s\|^{1/s}.$$

Proof. As

$$\sum_{\alpha\in\mathbb{Z}_+^n,|\alpha|=s}\frac{s!}{\alpha!}T^{*\alpha}T^\alpha\ge 0$$

we have

$$\begin{aligned}\Big\|\sum_{\alpha\in\mathbb{Z}_+^n,|\alpha|=s}\frac{s!}{\alpha!}T^{*\alpha}T^\alpha\Big\| &= \sup\Big\{\Big\langle\sum_{\alpha\in\mathbb{Z}_+^n,|\alpha|=s}\frac{s!}{\alpha!}T^{*\alpha}T^\alpha x,x\Big\rangle : x\in H,\ \|x\|=1\Big\}\\ &= \sup\Big\{\sum_{\alpha\in\mathbb{Z}_+^n,|\alpha|=s}\frac{s!}{\alpha!}\|T^\alpha x\|^2 : x\in H,\ \|x\|=1\Big\}=\|T^s\|^2.\end{aligned}$$

Therefore $r(T)=\inf_s\|T^s\|^{1/s}$.

Further,

$$\|T^s\|^2=\Big\|\sum_{\alpha\in\mathbb{Z}_+^n,|\alpha|=s}\frac{s!}{\alpha!}T^{*\alpha}T^\alpha\Big\|=\|M_T^s(I)\|\le\|M_T^s\|$$

and the limit $\lim_{s\to\infty}\|M_T^s\|^{1/s}$ exists and is equal to $r(M_T)=r(T)^2$. We conclude that $\lim_{s\to\infty}\|T^s\|^{1/s}$ exists and is equal to $r(T)$.

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Added in proof. We have been notified by Jerzy Trzeciak that the formula from Theorem 1 was proved in the paper: M. Chō and T. Huruya, *On the joint spectral radius*, Proc. Roy Irish Acad. Sect. A 91 (1991), 39–44, in the case of finite-dimensional Hilbert space operators.