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INSTITUT FÜR MATHEMATIK
UNIVERSITÄT WIEN
STRUDLHOFGASSE 4
A-1090 WIEN, AUSTRIA

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Conjugate martingale transforms

by

FERENC WEISZ (Budapest)

Abstract. Characterizations of H_1, BMO and VMO martingale spaces generated by bounded Vilenkin systems via conjugate martingale transforms are studied.

1. Introduction. A theory of H_p spaces of conjugate harmonic functions on Euclidean spaces was developed by Stein [22]. In particular, $H_1(\mathbb{R}^n)$ can be characterized via the Riesz transforms:

$$(*) \qquad H_1 = \{f \in L_1 : R_j f \in L_1,\ j = 1, \ldots, m\}.$$

Chao and Taibleson (see [6]–[10], [23]) have extended this theory to local fields. Moreover, for martingale spaces, Janson and Chao ([15], [8], [5]) studied transforms with matrix operators acting on the values of the difference sequences of q-martingales.

In this paper conjugate martingale transforms with matrix operators acting on the generalized Rademacher series of the difference sequences are investigated. These transforms were first introduced by Gundy [13]. Contrary to the statement in [13] Gundy only proved $(*)$ in the case when all matrices and martingales are real. This theorem is here extended to the complex case. More exactly, a necessary and sufficient condition for the transforms is given such that $(*)$ holds whenever the martingale H_1 space is generated by a bounded Vilenkin system. Note that this space is slightly more general than the H_1 space of q-martingales. We shall prove a version of F. and M. Riesz theorem. In the simplest case when all matrices are diagonal the transforms used in this paper are called multiplier transforms. Simon's question [20] whether H_1 can be characterized via a single multiplier transform if the multiplier has two values: -1 and 1, is answered. Moreover, a necessary and sufficient condition for $(*)$ to hold for multiplier transforms is also given. A family of integrable functions for which $\|f\|_{L_1} \sim \|f\|_{H_1}$ is obtained. Similarly to [4] we also introduce a transform in the dyadic case.

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In Section 5 a necessary and sufficient condition is given for $\mathrm{BMO} = L_\infty + \sum_{i=1}^m T_i L_\infty$ and $\mathrm{VMO} = C_W + \sum_{i=1}^m T_i C_W$ where C_W denotes the continuous functions on a Vilenkin group. The first result was known for q-martingales but for other type of transforms (see [15]). For $\mathrm{BMO}(\mathbb{T}^d)$ and $\mathrm{VMO}(\mathbb{T}^d)$ both results can be found in [11] and [16].

2. Preliminaries and notations. In this paper $\Omega = [0,1)$, $\mathcal{A}$ is the σ-algebra of Borel sets and P is Lebesgue measure. Let $(p_n, n \in \mathbb{N})$ be a sequence of natural numbers with $2 \le p_n \le N$ for a fixed N. Introduce notations $P_0 = 1$ and

$$P_{n+1} := \prod_{k=0}^{n} p_k \quad (n \in \mathbb{N}).$$

Every point $x \in [0,1)$ can be uniquely written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}}, \quad 0 \le x_k < p_k,\ x_k \in \mathbb{N};$$

if there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The functions

$$r_n(x) := \exp \frac{2\pi i x_n}{p_n}$$

are called *generalized Rademacher functions*. The product system generated by the generalized Rademacher functions is called a *Vilenkin system* (see [26]):

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n = \sum_{k=0}^{\infty} n_k P_k$, $0 \le n_k < p_k$ and $n_k \in \mathbb{N}$.

It is well known that each Vilenkin system is a complete orthonormal system. If $p_n = 2$ for every n then the Vilenkin system is said to be the *Walsh system*.

Let $\mathcal{F}_n$ be the σ-algebra generated by $\{r_0, \ldots, r_{n-1}\}$. It can easily be proved that

$$\mathcal{F}_n = \sigma\{[kP_n^{-1}, (k+1)P_n^{-1}) : 0 \le k < P_n\}$$

where $\sigma(\mathcal{B})$ denotes the σ-algebra generated by $\mathcal{B}$.

The conditional expectation operator with respect to $\mathcal{F}_n$ will be denoted by E_n. We write L_p and $\|\cdot\|_p$ for the space $L_p(\Omega, \mathcal{A}, P)$ and its norm; moreover, for $f \in L_1$ we set $E_0 f = 0$.

The *Vilenkin–Fourier series* of an integrable function f is given by

$$f(x) \sim \sum_{k=0}^{\infty} c_k w_k(x) \quad \text{where} \quad c_k := \widehat{f}(k) := E(f\overline{w}_k).$$

Let f_n be its P_nth partial sum. It is easy to see ([19]) that

$$f_n(x) = \sum_{k=0}^{P_n-1} c_k w_k(x) = P_n \int_{I_n(x)} f = E_n f(x)$$

where $I_n(x)$ denotes the atom of the σ-algebra $\mathcal{F}_n$ for which $x \in I_n(x)$ $(n \in \mathbb{N},\ x \in [0,1))$, that is to say, $(f_n, n \in \mathbb{N})$ is the martingale obtained from f. Moreover, a sequence of integrable and adapted (i.e. f_n is $\mathcal{F}_n$-measurable) functions $f = (f_n, n \in \mathbb{N})$ is a martingale if and only if there exist complex numbers c_k such that

$$f_n = \sum_{k=0}^{P_n-1} c_k w_k.$$

For a martingale f it is always assumed that $f_0 = 0$.

The martingale difference sequence is given by

$$d_{n+1} f := f_{n+1} - f_n = \sum_{k=P_n}^{P_{n+1}-1} c_k w_k, \quad d_0 f := 0.$$

This can be rewritten as

$$d_{n+1} f = \sum_{j=1}^{p_n-1} v_n^{(j)} r_n^j \tag{1}$$

where every $v_n^{(j)}$ is $\mathcal{F}_n$-measurable.

The following notations will be used for a martingale $f = (f_n, n \in \mathbb{N})$:

$$f^* := \sup_n |f_n|, \quad S(f) := \Big(\sum_{n=0}^{\infty} |d_n f|^2\Big)^{1/2},$$

$$s(f) := \Big(\sum_{n=0}^{\infty} E_n |d_{n+1} f|^2\Big)^{1/2}.$$

Since

$$E_n(r_n^j) = 0, \quad E_n(r_n^j \overline{r_n^l}) = \delta(j-l), \quad |r_n^j| = 1, \tag{2}$$

we obtain

$$s(f) = \Big(\sum_{n=0}^{\infty} \sum_{j=1}^{p_n-1} |v_n^{(j)}|^2\Big)^{1/2}.$$

It can easily be shown that the stochastic basis $\mathcal{F} := (\mathcal{F}_n, n \in \mathbb{N})$ is *regular* (see Garsia [12], p. 96), i.e.

$$f_{n+1} \le N f_n \quad (n \in \mathbb{N})$$

for every nonnegative martingale. (Recall that the sequence $(p_n, n \in \mathbb{N})$ is bounded by N.) Note that this definition is equivalent to the one in Gundy [13] (p. 273) (see Neveu [17], p. 72).

Let us introduce *martingale Hardy spaces* for $0 < p \le \infty$. Denote by H_p, H_p^- and $\mathbb{H}_p$ the spaces of martingales for which

$$\|f\|_{H_p} := \|S(f)\|_p < \infty,$$
$$\|f\|_{H_p^-} := \|s(f)\|_p < \infty$$

and

$$\|f\|_{\mathbb{H}_p} := \|f^*\|_p < \infty,$$

respectively. In martingale theory it is well known that if $f \in \mathbb{H}_p$ then f_n converges a.e. and in L^p norm as $n \to \infty$ (for $p \ge 1$; see [17]). Therefore, $\mathbb{H}_p$ can be identified with a certain subspace of L_p ($p \ge 1$). Moreover, a sharper assertion can be shown (see [2], [12], [13], [28]):

THEOREM A. *For every $0 < p < \infty$ one has $\mathbb{H}_p \sim H_p \sim H_p^-$ and $\mathbb{H}_p \sim L_p$ for $p > 1$ where $\sim$ denotes the equality of sets and the equivalence of norms.*

It is proved in [12] and in [28] that the dual of H_1 is BMO and the bounded linear functionals are given by

$$l_\phi(f) = E(f\overline{\phi}) \quad (f \in L_2)$$

where $\phi \in$ BMO is arbitrary and BMO denotes those functions $\phi \in L_2$ for which

$$\|\phi\|_{\mathrm{BMO}} := \sup_n \|(E_n|\phi - E_n\phi|^2)^{1/2}\|_\infty < \infty.$$

3. The transform. The transform introduced by Gundy [13] will be used. Let $A := (A_n, n \in \mathbb{N})$ be a sequence of matrices such that

$$A_n : \mathbb{C}^{p_n-1} \to \mathbb{C}^{p_n-1}.$$

If the differences of a martingale are written in the form (1) then the differences of the martingale transform are defined by

$$d_{n+1}(Tf) := \sum_{j=1}^{p_n-1} (A_n v_n)^{(j)} r_n^j$$

where $v_n := (v_n^{(j)})_{j=1}^{p_n-1}$. Define $(Tf)_n := \sum_{k=1}^n d_k(Tf)$. It is obvious that $((Tf)_n = Tf_n, n \in \mathbb{N})$ is a martingale.

The advantage of this transform is that if the matrices are diagonal then we obtain a multiplier transform. Other martingale transforms with matrix operators are investigated by Janson and Chao ([15], [8]). Our theorems are similar to those in [15] and [8].

We assume that the (euclidean) norms of A_n ($n \in \mathbb{N}$) are bounded.

PROPOSITION 1. *T is a bounded linear operator on* BMO, *on each H_p $(0 < p < \infty)$ and, consequently, on each L_p $(1 < p < \infty)$.*

Proof. Since

$$E_n|d_{n+1}(Tf)|^2 = \sum_{j=1}^{p_n-1} |(A_n v_n)^{(j)}|^2 \le C \sum_{j=1}^{p_n-1} |v_n^{(j)}|^2 = CE_n|d_{n+1}f|^2$$

one has $\|s(Tf)\|_p \le \|s(f)\|_p$. By Theorem A we find that T is bounded on each H_p $(0 < p < \infty)$ and on each L_p $(1 < p < \infty)$.

Now, $f \in$ BMO implies $f \in L_2$ and $Tf \in L_2$. We have

$$E_n|Tf - Tf_n|^2 = E_n\Big(\sum_{k=n}^\infty E_k|d_{k+1}(Tf)|^2\Big)$$
$$\le CE_n\Big(\sum_{k=n}^\infty E_k|d_{k+1}f|^2\Big) = CE_n|f - f_n|^2,$$

thus $\|Tf\|_{\mathrm{BMO}} \le \|f\|_{\mathrm{BMO}}$. ∎

For L_p spaces, this proposition can be found in a slightly more general form in [18].

4. A characterization of H_1. Assume that $A^{(1)}, \ldots, A^{(m)}$ are sequences of matrices described in Section 3 and let $T_1, \ldots, T_m$ be the corresponding martingale transforms. Proposition 1 shows that $f \in H_1$ implies that $T_1 f, \ldots, T_m f$ belong to H_1 and thus to L_1. To prove the converse we use the following very important lemma proved by Chao and Janson. Set

$$V_q := \Big\{x \in \mathbb{C}^q : \sum_{i=1}^q x_i = 0\Big\}.$$

Given a martingale f we can regard $d_{n+1}f$ on an atom of $\mathcal{F}_n$ as an element of V_{p_n}.

LEMMA 1 [8]. *Let W be a closed cone (i.e. $x \in W$ and $t \ge 0$ imply $tx \in W$) consisting of elements of the form $x = (x^{(0)}, \ldots, x^{(m)})$ where $x^{(i)} = (x_1^{(i)}, \ldots, x_q^{(i)}) \in V_q$ such that if $x^{(i)} = \eta_i(\lambda_1, \ldots, \lambda_q)$ for some $\eta_i \in \mathbb{C}$, $i = 0, 1, \ldots, m$ and $\lambda_k \in \mathbb{R}$, $k = 1, \ldots, q$, then $x = (0, \ldots, 0)$. Then there is a positive $p < 1$ such that*

$$\|a\|^p \le \frac{1}{q} \sum_{k=1}^q \|(a^{(i)} + x_k^{(i)})_{i=0}^m\|^p \tag{3}$$

for $a = (a^{(i)})_{i=0}^m \in \mathbb{C}^{m+1}$ and $x = (x^{(i)})_{i=0}^m \in W$, where $\|\cdot\|$ denotes the euclidean norm.

Returning to the martingale H_1 space we obtain

THEOREM 1. *Assume that, for each $n \in \mathbb{N}$, the matrices $A_n^{(1)}, \dots, A_n^{(m)}$ have no common eigenvector $(z_1, \dots, z_{p_n-1})$ with $\overline{z}_j = z_{p_n-j}$ for each $j = 1, \dots, p_n - 1$ and, moreover, (f_n) is a martingale such that $\|f_n\|_1$ and $\|T_i f_n\|_1$ $(i = 1, \dots, m)$ are uniformly bounded. Then (f_n) and $(T_i f_n)$ are martingales belonging to H_1. Furthermore,*

$$c \sum_{i=0}^{m} \|T_i f\|_1 \le \|f\|_{H_1} \le C \sum_{i=0}^{m} \|T_i f\|_1 \tag{4}$$

where $T_0 f_n := f_n$.

Proof. We are going to apply Lemma 1. Denote by $E_{i_0,\dots,i_{n-1}}$ the atoms of $\mathcal{F}_n$ such that

$$E_{i_0,\dots,i_{n-1}} = \bigcup_{i_n=1}^{p_n} E_{i_0,\dots,i_n}.$$

Set

$$a^{(i)} := T_i f_n(E_{i_0,\dots,i_{n-1}}) \quad \text{and} \quad x_k^{(i)} := d_{n+1}(T_i f)(E_{i_0,\dots,i_{n-1},k}).$$

It is easy to check that in this case W is a closed cone. Regard r_n^j as a p_n-dimensional vector. Since, by (2), $(r_n^j)_{j=0}^{p_n-1}$ is an orthogonal basis in $\mathbb{C}^{p_n}$ we deduce that the real, nonzero vector $(\lambda_1, \dots, \lambda_{p_n})$ can be uniquely written in the following way:

$$(\lambda_1, \dots, \lambda_{p_n}) = \sum_{j=1}^{p_n-1} z_j r_n^j. \tag{5}$$

(Recall that $\sum_{i=1}^{p_n} \lambda_i = 0$.) Since $\overline{r_n^j} = r_n^{p_n-j}$ we obtain $\overline{z}_j = z_{p_n-j}$ $(j = 1, \dots, p_n - 1)$. If for every $0 \le i \le m$

$$x^{(i)} = \eta_i(\lambda_1, \dots, \lambda_{p_n}) \quad (\eta_0 \ne 0)$$

then by the definition and by (5) we get

$$x^{(i)} = \eta_0 \sum_{j=1}^{p_n-1} (A_n^{(i)} z)^{(j)} r_n^j = \sum_{j=1}^{p_n-1} (\eta_i z_j) r_n^j$$

and, consequently, $z = (z_j)_{j=1}^{p_n-1}$ is a common eigenvector of $A_n^{(i)}$ $(0 \le i \le m)$, which is a contradiction. Hence $\eta_0 = 0$ and so $x = (0, \dots, 0)$ in Lemma 1. Thus the conditions of Lemma 1 are satisfied, so (3) holds. Now we apply a usual martingale majorant argument (see e.g. [15], [8]). Set

$$g_n := \|(T_i f_n)_{i=0}^m\|.$$

Then (3) shows that $g_n^p \le E_n(g_{n+1}^p)$ for some $p < 1$, thus g_n^p is a positive submartingale. From the second assumption of Theorem 1 we get for every $n \in \mathbb{N}$

$$\|g_n^p\|_{1/p} = \|g_n\|_1^p \le \Big(\sum_{i=0}^{m} \|T_i f_n\|_1 \Big)^p \le C. \tag{6}$$

Using the Doob inequality we conclude that $\sup_n g_n^p \in L_{1/p}$, thus $\sup_n g_n \in L_1$. Since $|T_i f_n| \le g_n$ we get $T_i f \in H_1$. The right hand inequality of (4) follows from

$$\|f\|_{H_1}^p \le \|\sup{}_n g_n\|_1^p = \|\sup{}_n g_n^p\|_{1/p} \le C_p \|g_n^p\|_{1/p}$$

and from (6). The other inequality of (4) comes trivially from Proposition 1. The proof of Theorem 1 is complete. ∎

This theorem can be found in [22] (p. 221) for $H_1(\mathbb{R}^n)$.

A finite measure ν on $(\Omega, \mathcal{A})$ defines a martingale (f_n) by

$$f_n(E_{i_0,\dots,i_{n-1}}) = P_n \nu(E_{i_0,\dots,i_{n-1}}).$$

Conversely, if (f_n) is a martingale then ν is a finite measure. Since $\|f_n\|_1 \le \|\nu\|$ we get the following F. and M. Riesz theorem.

COROLLARY 1. *Assume that $A_n^{(1)}, \dots, A_n^{(m)}$ satisfy the assumptions of Theorem 1. If ν and $T_i \nu$ are bounded measures then ν is absolutely continuous and belongs to H_1.*

In a special case the converse of Theorem 1 can also be proved. That $H_1 \subset \{f \in L_1 : T_i f \in L_1,\ i = 1, \dots, m\}$ follows from Proposition 1. If $A_n^{(1)}, \dots, A_n^{(m)}$ have no common eigenvector with the property as in Theorem 1, then the reverse inclusion is proved by Theorem 1.

THEOREM 2. *Assume that, for each $n \in \mathbb{N}$, the matrices $A_n^{(1)}, \dots, A_n^{(m)}$ have a common eigenvector $(z_{n;1}, \dots, z_{n;p_n-1})$ with $\overline{z}_{n;j} = z_{n;p_n-j}$ $(j = 1, \dots, p_n - 1)$ and $\sigma_n^{(i)} = \sigma^{(i)}$ $(n \in \mathbb{N},\ i = 1, \dots, m)$ for the corresponding eigenvalues. Then $H_1 \ne \{f \in L_1 : T_i f \in L_1,\ i = 1, \dots, m\}$.*

Proof. Since

$$(x_{n;1}, \dots, x_{n;p_n}) := \sum_{j=1}^{p_n-1} z_{n;j} r_n^j$$

is a real vector we can assume that $\min_k x_{n;k} = -1$ for all $n \in \mathbb{N}$.

Modifying slightly the proof of Lemma 6 in [15] we get

LEMMA 2. *If $(x_{n;1}, \dots, x_{n;p_n})$ are real numbers such that $\sum_{k=1}^{p_n} x_{n;k} = 0$ and $\min_k x_{n;k} = -1$ for all $n \in \mathbb{N}$, then there exists a function $f \in L_1$ such that $f \notin H_1$ and*

$$d_{n+1} f(E_{i_0,\dots,i_{n-1},k}) = \lambda_{i_0,\dots,i_{n-1}} x_{n;k}$$

where $\lambda_{i_0,\dots,i_{n-1}}$ are real numbers.

To prove Lemma 2 define

$$g_{n+1}(E_{i_0,\dots,i_n}) := \prod_{k=0}^{n}(1+x_{k;i_k}), \qquad g_0 := 0, \qquad f := \sum_{k=0}^{\infty}\frac{g_k}{k^2}.$$

Similarly to the proof of Lemma 6 in [15] it can be shown that $(g_n, n\in\mathbb{N})$ is a nonnegative martingale and $f - E_0 f \in L_1$ but $f - E_0 f \notin H_1$. ∎

To continue the proof of Theorem 2 take f constructed in Lemma 2. Then obviously $T_i f = \sigma^{(i)}\cdot f \in L_1$ but $f\notin H_1$, which shows the theorem. ∎

From this it follows that if, for each fixed i, every p_n and $A_n^{(i)}$ $(n\in\mathbb{N})$ are equal then the conditions in Theorem 1 are also necessary.

COROLLARY 2. *Suppose that $p_n = d$ and $B^{(i)} = A_n^{(i)}$ for every $n\in\mathbb{N}$ and $i=1,\dots,m$. Then $H_1 = \{f\in L_1 : T_i f\in L_1,\ i=1,\dots,m\}$ if and only if $B^{(1)},\dots,B^{(m)}$ have no common eigenvector $(z_1,\dots,z_{d-1})$ for which $\overline{z}_j = z_{d-j}$ $(j=1,\dots,d-1)$.*

Now we give some examples of such transforms. If the matrices $A^{(i)} = (A_n^{(i)}, n\in\mathbb{N})$ are all diagonal then the martingale transform is called a *multiplier transform*. Denote by $a_{n;k,l}^{(i)}$ the elements of $A_n^{(i)}$. Simon asked in [20] whether in case $a_{n;k,l}=0$ $(k\neq l)$, $a_{n;k} := a_{n;k,k} = -1$ $(1\le k\le [(p_n-1)/2])$, $a_{n;k} := a_{n;k,k} = 1$ $([(p_n-1)/2] < k \le p_n - 1)$ one has $H_1 = \{f\in L_1 : Tf\in L_1\}$ or not. These transforms are used in [20] to prove that the Vilenkin–Fourier series of $f\in L_p$ converges to f in L_p norm $(1<p<\infty)$. The results given below are more general and follow from Theorem 1 and Corollary 2.

COROLLARY 3. *Assume that T_i $(i=1,\dots,m)$ are multiplier transforms and for each $n\in\mathbb{N}$ and each j with $1\le j\le p_n-1$ there is some i such that $a_{n;j}^{(i)} \neq a_{n;p_n-j}^{(i)}$. Then $H_1 = \{f\in L_1 : T_i f\in L_1,\ i=1,\dots,m\}$.*

From this it follows that if, for each fixed i, every p_n is odd then the answer to Simon's question is yes.

COROLLARY 4. *Suppose that $p_n = d$ and $B^{(i)} = A_n^{(i)}$ are diagonal $(n\in\mathbb{N})$. Then $H_1 = \{f\in L_1 : T_i f\in L_1,\ i=1,\dots,m\}$ if and only if for each j with $1\le j\le d-1$ there is some i such that $a_j^{(i)}\neq a_{d-j}^{(i)}$.*

If d is even then for $j=d/2$ we get $a_j^{(i)} = a_{d-j}^{(i)}$ for all $1\le i\le m$. Hence, in this case, H_1 cannot be characterized by any finite number of multiplier transforms. Thus the answer to Simon's question is no.

Assume that every p_n $(n\in\mathbb{N})$ is even. For the eigenvalues σ_n and eigenvectors $z_n = (z_{n;1},\dots,z_{n;p_n-1})$ of Simon's matrices A_n mentioned in Theorem 2 we have $\sigma_n = a_{n;p_n/2} = 1$ and $z_{n;k} = 0$ $(k=1,\dots,p_n-1,$ $k\neq p_n/2)$, $z_{n;p_n/2}\in\mathbb{R}$. It follows from Theorem 2 that in this case $H_1 \neq \{f\in L_1 : Tf\in L_1\}$, so the answer to the question above is no.

If p_n is odd then let A_n be the diagonal matrix given by Simon. Let us modify this matrix for every even p_n. Set $a_{n;1,p_n/2} = -1$, $a_{n;k,k} = -1$ $(2\le k < p_n/2)$, $a_{n;p_n/2,1} = 1$, $a_{n;k,k} = 1$ $(p_n/2 < k \le p_n - 1)$, and else $a_{n;k,l} = 0$. It is easy to check that this matrix has no eigenvector with the property as in Theorem 1, when $p_n > 2$. If T denotes the corresponding transform then H_1 is characterized by a single transform.

COROLLARY 5. *If $p_n > 2$ for all $n\in\mathbb{N}$ and T denotes the last transform then $H_1 = \{f\in L_1 : Tf\in L_1\}$.*

The same corollary also holds for the following modification of Simon's matrix A_n for every even p_n: $a_{n;k,k} = -1$ $(1\le k < p_n/2)$, $a_{n;p_n/2,p_n-1} = 1$, $a_{n;k,k} = 1$ $(p_n/2 < k < p_n - 1)$, $a_{n;p_n-1,p_n/2} = -1$, and else $a_{n;k,l} = 0$.

Note that, for $p_n = 3$, this transform is the same as the so-called Hilbert transform H_3 and for $p_n = 2d+1$ it is different from H_{2d+1} (see [1]).

The next corollary follows easily from this.

COROLLARY 6. *Suppose that $f\in L_1$ and $\widehat{f}(k) = 0$ if $P_n \le k \le P_n + (p_n - 1 - [(p_n-1)/2])P_n - 1$ (or if $P_n + [(p_n-1)/2]P_n \le k \le P_{n+1} - 1$) for some $n\in\mathbb{N}$. Then $\|f\|_{H_1} \le C\|f\|_1$.*

Set $f = f_1 + f_2$ such that $\widehat{f_1}(k) = 0$ if $P_n\le k\le P_n + (p_n - 1 - [(p_n-1)/2])P_n - 1$ and $\widehat{f_2}(k) = 0$ if $P_n + [(p_n-1)/2]P_n \le k \le P_{n+1} - 1$ for some $n\in\mathbb{N}$. Thus, in case each p_n $(n\in\mathbb{N})$ is odd, then $f\in H_1$ if and only if $f_1\in L_1$ and $f_2\in L_1$, namely,

$$c(\|f_1\|_1 + \|f_2\|_1) \le \|f\|_{H_1} \le C(\|f_1\|_1 + \|f_2\|_1). \tag{7}$$

With the help of Corollary 5 a similar result can also be obtained if not every p_n is odd. (7) is analogous to a result due to Gundy and Varopoulos (see Theorem 2 in [14] and Corollary 4 in [8]).

Note that all the results of this paper can also be proved in the same way for the trigonometric model considered in [14].

If $p_n = 2$ for some $n\in\mathbb{N}$ then the results above are not usable. We define the transforms in this case similarly to [4]. If $p_n = 2$ but $p_{n-1}\neq 2$, $p_{n+1}\neq 2$ then we drop p_n and also f_{n+1} and $\mathcal{F}_{n+1}$, and let $p'_{n-1} := 2p_{n-1}$. If $p_{n-1}\neq 2$, $p_n = p_{n+1} = \dots = 2$ then we drop $p_n, p_{n+2},\dots$ and also $f_{n+1}, f_{n+3},\dots$ and $\mathcal{F}_{n+1},\mathcal{F}_{n+3},\dots$, and let $p'_{n+2k-1} = 2p_{n+2k-1}$ $(k=0,1,\dots)$. In any other case $p'_n = p_n$. Hence we get a martingale $(F_k := f_{n_k}, \mathcal{F}_{n_k})$ and a new sequence (p'_{n_k}) such that $p'_{n_k} > 2$. If f_{n+1} is dropped then $f_{n_{k_0}} := f_{n+2}$ is not dropped. Since $|f_{n+1}| \le N\cdot E_n|f_{n+1}| \le N\cdot E_n|f_{n+2}|$ we have

$$F^* \le f^* \le N\sup_k E_{n_{k-1}}|f_{n_k}|.$$

It is easy to prove (see Theorem 1 in [7]) that
$$\|\sup_k E_{n_{k-1}}|f_{n_k}|\|_1 \le N^2\|F^*\|_1 .$$
This yields that $f \in H_1$ if and only if $F \in H_1(\mathcal{F}_{n_k})$. Now let us transform F. From T_iF we define T_if by
$$T_if_l := E_l(T_iF_{n_k})$$
for $l \le n_k$. From Theorems 1 and 2 we obtain

THEOREM 3. *Assume that $p_n = 2$ for some $n \in \mathbb{N}$, $(F_k, \mathcal{F}_{n_k})$ is the martingale corresponding to (f_n) and the transforms $T_1, \ldots, T_m$ of (F_k) have the same property as in Theorem 1. Then $f \in H_1$ (or, equivalently, $F \in H_1(\mathcal{F}_{n_k})$) if $\|T_if_n\|_1$ (or, equivalently, if $\|T_iF_n\|_1$) are uniformly bounded $(i = 0, 1, \ldots, m)$. If every p_n is equal to 2 and $B^{(i)} = A_n^{(i)}$ then the above condition is also necessary.*

Note that Gundy [13] proved some results similar to Theorems 1 and 2. However, those results only hold when in (2) every $r_n^{(j)}$ is real. He proved Theorems 1 and 2 if $B^{(i)} = A_n^{(i)}$ have no common real eigenvector ($p_n = d$). He claims on p. 289 that for complex $r_n^{(j)}$, H_1 would be characterized by
$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
($p_n = 5$). This matrix has no real eigenvector, though it has an eigenvector with the property as in Corollary 2 (e.g. $[1, -i, i, 1]$). Consequently, by Corollary 2, H_1 cannot be characterized by this matrix.

5. A characterization of BMO **and** VMO. Denote by A_n^* the adjoint matrix of A_n and by T^* the corresponding martingale transform. It is easy to see that $E[(Tf)\overline{g}] = E(f\overline{T^*g})$. From the duality between H_1 and BMO we obtain

THEOREM 4. *Assume that, for each $n \in \mathbb{N}$, the matrices $A_n^{(1)*}, \ldots, A_n^{(m)*}$ have no common eigenvector $(z_1, \ldots, z_{p_n-1})$ with $\overline{z}_j = z_{p_n-j}$ $(j = 1, \ldots, p_n - 1)$. Then* $\mathrm{BMO} = L_\infty + \sum_{i=1}^m T_iL_\infty$ *and*
$$c\|\phi\|_{\mathrm{BMO}} \le \inf \sup_{0\le i\le m} \|g_i\|_\infty \le C\|\phi\|_{\mathrm{BMO}} \tag{8}$$
where $\phi = \sum_{i=0}^m T_ig_i$. If $p_n = d$ and $B^{(i)} = A_n^{(i)}$ $(n \in \mathbb{N},\ i = 0, 1, \ldots, m)$ then the above condition is also necessary.

Proof. The proof is similar to that of Corollary 2 in [15], so we only sketch it. It can easily be proved by the Hahn–Banach theorem that the dual of the space $\{f \in L_1 : T_i^*f \in L_1,\ i = 1, \ldots, m\}$ with norm
$$\|(f, T_1^*f, \ldots, T_m^*f)\| := \sum_{i=0}^m \|T_i^*f\|_1$$
is the space $\sum_{i=0}^m T_iL_\infty$ with norm
$$\|g\| := \inf \sup_{0\le i\le m} \|g_i\|_\infty$$
where $g = \sum_{i=0}^m T_ig_i$. The continuous linear functionals are given by
$$\sum_{i=0}^m E[(T_i^*f)\overline{g}_i] = \sum_{i=0}^m E(f\overline{T_ig_i}) . \tag{9}$$
Theorem 1 now proves the first part of the assertion.

To prove the necessity suppose that the condition of Theorem 4 does not hold, though $\mathrm{BMO} = \sum_{i=0}^m T_iL_\infty$ and (8) is true. Denote by L the vector space of Vilenkin step functions with zero mean, more exactly, the vector space
$$\{f : f \text{ is } \mathcal{F}_n\text{-measurable for any } n \in \mathbb{N} \text{ and } Ef = 0\} .$$
Then L is dense in $\{f \in L_1 : T_i^*f \in L_1,\ i = 1, \ldots, m\}$ because $T_i^*f_n \to T_i^*f$ in L_1 norm as $n \to \infty$ (see [17]) and $T_i^*f_n \in L$ for every $i = 0, 1, \ldots, m$ and $n \in \mathbb{N}$. Since
$$\sum_{i=0}^m \|T_i^*f\|_1 \le C\|f\|_{H_1}$$
(see (4)) we conclude from the next lemma that $H_1 = \{f \in L_1 : T_i^*f \in L_1,\ i = 1, \ldots, m\}$, which contradicts Corollary 2. ∎

This theorem was first proved by Fefferman and Stein [11] for $\mathrm{BMO}(\mathbb{R}^n)$. The following result is also interesting in its own right.

LEMMA 3. *Suppose that $M \subset Y \subset X$, X is a normed space, Y is a Banach space and their dual spaces are equivalent: $X^* \sim Y^*$, and, moreover, M is dense in X and also in Y. If*
$$c\|x\|_X \le \|x\|_Y \tag{10}$$
then $X \sim Y$.

Proof. For every $x \in M$ by the Hahn–Banach theorem there exists $z \in Y^*$ such that $\|z\|_{Y^*} = 1$ and $z(x) = \|x\|_Y$. Thus
$$\|x\|_Y = |z(x)| \le \|z\|_{X^*}\|x\|_X \le C\|x\|_X \tag{11}$$
for all $x \in M$. M is dense in X, so for every $x \in X$ there exists a sequence $x_n \in M$ $(n \in \mathbb{N})$ which converges to x in X. By (11), (x_n) is also a Cauchy sequence in Y. From (10) it follows that the limit of (x_n) in Y is x as well. Taking the limit in (11) completes the proof. ∎

Let VMO be the closure of L in BMO norm. It is proved in [28] that the dual of VMO is H_1 and the bounded linear functionals are given by

$$l_f(\phi) = E(\phi \bar{f}) \quad (\phi \in L)$$

where $f \in H_1$.

Let C_W represent the collection of functions $g : [0,1) \to \mathbb{C}$ which are continuous at every Vilenkin irrational point (i.e. at every point that cannot be written in the form k/P_n), continuous from the right on $[0,1)$, and have a finite limit from the left on $(0,1]$, all this in the usual topology. There is an isomorphism between $[0,1)$ and a Vilenkin group G (see e.g. [19]). G is compact and C_W is isomorphic to the space of continuous functions on G. It is well known that the dual of C_W is the space M of all bounded measures on $\mathcal{A}$.

Now we characterize the VMO space.

THEOREM 5. *Assume that, for each $n \in \mathbb{N}$, the matrices $A_n^{(1)*}, \ldots, A_n^{(m)*}$ have no common eigenvector $(z_1, \ldots, z_{p_n-1})$ with $\overline{z}_j = z_{p_n-j}$ $(j = 1, \ldots, p_n - 1)$. Then $\mathrm{VMO} = C_W + \sum_{i=1}^m T_i C_W$ and*

$$c\|\phi\|_{\mathrm{VMO}} \le \inf \sup_{0\le i\le m} \|g_i\|_\infty \le C\|\phi\|_{\mathrm{VMO}} \tag{12}$$

where $g_i \in C_W$ and $\phi = \sum_{i=0}^m T_i g_i$. If $p_n = d$ and $B^{(i)} = A_n^{(i)}$ $(n \in \mathbb{N}, i = 1, \ldots, m)$ then the above condition is also necessary.

Proof. First we show that the dual of the space $\sum_{i=0}^m T_i C_W$ with norm

$$\|g\| := \inf \sup_{0\le i\le m} \|g_i\|_\infty \quad \Big(g_i \in C_W,\ g = \sum_{i=0}^m T_i g_i\Big)$$

is

$$\Big\{(f_n) \text{ is a martingale} : \sup_{n\in\mathbb{N}} \sum_{i=0}^m \|T_i^* f_n\|_1 < \infty\Big\}.$$

From the Stone–Weierstrass theorem the Vilenkin polynomials are dense in C_W, thus L is dense in C_W, and hence also in $\sum_{i=0}^m T_i C_W$. If a linear functional l has a form similar to (9) for $g_i \in L$ then l is continuous and

$$\|l\| \le \sup_{n\in\mathbb{N}} \sum_{i=0}^m \|T_i^* f_n\|_1 .$$

Conversely, if l is a continuous linear functional on $\sum_{i=0}^m T_i C_W$ then it is bounded on C_W as well. Thus there exists $\nu \in M$ such that

$$l(g) = \int_\Omega g \, d\overline{\nu} \quad (g \in L)$$

and $\|\nu\| \le \|l\|$. So

$$l(T_i g_i) = \int_\Omega T_i g_i \, d\overline{\nu} = \int_\Omega g_i \, d\overline{T_i^* \nu} \quad (g_i \in L).$$

Consequently, $\|T_i^* \nu\| \le \|l\|$. If (f_n) is the martingale defined by ν then the proof of our statement is complete.

If the $B^{(i)*}$ do have a common eigenvector as in Theorem 5 then by Corollary 2

$$H_1 \ne \Big\{(f_n) \text{ is a martingale} : \sup_{n\in\mathbb{N}} \sum_{i=0}^m \|T_i^* f_n\|_1 < \infty\Big\},$$

hence VMO is not equivalent to $\sum_{i=0}^m T_i C_W$. Assume that the condition of Theorem 5 is satisfied. Then by Theorem 1 the dual of VMO and the dual of $\sum_{i=0}^m T_i C_W$ are equivalent. C_W is a Banach space, thus the same holds for $\sum_{i=0}^m T_i C_W$. Since $T_i : C_W \to \mathrm{VMO}$ are bounded we obtain

$$\|g\|_{\mathrm{VMO}} \le \sum_{i=0}^m \|T_i g_i\|_{\mathrm{VMO}} \le \sum_{i=0}^m \|g_i\|_\infty .$$

Finally, from Lemma 3 we get (12), which completes the proof. ∎

A similar result is proved in [16] for $\mathrm{VMO}(\mathbb{T}^d)$. Note that if $B^{(i)*}$ have a common eigenvector as in Theorem 5 then $\sum_{i=0}^m T_i C_W$ cannot be closed in BMO.

It is an open question whether these results can be extended to unbounded Vilenkin systems.

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DEPARTMENT OF NUMERICAL ANALYSIS
EÖTVÖS L. UNIVERSITY
BOGDÁNFY U. 10/B
H-1117 BUDAPEST, HUNGARY

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Erratum to the paper "On the reflexivity of pairs of isometries and of tensor products of some reflexive algebras"

(Studia Math. 83 (1986), 47–55)

by

MAREK PTAK (Kraków)

Abstract. A gap in the proof of [4, Theorem 1] is removed.

1. Introduction. Our purpose is to remove a gap in the proof of [4, Theorem 1]. All the notations are taken from [4]. Let us recall this theorem:

THEOREM 1. *Every pair $\{V_1, V_2\}$ of doubly commuting (V_1, V_2 commute and V_1, V_2^* commute) isometries on a Hilbert space H is reflexive.*

The main idea of the proof was to use the Wold-type decomposition (it exists for the above pair by [6, Theorem 3]): there are subspaces H_{uu}, H_{us}, H_{su}, H_{ss} such that

(1) $H = H_{uu} \oplus H_{us} \oplus H_{su} \oplus H_{ss}$, where all summands reduce V_1 and V_2,
(2) $V_1|_{H_{uu}}$ and $V_2|_{H_{uu}}$ are unitary operators,
(3) $V_1|_{H_{us}}$ is a unitary operator, $V_2|_{H_{us}}$ is a shift,
(4) $V_1|_{H_{su}}$ is a shift, $V_2|_{H_{su}}$ is a unitary operator,
(5) $V_1|_{H_{ss}}$, $V_2|_{H_{ss}}$ are shifts.

After proving the reflexivity of each component, the last step was to sum them up. This requires some extra property for each component besides the reflexivity. Property C was used (for definition see [4]). The gap was in the proof of this property for cases (3), (4), (5). For cases (3), (4), the idea of the proof is correct but the details are not straightforward. These are given in Section 2. In fact, to sum up reflexive components we need (see [2, Theorem 3.8]) to prove a weaker (see [2, Proposition 2.5, (2)]) property than C, namely property D(1) (introduced in [2] and now known as property

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