$M'_{(A\omega)}((0,t])$. From (21) we get

$$\mathcal{M}(u\circ S)(\zeta)=u\circ S[y^{-\zeta-1}]=u\left[(S^{-1}(x))^{-\zeta-1}\frac{1}{|\det A|}(S^{-1}(x))^{1}x^{-1}\right]$$
$$=\frac{1}{|\det A|}u[x^{-A^{-1}\zeta-1}]=\frac{1}{|\det A|}(\mathcal{M}u)\circ A^{-1}\zeta\,,$$

which ends the proof.

Remark 3. After the change of variables $\mathbb{R}^n\ni y\mapsto e^y\in\mathbb{R}^n_+$, Theorem 3′ (and hence Theorem 3) extends Theorem 1 to the case of the non-compact set $A^{\mathrm{tr}}(\ln(0,t])$.

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Weighted inequalities for square and maximal functions in the plane

by

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Abstract. We prove weighted inequalities for square functions of Littlewood–Paley type defined from a decomposition of the plane into sectors of lacunary aperture and for the maximal function over a lacunary set of directions. Some applications to multiplier theorems are also given.

1. Introduction. Square functions are often used in Harmonic Analysis because their action on a function gives a new one with equivalent L^p-norm. They can be viewed in some sense as a substitute of Plancherel's theorem in L^p, $p\neq 2$.

In this paper we consider two such square functions associated with a decomposition of $\mathbb{R}^2$ into angles of lacunary aperture. Let us take the lines through the origin with slope $\pm 2^j$, $j\in\mathbb{Z}$, and consider the angular sectors they determine. More precisely, we set

$$\Delta_j=\{(x_1,x_2)\in\mathbb{R}^2\mid 2^{-j}\le|x_2/x_1|<2^{-j+1}\}$$

and define the multiplier operator S_j as $(S_jf)^\wedge=\chi_{\Delta_j}\widehat{f}$ (we denote by χ_A the characteristic function of A). Our first square function will be

$$g(f)=\Big(\sum_{j=-\infty}^{\infty}|S_jf|^2\Big)^{1/2}.$$

We shall also consider a smooth decomposition defined as follows: let φ_j be a homogeneous function of degree zero, supported on $\Delta_j\cup\Delta_{j+1}$ and such that the restriction to the unit circle S^1 (denoted again by φ_j) is C^∞ and satisfies

$$|D^\alpha\varphi_j(\theta)|\le C2^{-|j|\alpha}\quad(C\text{ independent of }j)\,.$$

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Denoting by $\widetilde{S}_j$ the corresponding multiplier operator, $(\widetilde{S}_j f)^\wedge = \varphi_j \widehat{f}$, the associated square function will be

$$\widetilde{g}(f) = \Big(\sum_{j=-\infty}^{\infty} |\widetilde{S}_j f|^2\Big)^{1/2}.$$

Closely related to these square functions is the maximal operator

$$M^* f(x) = \sup_j M_{\theta_j} f(x)$$

where θ_j is the intersection of the line with slope 2^j and S^1, and given $\theta \in S^1$, the directional Hardy–Littlewood maximal function M_θ is defined as

$$M_\theta f(x) = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x - t\theta)|\, dt.$$

Nagel, Stein and Wainger proved the boundedness in L^p of M^* for $1 < p \le \infty$ ([NSW]) and it is easy to deduce that g is also bounded in L^p for $1 < p < \infty$. There is a previous partial result in [CF]. The boundedness of $\widetilde{g}$ can be deduced from the Marcinkiewicz multiplier theorem.

We are interested in weighted norm inequalities for g, $\widetilde{g}$ and M^*, i.e., in their boundedness in $L^p(w)$ (the L^p space with respect to the measure $w(x)\,dx$, whose norm we denote by $\|\cdot\|_{p,w}$) for w locally integrable and nonnegative.

As usual, A_p (resp. A_p^*) denotes the class of weights for which the Hardy–Littlewood maximal function M (resp. the strong maximal function M_S) is bounded in $L^p(w)$, $1 < p < \infty$, characterized by

$$\sup_Q \Big(|Q|^{-1} \int_Q w\Big)\Big(|Q|^{-1} \int_Q w^{-1/(p-1)}\Big)^{p-1} \le C$$

where Q ranges over all squares in $\mathbb{R}^2$ (resp. all rectangles with sides parallel to the coordinate axes). We refer to [GR] for details concerning weights.

In Section 2 we prove the boundedness of $\widetilde{g}$ in $L^p(w)$, $\forall w \in A_p^*$. In Section 3 we prove the boundedness of M^* for the weights which satisfy uniform inequalities with respect to each one of the M_{θ_j}. The extrapolation theorem provides then the weighted inequalities for g in a very simple way. We end the paper with some results for multipliers deduced from the square functions.

The use of dyadic decompositions is just for simplicity. The results are still true with the obvious modifications if the decomposition of each quadrant is defined through an arbitrary lacunary sequence. Trivial modifications are similarly needed when the "limit directions" of the decomposition are not the coordinate axes.

We are grateful to José L. Torrea for an interesting suggestion concerning Section 4.

2. The smooth decomposition. We start with a vector-valued inequality for the operators $\widetilde{S}_j$ defined above.

LEMMA 1. *The vector-valued inequality*

$$\Big\|\Big(\sum_{j=-\infty}^{\infty} |\widetilde{S}_j f_j|^2\Big)^{1/2}\Big\|_{p,w} \le C\Big\|\Big(\sum_{j=-\infty}^{\infty} |f_j|^2\Big)^{1/2}\Big\|_{p,w}$$

holds whenever $w \in A_p^*$.

Proof. Define $\psi_j(\xi_1,\xi_2) = \varphi_j(2^{j/2}\xi_1, 2^{-j/2}\xi_2)$. Then ψ_j is homogeneous of degree zero, C^∞ on S^1 and supported in $1/2 \le |\xi_2|/|\xi_1| \le 2$, so that the operator T_j with multiplier ψ_j satisfies (see [GR], p. 411)

$$\int |T_j g|^2 w \le C \int |g|^2 w \quad \forall w \in A_2$$

with C depending on w but not on j. Since φ_j is obtained from ψ_j by dilating each variable in a different way, we get

$$(*) \qquad \int |\widetilde{S}_j g|^2 w \le C \int |g|^2 w \quad \forall w \in A_2^*$$

and the lemma holds for $p = 2$. For other values of p the result follows by the extrapolation theorem of Rubio de Francia (see [GR], p. 461 or [R]). ■

THEOREM 2. *Let $\widetilde{g}$ be the square function defined above. Then*

$$\|\widetilde{g}(f)\|_{p,w} \le C\|f\|_{p,w} \quad \forall w \in A_p^*.$$

Proof. Decompose the plane into dyadic rectangles (i.e., cartesian products of the usual one-dimensional dyadic intervals). Denote by E_j the union of the rectangles which have nonempty intersection with the support of φ_j, so that $\chi_{E_j}\varphi_j = \varphi_j$, and let $\widehat{f}_j = \widehat{f}\chi_{E_j}$. Then $\widetilde{S}_j f = \widetilde{S}_j f_j$ and by Lemma 1, it suffices to show that

$$\Big\|\Big(\sum_{j=-\infty}^{\infty} |f_j|^2\Big)^{1/2}\Big\|_{p,w} \le C\|f\|_{p,w}.$$

If $\{\varepsilon_j\}$ is an arbitrary sequence of signs ± 1, $\sum_{j=-\infty}^{\infty} \varepsilon_j \chi_{E_j}$ is constant on each dyadic rectangle and its absolute value is bounded by 3. From the weighted Littlewood–Paley inequalities in the plane (see [K]) we have

$$\Big\|\sum_{j=-\infty}^{\infty} \varepsilon_j f_j\Big\|_{p,w} \le C\|f\|_{p,w}$$

and the theorem follows as usual by using Rademacher functions (see [S]). ■

Observation. If we consider only positive values of j (i.e., only sectors approaching the OX_1 axis), inequality $(*)$ holds whenever w is a weight for the maximal function over rectangles with smaller basis than height (the dilations in the multiplier have a factor $2^{-j/2}$ in the first variable, $2^{j/2}$ in the second variable). Those weights can be characterized as the usual A_p or A_p^* weights by modifying the corresponding family of rectangles.

3. Maximal function along dyadic directions. We prove first a pointwise inequality for a fixed directional maximal function. Given $\theta \in S^1$ and $0 < \delta < 1$ we define a function $\omega_{\theta,\delta}$, homogeneous of degree zero, smooth on S^1, identically equal to 1 in a sector of width δ and vanishing outside a sector of width 2δ, both centered in the direction orthogonal to θ. Let $(R_{\theta,\delta}f)^\wedge = \omega_{\theta,\delta}\widehat{f}$.

LEMMA 3. *Let $\theta \in S^1$ and $0 < \delta < 1$. With the above notation we have*

$$M_\theta f(x) \le C\mathcal{M}_{\theta,\delta}f(x) + MM_\theta(R_{\theta,\delta}f)(x)$$

where $\mathcal{M}_{\theta,\delta}$ is the maximal function over rectangles with sides h (parallel to θ) and δh (orthogonal to θ).

Proof. Without loss of generality we can assume $\theta = 0$ and write $\mathcal{M}_\delta$, ω_δ and R_δ. Let $\psi \in C^\infty(\mathbb{R})$ with compact support, nonnegative and such that $\int \psi = 1$ and $\psi(0) \ne 0$. For $f \ge 0$ we have

$$M_0 f(x) \le C \sup_{h>0} \int_{\mathbb{R}} \frac{1}{h}\psi\left(\frac{t}{h}\right) f(x_1 - t, x_2)\,dt = C \sup_{h>0} N_h f(x)$$

where $(N_h f)^\wedge(\xi) = \widehat{\psi}(h\xi_1)\widehat{f}(\xi)$.

Take an auxiliary function $\widehat{\Phi} \in C^\infty(\mathbb{R}^2)$, compactly supported and equal to 1 in $|\xi| \le 1$, and decompose $\widehat{\psi}$ as

$$\begin{aligned}\widehat{\psi}(\xi_1) = {} & \widehat{\psi}(\xi_1)\widehat{\Phi}(\delta\xi) + \widehat{\psi}(\xi_1)(1-\widehat{\Phi}(\delta\xi))(1-\omega_\delta(\xi)) \\ & + \widehat{\psi}(\xi_1)(1-\widehat{\Phi}(\delta\xi))\omega_\delta(\xi).\end{aligned}$$

Differentiating with respect to ξ_1 and ξ_2 we see that the first two terms in this sum have Fourier transforms K_1, K_2 satisfying

$$|x_1|^\alpha |x_2|^\beta |K_i(x)| \le C\delta^{-1+\beta}, \quad i = 1, 2,$$

so that they give rise to operators bounded by $\mathcal{M}_\delta$ and the same is true when we introduce the dilation factor h. The third term is clearly controlled by MM_0R_δ. ■

Observe that if $|\theta - j\pi/2| \le C\delta$ for some $j = 0, 1, 2, 3$, the rectangles in the definition of $\mathcal{M}_\delta$ can be included in rectangles of comparable area and sides parallel to the coordinate axes. Then if we consider the M_{θ_j} defined in the introduction and take $\delta \sim 2^{-|j|}$ we have

$$M_{\theta_j} f(x) \le CM_S f(x) + MM_{\theta_j}(\widetilde{S}_{-j}f)(x)$$

where we use the notation $\widetilde{S}_j$ because the operators become similar to those of Section 2.

THEOREM 4. *Let $W_p = \{w \mid M_{\theta_j}$ is bounded in $L^p(w)$ with constant uniform in $j\}$. Then*

$$\|M^*f\|_{p,w} \le C\|f\|_{p,w} \quad \textit{if and only if} \quad w \in W_p.$$

Proof. The necessity is immediate. Let us prove that the condition is also sufficient.

Notice first that also the "limit" maximal functions M_0, $M_{\pi/2}$ are bounded in $L^p(w)$, $\forall w \in W_p$, so that $W_p \subset A_p^*$ since $M_S f(x) \le M_0 M_{\pi/2} f(x)$. From the inequality above we have

$$M^*f(x) \le CM_S f(x) + \sup_j MM_{\theta_j}(\widetilde{S}_{-j}f)(x).$$

Since M and M_S are bounded for $w \in A_p^*$ it is enough to prove

$$\|\sup_j M_{\theta_j}(\widetilde{S}_{-j}f)\|_{p,w} \le C\|f\|_{p,w}, \qquad w \in W_p.$$

Let $p \ge 2$. Then

$$\begin{aligned}\|\sup_j M_{\theta_j}(\widetilde{S}_{-j}f)\|_{p,w}^p &\le \sum_j \int_{\mathbb{R}^2} |M_{\theta_j}(\widetilde{S}_{-j}f)|^p w \\ &\le C\int_{\mathbb{R}^2} \sum_j |\widetilde{S}_j f|^p w \le C\int_{\mathbb{R}^2} \Big(\sum_j |\widetilde{S}_j f|^2\Big)^{p/2} w \le C\|f\|_{p,w}^p\end{aligned}$$

from Theorem 2.

For $p \le 2$ we use a method due to M. Christ (see [Ca]). For each N we have

$$\|\sup_{-N\le j\le N} |M_{\theta_j} f|\|_{p,w} \le B(N)\|f\|_{p,w}$$

where $B(N)$ denotes the norm of the operator (i.e., the smallest constant on the right-hand side). Then

$$\|\sup_{-N\le j\le N} |M_{\theta_j} g_j|\|_{p,w} \le B(N)\|\sup_{-N\le j\le N} |g_j|\|_{p,w}$$

and also

$$\begin{aligned}\|\sup_{-N\le j\le N} |M_{\theta_j} g_j|\|_{p,w} &\le \Big\|\Big(\sum_{j=-N}^{N} |M_{\theta_j} g_j|^p\Big)^{1/p}\Big\|_{p,w} \\ &\le C\Big\|\Big(\sum_{j=-N}^{N} |g_j|^p\Big)^{1/p}\Big\|_{p,w}.\end{aligned}$$

Interpolating we obtain

$$\Big\| \sup_{-N\le j\le N} |M_{\theta_j} g_j| \Big\|_{p,w} \le CB(N)^{1-p/2} \Big\| \Big(\sum_{j=-N}^{N} |g_j|^2 \Big)^{1/2} \Big\|_{p,w}.$$

With $g_j = \widetilde{S}_{-j} f$ we get

$$\Big\| \sup_{-N\le j\le N} |M_{\theta_j} f| \Big\|_{p,w} \le CB(N)^{1-p/2} \|f\|_{p,w}$$

so that $B(N) \le CB(N)^{1-p/2}$ and $B(N)$ is bounded uniformly in N. ∎

If we consider only a set of directions approaching the OX_1 axis for example, the theorem remains true with some modifications in the proof. The inclusion $W_p \subset A_p^*$ is no longer true but we can use instead the weights described in the observation following Theorem 2 because the maximal function over rectangles with longer basis than height is controlled by $M_0 M_{\pi/4}$ for example.

4. The decomposition with characteristic functions. Let P_θ denote the half-plane $P_\theta = \{\xi \in \mathbb{R}^2 \mid \langle \xi, \theta \rangle \ge 0\}$. The multiplier operator defined by χ_{P_θ} is essentially a Hilbert transform so that it is bounded in $L^p(w)$ for the same weights as M_θ.

THEOREM 5. *Let W_p be as in Theorem 4. Then*

$$\|g(f)\|_{p,w} \le C\|f\|_{p,w} \quad \forall w \in W_p.$$

Proof. If P_j denotes the half-plane defined by $\theta_j + \pi/2$, then the characteristic function of the (double) sector determined by the lines through θ_j and θ_{j+1} coincides with $(\chi_{P_j} - \chi_{P_{j+1}})^2$, so that

$$\int |S_j f|^2 w \le C \int |f|^2 w \quad \forall w \in W_2.$$

Let $\{\widetilde{S}_j\}$ be a smooth decomposition with $\varphi_j(\xi) = 1$ for $\xi \in \Delta_j$, i.e., $S_j \widetilde{S}_j = S_j$. Applying Theorem 2 we have for $w \in W_2$

$$\int \sum_{j=-\infty}^{\infty} |S_j f|^2 w \le C \int \sum_{j=-\infty}^{\infty} |\widetilde{S}_j f|^2 w \le C\|f\|_{2,w}^2.$$

The result for other values of p can be obtained by extrapolation (see [R], p. 539). ∎

Following [R] we can also give a factorization theorem for the weights in W_p: let $W_1 = \{w \mid M^* w(x) \le Cw(x) \text{ a.e.}\}$; then $w \in W_p$ if and only if there exist $w_0, w_1 \in W_1$ such that $w = w_0 w_1^{1-p}$.

A radial function $w(x) = w_0(|x|)$ where either $w_0^2 \in A_1(\mathbb{R}^+)$ or w_0 is decreasing and $w_0 \in A_1(\mathbb{R}^+)$ is a uniform weight for all M_θ (see [D]) so that it is in W_1. For example, $|x|^\alpha \in W_1$ if $-1 < \alpha \le 0$. By using the above factorization result we can construct weights in W_p, in particular, $|x|^\alpha \in W_p$ if $-1 < \alpha < p - 1$.

5. Some multiplier theorems. The next theorem gives weighted inequalities for a homogeneous multiplier operator assuming a Hörmander type condition on the unit circle. In the proof we use the following fact (which can be deduced from [S], p. 73, Theorem 5). Let m be a bounded homogeneous function of degree zero with $m' \in L^q(S^1)$, $1 < q < \infty$. Then there exist $a \in \mathbb{C}$ and $\Omega \in L^q(S^1)$ such that $(Tf)^\wedge = m\widehat{f}$ is equivalent to

$$Tf(x) = af(x) + \text{ p.v.} \int_{\mathbb{R}^2} \frac{\Omega(y')}{|y|^2} f(x-y)\, dy$$

$(y' = y|y|^{-1})$.

THEOREM 6. *Let m be a bounded function with compact support in $[0, \pi/4]$ such that for some $q \ge 2$*

$$\sup_{0 \le t \le \pi/4} \int_t^{2t} |\theta m'(\theta)|^q \frac{d\theta}{\theta} < \infty.$$

Then the homogeneous extension of m defines a bounded operator in $L^p(w)$, $\forall w \in A_{p/q}^*$, $p \ge q'$.

Proof. Take a function ϕ, C^∞ in $\mathbb{R}^+$, $\phi(t) = 1$ if $0 < t \le \pi/4$, $\phi(t) = 0$ if $t \ge \pi/2$, and define $\psi(t) = \phi(t) - \phi(2t)$. Then $\sum_{j=0}^\infty \psi(2^j t) = \phi(t)$.

Decompose m as $\sum_{j=0}^\infty m_j$ where $m_j = m\psi(2^j \cdot)$ and let T_j be the multiplier operator associated with m_j. The derivative of the multiplier $m(2^{-j}\cdot)\psi$ is uniformly in L^q according to the hypothesis so that it is of the type described above. Following [D] or [Wa], it defines a bounded operator in $L^p(w)$, $\forall w \in A_{p/q'}$, $p \ge q'$. The same dilation argument used in Theorem 2 implies that m_j is bounded in $L^p(w)$, $\forall w \in A_{p/q'}^*$, $p \ge q'$. Take now a smooth decomposition as in Section 2 with $\varphi_j \equiv 1$ on $\operatorname{supp} \psi(2^j \cdot)$. Then $Tf = \sum_{j=0}^\infty T_j \widetilde{S}_j^2 f$ and

$$\begin{aligned} \|Tf\|_{2,w} &\le C \Big\| \Big(\sum_{j=0}^\infty |T_j \widetilde{S}_j f|^2 \Big)^{1/2} \Big\|_{2,w} \\ &\le C \Big\| \Big(\sum_{j=0}^\infty |\widetilde{S}_j f|^2 \Big)^{1/2} \Big\|_{2,w} \le C\|f\|_{2,w} \quad \forall w \in A_{2/q'}^*. \end{aligned}$$

For other values of p the theorem follows by extrapolation. ∎

COROLLARY 7. *If m is bounded and continuous outside the origin and $|m'(\theta)| \le C|\theta|^{-1}$, then its homogeneous extension defines a bounded operator in $L^p(w)$, $\forall w \in A_p^*$, $1 < p < \infty$.*

Under these conditions m satisfies Theorem 6 for all $q < \infty$ so that we get the weights in $\bigcup_{s<p} A_s^*$, but this coincides with A_p^* (see [GR]).

We can also give weighted inequalities for multipliers corresponding to characteristic functions of polygons with infinitely many sides like the one considered in [CF]. Take the points $A_j = (2^{-j}, j)$, $j \in \mathbb{Z}$, in the plane and consider the polygonal line with sides $A_j A_{j-1}$. Let P be the region of the plane above this polygonal line.

THEOREM 8. *Let W_p be as in Theorem 4. Then χ_P defines a bounded operator in $L^p(w)$, $\forall w \in W_p$, $1 < p < \infty$.*

Proof. Let H_j be the operator whose multiplier is the characteristic function of the half-plane determined by the line through A_j and A_{j-1} and R_j given by

$$(R_j f)^\wedge(\xi) = \chi_{[2^{-j}, 2^{-j+1}]}(\xi_1)\widehat{f}(\xi)\,.$$

Then the operator T associated with χ_P is given by $Tf = \sum_{j\in\mathbb{Z}} R_j^2 H_j f$. Again the result for $p = 2$ is easy because the H_j are uniformly bounded for $w \in W_p$, and for $p \ne 2$ we extrapolate. ∎

In [St] compact polygons with infinitely many sides are considered. Take a lacunary sequence approaching 0 and consider the polygon inscribed in the unit circle it determines. Proceeding as in the preceding theorem we can obtain weighted inequalities. The details are left to the reader.

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