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A strong mixing condition for second-order stationary random fields

by

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Abstract. Let $\{X_{mn}\}$ be a second-order stationary random field on $\mathbb{Z}^2$. Let $\mathcal{M}(L)$ be the linear span of $\{X_{mn} : m \le 0,\ n \in \mathbb{Z}\}$, and $\mathcal{M}(R_N)$ the linear span of $\{X_{mn} : m \ge N,\ n \in \mathbb{Z}\}$. Spectral criteria are given for the condition $\lim_{N\to\infty} c_N = 0$, where c_N is the cosine of the angle between $\mathcal{M}(L)$ and $\mathcal{M}(R_N)$.

1. Introduction.

Suppose that $\{X_n\}_{n=-\infty}^{\infty}$ is a stationary process on the probability space $(\Omega, \mathcal{B}, \nu)$. A classical (linear) prediction problem is to estimate X_n, $n \ge 1$, based on the past of the process; that is, to find X in the linear span $\mathcal{P}$ of $\{\ldots, X_{-2}, X_{-1}, X_0\}$ for which the mean error $(\int |X - X_n|^2\, d\nu)^{1/2}$ is a minimum (see [4], [5], [17]). A variation on this idea is to replace X_n by the span $\mathcal{F}_n$ of $\{X_n, X_{n+1}, X_{n+2}, \ldots\}$, and to investigate the linear dependence between the subspaces $\mathcal{P}$ and $\mathcal{F}_n$. This class of problems is addressed in, for instance, [6], [8]–[11], [16], [18], [20]. These concerns, in turn, admit a multitude of generalizations.

In this article, we consider prediction problems in which the process is replaced by a random field, $\{X_{mn}\}_{\mathbb{Z}^2}$. For any subset S of $\mathbb{Z}^2$, we define $\mathcal{M}(S)$ to be the linear span of $\{X_{mn} : (m,n) \in S\}$; such spaces play roles analogous to $\mathcal{P}$ and $\mathcal{F}_n$. Now the issue is to understand the dependence between $\mathcal{M}(S_1)$ and $\mathcal{M}(S_2)$. In particular, we seek descriptions of those fields for which the dependence tends to zero as the distance between the generating sets S_1 and S_2 increases to infinity in some way—a sort of "strong mixing" condition. As in the case of processes on $\mathbb{Z}$, we pass to the spectral domain and apply techniques from function theory. This yields spectral criteria for strong mixing to occur.

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Related results in the prediction theory of stationary random fields can be found in [7], [12], [14], [15], and [19].

2. Preliminaries. Let $\{X_{mn}\}_{\mathbf{Z}^2}$ be a complex-valued, zero-mean, wide-sense stationary random field on $\mathbf{Z}^2$. Its spectral measure μ is a finite nonnegative Borel measure on the torus $\mathbf{T}^2$. We now identify the space $\mathcal{M}(S)$ with its spectral isomorph, the span in $L^2(\mu)$ of the functions $\{e^{ims+int} : (m,n) \in S\}$. As the measure of linear dependence between $\mathcal{M}(S_1)$ and $\mathcal{M}(S_2)$, we take the cosine $c(S_1, S_2)$ of the angle between them:

$$c(S_1, S_2) = \sup\left\{\left| \int f_1\overline{f}_2\, d\mu\right| : f_j \in \mathcal{M}(S_j),\ \|f_j\| \le 1\right\}.$$

Among natural choices of generating sets are the left halfplane

$$L = \{(m,n) \in \mathbf{Z}^2 : m \le 0\}$$

and the right halfplanes

$$R_N = \{(m,n) \in \mathbf{Z}^2 : m \ge N\}.$$

The goal is to describe those μ for which $\lim_{N\to\infty} c(L, R_N) = 0$, a program generalizing that of Helson and Sarason in [8]. In the analysis that follows, the principal tools include function theory on the unit circle $\mathbf{T}$, and on the unit disc $\mathbf{D}$. (Duren [3] served as the reference.) It will be convenient to identify a function $f(e^{i\theta})$ on $\mathbf{T}$ with its harmonic extension $f(z)$ into $\mathbf{D}$, and likewise a function $g(z)$ on $\mathbf{D}$ with its radial limit $g(e^{i\theta})$, whenever these make sense. Normalized Lebesgue measure on $\mathbf{T}$ will be denoted by σ.

3. Principal results. In [8] (along with [20]) it was shown that the cosine of the angle between the past and future of a stationary process tends to zero if and only if the spectral measure of the process is of the form $|P(e^{i\theta})|^2 \exp\varphi(e^{i\theta})\, d\sigma(e^{i\theta})$, where P is a polynomial and φ is a real function of vanishing mean oscillation on $\mathbf{T}$. This turns out to have a close analogue in the random field picture.

The axial alignment of L and R_N in $\mathbf{Z}^2$ allows a "separation of variables" technique. That is, with $\mathbf{Z}^2$ parametrized by (m,n), and $\mathbf{T}^2$ by (e^{is}, e^{it}), the shifting occurs only in the m-direction; hence the variable e^{is}—which is coupled with m—is the important one in determining the mixing behavior. Thus, one might expect the condition $c(L, R_N) \to 0$ to occur exactly when μ is sufficiently well-behaved in the variable e^{is}, uniformly (in some sense) in e^{it}. This is, in fact, the case.

The search for a precise statement enjoys a first reduction via an extension of Szegö's alternative. Here, μ_2 is the second marginal of μ.

3.1. THEOREM. *The space $\bigcap_{N=0}^{\infty} \mathcal{M}(R_N)$ is trivial if and only if*

$$\mu \ll \sigma \times \mu_2\,, \quad \textit{and}$$

$$\text{(3-1)}\qquad \int_{\mathbf{T}} \log[d\mu/d(\sigma\times\mu_2)](e^{is}, e^{it})\, d\sigma(e^{is}) > -\infty\,, \qquad a.e.\,[d\mu_2(e^{it})]\,.$$

Proof. See [13, Theorem 3]. ∎

3.2. LEMMA. *In order that $c(L, R_M) < 1$ for some M, it is necessary that $\bigcap_{N=0}^{\infty} \mathcal{M}(R_N) = (0)$.*

Proof. Fix $\varepsilon > 0$ and $M \in \mathbf{Z}_+$. Suppose that $\mathcal{M}_\infty = \bigcap_{N=0}^{\infty} \mathcal{M}(R_N)$ contains a nonzero vector f. There exists a finite trigonometric sum p for which $\|f - p\| < \varepsilon$. Choose an integer m such that $e^{ims}p(e^{is}, e^{it})$ is in $\mathcal{M}(L)$. Note that $e^{ims}f(e^{is}, e^{it})$ is in $\mathcal{M}_\infty$, and hence in $\mathcal{M}(R_M)$. And now

$$\begin{aligned} c(L, R_M) &\ge \|e^{ims}f\|^{-1}\|e^{ims}p\|^{-1}|\langle e^{ims}f, e^{ims}p\rangle| \\ &\ge \|f\|^{-1}(\|f\|+\varepsilon)^{-1}(\|f\|^2 - \varepsilon\|f\|) = \frac{\|f\|-\varepsilon}{\|f\|+\varepsilon}. \end{aligned}$$

This forces $c(L, R_M) = 1$. ∎

Accordingly, we can assume that the restrictions on μ in (3-1) hold. With that, we define

$$\text{(3-2)}\qquad h(z, e^{it}) = \exp\frac{1}{2}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log w(e^{i\theta}, e^{it})\, d\sigma(e^{i\theta})\,,$$

where $w = [d\mu/d(\sigma\times\mu_2)]$. The radial limit function $h(e^{is}, e^{it})$ is outer in e^{is} for μ_2-almost every e^{it}, and $|h|^2 = w$ a.e. $[\sigma\times\mu_2]$ on the torus. This provides the needed spectral factorization for passing to the Lebesgue space, as has been done so successfully in the univariate problem. For, suppose $\mathcal{L}^p(S)$ is the span in $L^p(\sigma\times\mu_2)$ of $\{e^{ims+int} : (m,n) \in S\}$.

3.3. LEMMA. *If h exists, then $h^{-1} \in \mathcal{M}(R_0)$, and $h\mathcal{M}(R_0) = \mathcal{L}^2(R_0)$.*

Proof. This is the content of [1, 2.2 and 2.3]. ∎

The next assertion, a consequence of the spectral factorization and some duality theory, effects the separation of variables. As such, it makes use of the univariate cosine

$$\varrho_N(e^{it}) = \sup\left\{\int f(e^{is})\overline{g}(e^{is})w(e^{is}, e^{it})\, d\sigma(e^{is}) : (*)\right\},$$

$$(*)\qquad \begin{cases} f \in L^2(w(e^{is}, e^{it})\, d\sigma(e^{is})) - \operatorname{span}\{e^{ims} : m \ge N\}\,, \\ g \in L^2(w(e^{is}, e^{it})\, d\sigma(e^{is})) - \operatorname{span}\{e^{ims} : m \le 0\}\,, \\ \int |f(e^{is})|^2 w(e^{is}, e^{it})\, d\sigma(e^{is}) = 1\,, \\ \int |g(e^{is})|^2 w(e^{is}, e^{it})\, d\sigma(e^{is}) = 1\,, \end{cases}$$

and a dual extremal quantity

$$K_N = \inf\left\{\left\|A(e^{is},e^{it}) - \frac{\overline{h}(e^{is},e^{it})}{h(e^{is},e^{it})}e^{i(N-1)s}\right\|_{L^\infty(\sigma\times\mu_2)} : A \in \mathcal{L}^\infty(R_0)\right\}.$$

3.4. THEOREM. *If* (3-1) *holds, then* $c(L,R_N) = K_N = \|\varrho_N(e^{it})\|_{L^\infty(\mu_2)}$.

Proof. First, note that $\varrho_N(e^{it})$ is bounded by 1. Moreover, it is measurable, since the set $\{e^{it} : \varrho_N(e^{it}) < \alpha\}$ can be expressed as the countable intersection

$$\bigcap\Big\{e^{it} : \alpha > \Big|\int f(e^{is})\overline{g}(e^{it})w(e^{is},e^{it})\,d\sigma(e^{is})\Big|,$$
$$\int |f(e^{is})|^2 w(e^{is},e^{it})\,d\sigma(e^{is}) \le 1,\ \int |g(e^{is})|^2 w(e^{is},e^{it})\,d\sigma(e^{is}) \le 1,$$
$$\text{and } f \text{ and } g \text{ are polynomials with complex rational coefficients in } \mathcal{M}(R_N) \text{ and } \mathcal{M}(L), \text{ respectively}\Big\}.$$

Hence $\varrho_N(e^{it}) \in L^\infty(\mu_2)$.

Now for $F \in \mathcal{M}(R_N)$ and $G \in \mathcal{M}(L)$,

$$\|F\|^{-1}\|G\|^{-1}\Big|\int F\overline{G}w\,d(\sigma\times\mu_2)\Big|$$
$$\le \int\Big\{\varrho_N(e^{it})\cdot\Big[\int |F(e^{ix},e^{it})|^2 w(e^{ix},e^{it})\,d\sigma(e^{ix})\Big]^{1/2}$$
$$\times\Big[\int |G(e^{iy},e^{it})|^2 w(e^{iy},e^{it})\,d\sigma(e^{iy})\Big]^{1/2}\cdot\|F\|^{-1}\|G\|^{-1}\Big\}\,d\mu_2(e^{it})$$
$$\le \|\varrho_N(e^{it})\|_{L^\infty(\mu_2)}.$$

Taking a supremum over F and G gives

$$c(L,R_N) \le \|\varrho_N(e^{it})\|_{L^\infty(\mu_2)}.$$

Next, let $\varepsilon > 0$, and choose A in $\mathcal{L}^\infty(R_0)$ satisfying

$$\|A - \overline{h}h^{-1}e^{i(N-1)s}\| \le K_N + \varepsilon.$$

For μ_2-almost every e^{it}, $A(\cdot,e^{it})$ lies in $H^\infty(\mathsf{T})$. It follows that

$$\inf\left\{\left\|a(e^{is}) - \frac{\overline{h}(e^{is},e^{it})}{h(e^{is},e^{it})}e^{i(N-1)s}\right\|_{L^\infty(\sigma)} : a \in H^\infty(\mathsf{T})\right\}$$
$$\le K_N + \varepsilon, \quad \text{a.e.}\,[\mu_2(e^{it})].$$

But this l.h.s. is just $\varrho_N(e^{it})$, by [8, Theorem 3]. This yields

$$\|\varrho_N(e^{it})\|_{L^\infty(\mu_2)} \le K_N.$$

Finally, observe that K_N is the norm of $\overline{h}e^{i(N-1)s}/h$, as a bounded linear functional on $\mathcal{L}^1(R_1)$, the subspace of $L^1(\sigma\times\mu_2)$ annihilated by $\mathcal{L}^\infty(R_0)$.

That is,

$$K_N = \sup\Big\{\Big|\int H\overline{h}h^{-1}e^{i(N-1)s}\,d(\sigma\times\mu_2)\Big| : H \in \mathcal{L}^1(R_1),\ \|H\|_1 \le 1\Big\}.$$

For such H, define

$$\psi(z,e^{it}) = \exp\frac{1}{2}\int\frac{e^{i\vartheta}+z}{e^{i\vartheta}-z}\log|H(e^{i\vartheta},e^{it})|\,d\sigma(e^{i\vartheta}),$$
$$\Phi = H\psi^{-2}.$$

Note that $|\psi|^2 = |H|$, a.e. $[\sigma\times\mu_2]$; $|\Phi| = 1$, a.e. $[\sigma\times\mu_2]$; $\psi(\cdot,e^{it})$ is outer a.e. $[\mu_2]$; and $e^{-is}H(e^{is},e^{it})$ is in $H^1(\mathsf{T})$ in e^{is}, a.e. $[\mu_2]$. Therefore,

$$K_N = \sup\Big\{\Big|\int(\Phi\psi)\psi\overline{h}h^{-1}e^{i(N-1)s}d(\sigma\times\mu_2)\Big| : \Phi\psi^2 = H \in \mathcal{L}^1(R_1), \|H\|_1 \le 1\Big\}$$
$$\le \sup\Big\{\Big|\int\Big(\frac{u}{h}e^{-is}\Big)\Big(\frac{v}{h}e^{i(N-1)s}\Big)|h|^2\,d(\sigma\times\mu_2)\Big| :$$
$$u,v \in \mathcal{L}^2(R_1),\ \int |u|^2\,d(\sigma\times\mu_2) \le 1,\ \int |v|^2\,d(\sigma\times\mu_2) \le 1\Big\}.$$

By 3.3, $(ue^{-is}/h)^- \in \mathcal{M}(L)$ and $(ve^{i(N-1)s}/h) \in \mathcal{M}(R_N)$. We conclude that

$$K_N \le c(L,R_N),$$

completing a chain of inequalities which establishes 3.4. ∎

This provides a way to relate the present problem to the univariate version. We adapt the solution of the latter to obtain, through 3.4, structural information about those μ satisfying $\lim_{N\to\infty} c(L,R_N) = 0$.

The following constructs will be needed. Let $\mathcal{G}_0$ be the collection of polynomials in e^{is} with coefficients in $L^\infty(\mu_2(e^{it}))$. Next, if $f(\cdot,e^{it})$ is integrable a.e. $[\mu_2(e^{it})]$, let $\widetilde{f}$ be the conjugate of f with respect to the first variable. Take $\mathcal{W}_0$ to be the collection of nonnegative, $[\sigma\times\mu_2]$-measurable functions $w_0(e^{is},e^{it})$ which satisfy the condition:

(3-3) For any $\varepsilon > 0$, there exist real functions r_ε and S_ε in $L^\infty(\sigma\times\mu_2)$ with $\|r_\varepsilon\|_\infty + \|S_\varepsilon\|_\infty < \varepsilon$, and a polynomial Q_ε in e^{is} with $[\mu_2(e^{it})]$-measurable coefficients, such that $Q_\varepsilon(z,e^{it})$ is nonvanishing for z in the closed disc, a.e. $[\mu_2(e^{it})]$, and

$$w_0 = |Q_\varepsilon|^2\exp(r_\varepsilon + \widetilde{S}_\varepsilon), \quad \text{a.e.}\,[\sigma\times\mu_2].$$

3.5. THEOREM. *The strong mixing condition*

$$\lim_{N\to\infty} c(L,R_N) = 0$$

holds if and only if

(a) $\mu \ll \sigma\times\mu_2$;

(b) $\int \log w(e^{is}, e^{it})\, d\sigma(e^{is}) > -\infty$, *a.e.* $[\mu_2(e^{it})]$ *and*
(c) $w = |P|^2 w_0$ *for some* $P \in \mathcal{G}_0$ *and* $w_0 \in \mathcal{W}_0$;

where $w = [d\mu/d(\sigma \times \mu_2)]$.

Thus, in accord with the univariate case, strong mixing occurs exactly when the spectral measure is continuous in the appropriate variable, its density is logarithmically integrable in that variable, the zero set of that density is removable by a type of polynomial, and the remaining factor satisfies a smoothness condition. Efforts here to express the smoothness condition in terms of VMO, as was done in [20], have been unsuccessful. The principal obstacle is that the smoothness condition needs to be uniform in the variable e^{it}, in a sense which is difficult to control with norms. The example in 4.3 shows that without this uniformity, nonmixing can occur even if $w(\cdot, e^{it})$ is analytic for almost every fixed e^{it}.

Proof of 3.5. Suppose that (a), (b) and (c) hold. Fix ε, $0 < \varepsilon < \pi/8$, and let Q_ε, r_ε and S_ε be the functions associated with w_0 through (3-3). Put

$$N_\varepsilon(e^{it}) = \deg Q_\varepsilon(z, e^{it}) = \lim_{p\to\infty} \frac{1}{2\pi i} \int_{|z|=p} \frac{Q'_\varepsilon(z, e^{it})}{Q_\varepsilon(z, e^{it})}\, dz;$$

$$A(e^{is}, e^{it}) = \exp[-r_\varepsilon(e^{is}, e^{it}) - i\widetilde{r}_\varepsilon(e^{is}, e^{it})]; \quad \text{and}$$
$$B(e^{is}, e^{it}) = \exp[iN_\varepsilon(e^{it})]\widetilde{Q}_\varepsilon(e^{is}, e^{it})/Q_\varepsilon(e^{is}, e^{it}).$$

Observe that A is bounded, and $A(z, e^{it})$ is analytic in $z \in \mathbb{D}$. Also, $B(z, e^{it})$ is a Blaschke product in z with $N_\varepsilon(e^{it})$ factors. Moreover, its zeros are the reciprocal complex conjugates of those of $Q_\varepsilon(z, e^{it})$. To see this, compare the expression for B with the equation

$$\frac{\overline{\alpha}}{\alpha} \cdot \frac{z - \left(\frac{1}{\alpha}\right)^{-}}{1 - \left(\frac{1}{\alpha}\right) z} = z \cdot \frac{\overline{z} - \overline{\alpha}}{z - \alpha},$$

with $|z| = 1$ and $|\alpha| > 1$.

From the definition of $\mathcal{W}_0$, we see that $\log w_0(\cdot, e^{it})$ is integrable, a.e. $[\sigma_2(e^{it})]$. Hence we can define

$$h(z, e^{it}) = \exp \frac{1}{2} \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log w_0(e^{i\theta}, e^{it})\, d\sigma(e^{i\theta}).$$

Now, for fixed e^{it} and computation modulo 2π,

$$\begin{aligned}\arg(ABh^2 e^{-iN_\varepsilon s}) &= \arg A + \arg h^2 + \arg(Be^{-iN_\varepsilon s}) \\ &= \arg A + (\log w_0) + \arg(Be^{-iN_\varepsilon s}) \\ &= \arg A + \widetilde{r}_\varepsilon + S_\varepsilon - \int S_\varepsilon\, d\sigma(e^{is}) + \arg Q_\varepsilon^2 + \arg(Be^{-iN_\varepsilon s}) \\ &= S_\varepsilon - \int S_\varepsilon\, d\sigma(e^{is}).\end{aligned}$$

(Here, choose the branch of the argument function which vanishes at $z = 1$.) It follows that

$$|\arg(ABh^2 e^{-iN_\varepsilon s})| \le 2\varepsilon.$$

This, together with

$$|\log |ABe^{-iN_\varepsilon s} h/\overline{h}|\,| = |\log |A|\,| = |r| < \varepsilon,$$

yields

$$\left\| \frac{\overline{h}}{h} \cdot e^{i(N-1)s} - ABe^{i(N-N_\varepsilon-1)s} \right\|_{L^\infty(\sigma\times\mu_2)} < 2\varepsilon,$$

where $N = 1 + \|N_\varepsilon(e^{it})\|_{L^\infty(\sigma\times\mu_2)}$.

Let $\Gamma(e^{it}) = \min\{1, [\int w_0\, d\sigma(e^{is})]^{-1}\}$, so that $\Gamma(e^{it}) w_0(e^{is}, e^{it})$ is $[\sigma \times \mu_2]$-integrable. Applying 3.4 to the last inequality shows that under the measure $w_0 d(\sigma \times \mu_2)$, the cosine for L and R_N is less than 2ε.

Finally, let d be the degree of P as a member of $\mathcal{G}_0$. Choose any $f \in \mathcal{M}(R_{N+d})$ and $g \in \mathcal{M}(L)$ with $\|f\| \le 1$ and $\|g\| \le 1$, as objects associated with $L^2(\mu)$. Then

$$\int f\overline{g}\, d\mu = \int (\Gamma^{-1/2} Pf)(\Gamma^{-1/2} \overline{Pg})(\Gamma w_0)\, d(\sigma \times \mu_2).$$

But $\Gamma^{-1/2} Pfe^{-ids}$ and $\Gamma^{-1/2} Pge^{-ids}$ lie in the subspaces of $L^2(\Gamma w_0 d(\sigma \times \mu_2))$ generated by R_N and L, respectively. We conclude that

$$c(L, R_{N+d}) \le 2\varepsilon.$$

This proves the sufficiency assertion.

Conversely, suppose $\lim_{N\to\infty} c(L, R_N) = 0$. Then (a) and (b) must hold. From 3.4 it follows that for each positive integer k, there exist A_k in $\mathcal{L}^\infty(R_0)$ and a positive integer N_k such that

$$\|1 - A_k e^{-i(N_k-1)s} h/\overline{h}\|_{L^\infty(\sigma\times\mu_2)} < \frac{\pi}{4} \cdot 2^{-k}.$$

Put $r_k = -\log|A_k|$ and $S_k = -\arg(Ah^2 e^{-iN_k s})$, so that $\|r_k\| + \|S_k\|_{L^\infty(\sigma\times\mu_2)} < (\pi/2)2^{-k}$. Consider the function

$$u = Ah^2 \exp(-iN_k s - \widetilde{S}_k + iS_k).$$

Note that u is nonnegative on $\mathbb{T}^2$, and $u(z, e^{it})$ is analytic in $z \in \mathbb{D}$. As in [8, p. 10], it follows that $u(z, e^{it})$ has an analytic continuation across $|z| = 1$. And now the reflection principle asserts that for μ_2-almost every fixed e^{it}, $u(e^{is}, e^{it})$ is the squared modulus of a polynomial P_k in e^{is}, of degree at most N_k. Observe that

$$|P_j(e^{is}, e^{it})/P_k(e^{is}, e^{it})|^2 = \exp[(r_k - r_j) + (\widetilde{S}_k - \widetilde{S}_j)]$$

is $[\sigma(e^{is})]$-integrable, a.e. $[\mu_2(e^{it})]$. Therefore, the unimodular roots of each $P_k(z,e^{it})$, $k=1,2,\ldots$, must coincide.

To construct P as in (c), choose any P_k and define

$$\lambda(e^{it}) = \lim_{\varrho\to\infty} \frac{1}{2\pi i} \int_{\{|z|=\varrho\}} \frac{P_k'(z,e^{it})}{P_k(z,e^{it})}\,dz\,,$$

$$\lambda_1(e^{it}) = \lim_{\varrho\uparrow 1} \frac{1}{2\pi i} \int_{\{|z|=\varrho\}} \frac{P_k'(z,e^{it})}{P_k(z,e^{it})}\,dz\,,$$

$$\lambda_0(e^{it}) = \lim_{\varrho\downarrow 0} \frac{1}{2\pi i} \int_{\{|z|=\varrho\}} \frac{P_k'(z,e^{it})}{P_k(z,e^{it})}\,dz;$$

these give the numbers of roots of $P(\cdot,e^{it})$ in $\mathbf{C}$, in $\mathbf{D}$, and at 0, respectively.

For each e^{it}, factor P_k into $P_-P_0P_+$, such that the roots of P_-, P_0 and P_+ lie in $\mathbf{D}$, on $\mathbf{T}$, and outside $\mathbf{T}$, respectively, such that $P_0(0,e^{it})=1$ and

$$\int \log|P_+(e^{is},e^{it})|\,d\sigma(e^{is}) = 0\,,\qquad \text{a.e.}\,[\mu_2(e^{it})]\,.$$

We establish the measurability of P_-, P_0, and P_+ as follows. First let

$$H(z,e^{it}) = \exp \int \frac{e^{i\theta}+z}{e^{i\theta}-z} \log|P_k(e^{i\theta},e^{it})|\,d\sigma(e^{i\theta})\,,$$

$$J(z,e^{it}) = P_k(z,e^{it})/H(z,e^{it})\,,$$

$$C(e^{it}) = \frac{1}{2\pi i}\int_{\mathbf{T}} P_k(z,e^{it}) z^{-[\lambda(e^{it})+1]}\,dz\,.$$

We find that

$$J(z,e^{it}) z^{[\lambda_1(e^{it})-\lambda_0(e^{it})]} = \frac{\overline{C}(e^{it})}{|C(e^{it})|}\cdot\frac{P_-(z,e^{it})}{\overline{P}_-(z,e^{it})}\cdot\frac{\overline{z}^{\lambda_0(e^{it})}}{z^{\lambda_0(e^{it})}}\,.$$

(To see this, compare with

$$\frac{-\alpha}{|\alpha|}\cdot\frac{z-\alpha}{1-\overline{\alpha}z}\cdot z = \frac{-\alpha}{|\alpha|}\cdot\frac{z-\alpha}{\overline{z}-\overline{\alpha}}$$

for $|z|=1$, $0<|\alpha|<1$.) It follows that $\arg(P_0P_+)$ is measurable, and hence P_0P_+ is measurable. Repeating this argument with P_k replaced by $z^{\lambda_0(e^{it})}P_k(1/z,e^{it})$ shows that P_-P_0 is measurable, as then are P_-, P_0, and P_+ separately.

Now $P_0(e^{is},e^{it})$ is of the form

$$P_0(e^{is},e^{it}) = \sum_{m=0}^{J} a_m(e^{it}) e^{ims}\,,$$

where each $a_m(e^{it})$ is μ_2-measurable. Define

$$P(e^{is},e^{it}) = A(e^{it})P_0(e^{is},e^{it})\,,$$

with

$$A(e^{it}) = \Big[1+\sum_{m=0}^{J}|a_m(e^{it})|\Big]^{-1}.$$

Then $P(z,e^{it})\in\mathcal{G}_0$, and its roots are exactly those unimodular roots common to all the P_j, $j=1,2,\ldots$ Finally, let

$$Q_k(z,e^{it}) = \exp\int\frac{e^{i\theta}+z}{e^{i\theta}-z}\log\left|\frac{P_k(e^{i\theta},e^{it})}{P(e^{i\theta},e^{it})}\right|\,d\sigma(e^{i\theta})\,.$$

Observe that $|Q_k| = |P_k/P|$ on $\mathbf{T}^2$, and $Q_k(z,e^{it})$ is a polynomial whose roots all lie outside $\mathbf{T}$.

We have, at last, the representation

$$w = |P|^2 w_0\,,$$

where

$$w_0 = |Q_k|^2\exp(r_k+\widetilde{S}_k)$$

satisfies (3-3) with $\varepsilon = (\pi/2)2^{-k}$, $k=1,2,\ldots$ ∎

4. Further developments. The separation of variables approach, as realized through 3.4, has other consequences as well. First, following a course parallel to that of [8], we find spectral criteria for the condition $c(L,R_N)<1$.

4.1. THEOREM. *In order that $c(L,R_N)<1$, it is necessary that $d\mu(e^{is},e^{it})$ be of the form*

$$\Gamma(e^{it})|P(e^{is},e^{it})|^2\exp[r(e^{is},e^{it})+\widetilde{S}(e^{is},e^{it})]\,d[\sigma(e^{is})\times\mu_2(e^{it})]$$

where $P\in\mathcal{G}_0$ is of degree less than N, r and S are real functions in $L^\infty(\sigma\times\mu_2)$, $\|S\|<\pi/2$, and Γ is μ_2-measurable.

Proof. If $c(L,R_N)<1$, then the restrictions (3-1) on μ hold. Hence 3.4 applies; given ε, $c(L,R_N)<1-\varepsilon<1$, there exists $A\in\mathcal{L}^\infty(R_0)$ such that

$$\left\|A-\frac{\overline{h}}{h}e^{i(N-1)s}\right\|_\infty \le c(L,R_N)+\varepsilon<1\,.$$

(As before, h is defined through (3-2).) This implies that for some constants $K<\pi/2$ and C,

$$|\arg(Ah^2e^{-i(N-1)s})|\le K<\pi/2\,,\qquad \text{a.e.}\,[\mu_2]\,,$$

and

$$|\log|A||\le C\,,\qquad \text{a.e.}\,[\mu_2]\,.$$

Put $r = -\log|A|$, $S = -\arg(Ah^2e^{-iNs})$ and $u = Ah^2\exp(-iNs - \widetilde{S} + iS)$. As in the proof of 3.5, u turns out to be $\Gamma(e^{it})|P(e^{is}, e^{it})|^2$, where $P \in \mathcal{G}_0$, $\deg P < N$, and $\Gamma(e^{it})$ is a nonnegative μ_2-measurable function of e^{it}. Now, if $w = |h|^2 = d\mu/d(\sigma \times \mu_2)$,

$$\begin{aligned} w = h^2 \cdot \exp(-2i\arg h) &= \Gamma|P|^2A^{-1}\exp[i(N-1)s + \widetilde{S} - iS - 2i\arg h] \\ &= \Gamma|P|^2\exp[r + \widetilde{S} - iS - i\arg(Ah^2e^{-i(N-1)s})] = \Gamma|P|^2\exp[r + \widetilde{S}]. \ \blacksquare \end{aligned}$$

4.2. Theorem. *Suppose that $d\mu(e^{is}, e^{it})$ is of the form*

$$\Gamma(e^{it})|P(e^{is}, e^{it})|^2\exp[r(e^{is}, e^{it}) + \widetilde{S}(e^{is}, e^{it})]\, d[\sigma(e^{is}) \times \mu_2(e^{it})],$$

where $P \in \mathcal{G}_0$, $\deg P = N$, r and S are real functions in $L^\infty(\sigma \times \mu_2)$, and $\Gamma(e^{it})$ is a nonnegative μ_2-measurable function of e^{it}. If $\|r\|_\infty$ and $\|S\|_\infty$ are sufficiently small, then $c(L, R_m) < 1$ whenever $m > N$.

Proof. Consider the univariate weight function $\phi(\cdot) = \exp[r(\cdot, e^{it}) + \widetilde{S}(\cdot, e^{it})]$ for fixed e^{it}. It has a factorization $\phi = |f|^2$, where $f \in H^2(\mathbf{T})$. We can choose $f = \exp\frac{1}{2}(r + i\widetilde{r} + \widetilde{S} - iS)$.

Let $0 < \varepsilon < 1$, and assume that $\|r\|_\infty$ and $\|S\|_\infty$ are small enough to allow

$$[e^{\|r\|_\infty}\sin\|S\|_\infty]^2 + [e^{\|r\|_\infty}\cos\|S\|_\infty - 1]^2 \le (1-\varepsilon)^2.$$

Then there exists $A(e^{is})$ in $H^\infty(T)$ such that

$$\|1 - A(\cdot)f(\cdot, e^{it})/\overline{f}(\cdot, e^{it})\|_{L^\infty(\sigma)} \le 1 - \varepsilon,$$

namely $A(e^{is}) = \exp[-r(e^{is}, e^{it}) - i\widetilde{r}(e^{is}, e^{it})]$. (For then $e^{-\|r\|_\infty} \le |A| \le e^{\|r\|_\infty}$, and $\arg(Af^2) = -\widetilde{r} + 2\cdot\frac{1}{2}(\widetilde{r} - S) = -S$.) It follows that for μ_2-almost every e^{it}, $\varrho_{N+1}(e^{it})$ is at most $1-\varepsilon$. By 3.4, this yields $c(L, R_{N+1}) \le 1-\varepsilon$. $\blacksquare$

In the following example, 3.4 is used to calculate $c(L, R_N)$ explicitly. It illustrates that random fields with rational spectral densities can exhibit nonmixing behavior. This stands in contrast to the case of processes on $\mathbf{Z}$: if such a process has a rational spectral density, then its past and N-step future are asymptotically orthogonal.

4.3. Proposition. *Let $d\mu = |h|^2\, d\sigma^2$, where*

$$h(e^{is}, e^{it}) = \frac{e^{is} - e^{it}}{2 - e^{is} - e^{it}}.$$

Then $c(L, R_N) = 1$ for all N.

Proof. First we check that $|h|^2$ is σ^2-integrable. For $e^{it} \ne 1$,

$$\int |h(e^{is}, e^{it})|^2\, d\sigma(e^{is}) = \frac{1}{2\pi i}\int_{\mathbf{T}} \frac{z - e^{it}}{2 - z - e^{it}} \cdot \frac{1 - e^{-it}z}{2z - 1 - e^{-it}z}\, \frac{dz}{z}$$

$$\begin{aligned} &= \frac{-e^{it}}{2 - e^{it}} + \frac{(2-e^{-it})^{-1} - e^{it}}{2 - e^{-it} - (2-e^{it})^{-1}} \cdot \frac{1 - e^{-it}(2-e^{-it})^{-1}}{2 - e^{-it}} \\ &\le 1 + \frac{2|1 - e^{-it}|^2}{|2 - e^{-it}|^2 - 1}. \end{aligned}$$

Therefore

$$\begin{aligned} \int |h|^2\, d\sigma^2 &\le \int_{\mathbf{T}} \left(1 + \frac{2|1 - e^{-it}|^2}{|2 - e^{-it}|^2 - 1}\right) d\sigma(e^{it}) \\ &\le 1 + \int_{-\pi}^{\pi} \frac{2(2 - 2\cos t)}{5 - 4\cos t - 1}\, \frac{dt}{2\pi} = 2. \end{aligned}$$

Next, note that $h(\cdot, e^{it})$ is outer in $H^2(\mathbf{T})$ for e^{it} fixed. Accordingly,

$$\begin{aligned} \varrho_N(e^{it}) &= \sup\left\{\left|\int \frac{\overline{h}}{h} e^{i(N-1)s} F(e^{is})\, d\sigma(e^{is})\right| : F(e^{is}) \in e^{is}H^1(\mathbf{T}), \|F\|_1 \le 1\right\} \\ &= \sup\left\{\left|\int \frac{2 - e^{is} - e^{it}}{2 - e^{-is} - e^{-it}}(-e^{-is-it})e^{i(N-1)s}F(e^{is})\, d\sigma(e^{is})\right| : F(e^{is}) \in e^{is}H^1(\mathbf{T}), \|F\|_1 \le 1\right\} \\ &= \sup\left\{\left|\int_{\mathbf{T}} \frac{1}{2 - e^{-it}} \cdot \frac{2 - e^{it} - z}{z - (2 - e^{-it})^{-1}} \cdot (z^{N-1})F(z)\, \frac{dz}{2\pi i}\right| : F \in H^1(\mathbf{T}), \|F\|_1 \le 1\right\} \\ &= \frac{|2 - e^{it} - (2 - e^{-it})^{-1}|}{|2 - e^{-it}|^N} \cdot \sup\left\{\left|F\left(\frac{1}{2 - e^{-it}}\right)\right| : F \in H^1(\mathbf{T}), \|F\|_1 \le 1\right\}. \end{aligned}$$

Put $\omega = (2 - e^{-it})^{-1}$, and $F(z) = (1 - \overline{\omega}z)^{-2}$. Then $\|F\|_1 = (1 - |\omega|^2)^{-1}$, and

$$\varrho_N(e^{it}) \ge \frac{|\overline{\omega}^{-1} - \omega|}{|\omega|^{-N}} \cdot \frac{(1 - |\omega|^2)^{-2}}{(1 - |\omega|^2)^{-1}} = |\omega|^{N-1} = |2 - e^{-it}|^{1-N}.$$

By 3.4, $c(L, R_N) = \|\varrho_N(e^{it})\|_\infty \ge |2 - e^0|^{1-N} = 1$. This proves the claim. $\blacksquare$

Of course, there do exist μ for which $c(L, R_N) \to 0$. The next result exhibits a large class of examples. Here, let $\mathcal{K}_0$ be the collection of finite trigonometric sums in e^{is} with coefficients in $L^\infty(\sigma(e^{it}))$, and let $\mathcal{K}_1$ be the closure in $L^\infty(\sigma^2)$ of $\operatorname{Re}\mathcal{K}_0$.

4.4. PROPOSITION. *Suppose that $d\mu$ is of the form $|P|^2\exp(U+\widetilde{V})\,d\sigma^2$, where $P\in\mathcal{K}_0$, and U, $V\in\mathcal{K}_1$. Then $c(L,R_N)\to 0$.*

Proof. It suffices to show that $\exp(U+\widetilde{V})\in\mathcal{W}_0$. Let $\varepsilon>0$. There exist real functions U_0 and V_0 in $\mathcal{K}_0$ such that

$$\|U-U_0\|_{L^\infty(\sigma^2)}+\|V-V_0\|_{L^\infty(\sigma^2)}<\varepsilon.$$

Note that $\widetilde{V}_0$ is again a real function in $\mathcal{K}_0$. Now $\exp(U_0+\widetilde{V}_0)$ has a series expansion in e^{is} which converges in $L^\infty(\sigma^2)$. In particular, it can be expressed in the form $|Q_0|^2\exp\psi$ where $Q_0(e^{is},e^{it})$ is a polynomial in e^{is} with no roots on the circle a.e. $[\sigma(e^{it})]$, and ψ is a bounded real function satisfying $\|\psi\|_{L^\infty(\sigma^2)}<\varepsilon$. Put $r=U+\psi-U_0$, $S=V-V_0$, and

$$Q(z,t)=\exp\int\frac{e^{i\theta}+z}{e^{i\theta}-z}\log|Q_0(e^{i\theta},e^{it})|\,d\sigma(e^{i\theta}).$$

Then

$$\begin{aligned}\exp(U+\widetilde{V})&=\exp(U_0+\widetilde{V}_0)\exp(U-U_0)\exp(\widetilde{V}-\widetilde{V}_0)\\&=|Q|^2\exp\psi\exp(r-\psi)\exp\widetilde{S}=|Q|^2\exp(r+\widetilde{S}).\end{aligned}$$

This shows that $\exp(U+\widetilde{V})\in\mathcal{W}_0$. ■

The representation of $\mathcal{M}(R_0)$ in 3.3 makes possible a formula for the distance from the function 1 to the space $\mathcal{M}(R_N)$, an N-step prediction error for halfplanes of a random field. For $N=1$, this was done in [12], and for a process on $\mathbf{Z}$, see [5], [17]. In the present situation, let $d\mu=w\,d(\sigma\times\mu_2)+d\lambda$ be the Lebesgue decomposition of μ with respect to $\sigma\times\mu_2$. There is a measurable subset A of the circle such that

$$\int_{\mathbf{T}}\log w(e^{i\theta},e^{it})\,d\sigma(e^{i\theta})>-\infty$$

if and only if $e^{it}\in A$. For such e^{it}, define h on $\mathbf{T}$ via (3-2).

4.5. THEOREM.

$$\inf\{\|1+f\|^2_{L^2(\mu)}:f\in\mathcal{M}(R_N)\}=\sum_{m=0}^{N-1}\int_A\Big|\int_{\mathbf{T}}h(e^{is},e^{it})e^{-ims}\,d\sigma(e^{is})\Big|^2\,d\mu_2(e^{it}).$$

Proof. There is a Borel subset Ω of the torus such that $\lambda(\Omega^c)=0=(\sigma\times\mu_2)(\Omega)$. Put $E=(\mathbf{T}\times A)\cap\Omega^c$, so that $\mu=\mu_a+\mu_b$, where

$$\begin{aligned}d\mu_a&=1_E\,d\mu=1_E|h|^2d(\sigma\times\mu_2),\\d\mu_b&=1_{E^c}\,d\mu=1_{E^c}w\,d(\sigma\times\mu_2)+d\lambda.\end{aligned}$$

Also, let $\mathcal{M}_{(\cdot)}(R_N)$ be the subspace of $L^2(\mu_{(\cdot)})$ generated by R_N. It is known (see [12] or [13]) that

$$\bigcap_{m=1}^\infty\mathcal{M}_a(R_m)=(0),\qquad\bigcap_{m=1}^\infty\mathcal{M}(R_m)=\bigcap_{m=1}^\infty\mathcal{M}_b(R_m)=L^2(\mu_b).$$

Fix N and suppose that P is the projection operator of $L^2(\mu)$ onto $\mathcal{M}(R_N)$. By the above observations, $1_{E^c}\in L^2(\mu_b)\subseteq\mathcal{M}(R_N)$, hence $P1_{E^c}=1_{E^c}$.

Now for any f in $L^2(\mu)$, observe that

$$Pf=1_EPf+1_{E^c}Pf$$

is its representation with respect to $\mathcal{M}_a(R_N)\oplus L^2(\mu_b)$. (The second term $1_{E^c}Pf$ clearly lies in $L^2(\mu_b)$, and the first in $L^2(\mu_a)$; moreover, $1_EPf=Pf-1_{E^c}Pf$ belongs to $\mathcal{M}(R_N)$.) In particular, $(1_EP1_E)|L^2(\mu_a)$ is the projection operator of $L^2(\mu_a)$ onto $\mathcal{M}_a(R_N)$. To see this, let $f\in L^2(\mu_a)$ and check

$$1_EP1_E(1_EP1_Ef)=1_EP1_Ef;$$

and for f, $g\in L^2(\mu_a)$

$$\langle 1_EP1_Ef,g\rangle=\langle 1_EP1_Ef,1_Eg\rangle_\mu=\langle 1_Ef,1_EP1_Eg\rangle_\mu=\langle f,1_EP1_Eg\rangle_{\mu_a}.$$

Therefore

$$\begin{aligned}\|1-P1\|^2_{L^2(\mu)}&=\langle 1-P1,1-P1\rangle_\mu=\langle 1-P1,1\rangle_\mu\\&=\langle(1-P)1_E,1\rangle_\mu+\langle(1-P)1_{E^c},1\rangle_\mu=\langle(1-P)1_E,1_E+1_{E^c}\rangle_\mu+0\\&=\langle(1-P)1_E,1_E\rangle_\mu+\langle 1_E,(1-P)1_{E^c}\rangle_\mu=\langle 1_E-1_EP1_E,1_E\rangle_{\mu_a}+0\\&=\langle 1_E-1_EP1_E,1_E-1_EP1_E\rangle_{\mu_a}=\|1_E-1_EP1_E\|^2_{\mu_a}\\&=\inf\Big\{\int_E|1+f|^2|h|^2\,d(\sigma\times\mu_2):f\in\mathcal{M}_a(R_N)\Big\}\\&=\inf\Big\{\int_E|h+\phi|^2d(\sigma\times\mu_2):\phi\in 1_E\cdot\mathcal{L}^2(R_N)\Big\}\\&\le\sum_{m=0}^{N-1}\int_A\Big|\int_{\mathbf{T}}h(e^{is},e^{it})\,d\sigma(e^{is})\Big|^2\,d\mu_2(e^{it}).\end{aligned}$$

In the last step we took

$$\phi(e^{is},e^{it})=h(e^{is},e^{it})-\sum_{m=0}^{N-1}e^{ims}\int_{\mathbf{T}}h(e^{i\theta},e^{it})e^{-im\theta}\,d\sigma(e^{i\theta}),$$

which lies in $1_E \cdot \mathcal{L}^2(R_N)$.

For the reverse inequality, note that

$$\inf\Big\{ \int_E |h+\phi|^2 \, d(\sigma \times \mu_2) : \phi \in \mathcal{L}^2(R_N)\Big\}$$
$$\geq \int_A \inf\Big\{ \int_{\mathbf{T}} |h(e^{is}, e^{it}) - \Phi(e^{is})|^2 \, d\sigma(e^{is}) : \Phi \in e^{iNs} H^2(\mathbf{T})\Big\} \, d\mu_2(e^{it})$$
$$= \int_A \Big(\sum_{m=0}^{N-1} \Big| \int_{\mathbf{T}} h(e^{is}, e^{it}) e^{-ims} \, d\sigma(e^{is}) \Big|^2 \Big) \, d\mu_2(e^{it}) . \blacksquare$$

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