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Isomorphy classes of spaces of holomorphic functions on open polydiscs in dual power series spaces

by

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Abstract. Let $\Lambda_R(\alpha)$ be a nuclear power series space of finite or infinite type with $\lim_{j\to\infty}(1/j)\log\alpha_j = 0$. We consider open polydiscs $\mathbf{D}_a$ in $\Lambda_R(\alpha)'_b$ with finite radii and the spaces $H(\mathbf{D}_a)$ of all holomorphic functions on $\mathbf{D}_a$ under the compact-open topology. We characterize all isomorphy classes of the spaces $\{H(\mathbf{D}_a) \mid a \in \Lambda_R(\alpha), a > 0\}$. In the case of a nuclear power series space $\Lambda_1(\alpha)$ of finite type we give this characterization in terms of the invariants $(\overline{\Omega})$ and $(\widetilde{\Omega})$ known from the theory of linear operators between Fréchet spaces.

Introduction. Let $\Lambda_R(\alpha)$ be a nuclear power series space. For $a \in \Lambda_R(\alpha)$, $a > 0$, the set

$$\mathbf{D}_a := \{z \in \Lambda_R(\alpha)' \mid \sup_{j\in\mathbf{N}} |z_j| a_j < 1\}$$

is called an *open polydisc* (with finite radii) in $\Lambda_R(\alpha)'_b$. Then the space $H(\mathbf{D}_a)$ of all holomorphic functions on $\mathbf{D}_a$ endowed with the compact-open topology is a nuclear Fréchet space and the monomials in the coordinate functions form an absolute basis of $H(\mathbf{D}_a)$ by a theorem of Boland and Dineen [1]. Using this result and the linear topological invariants $(\overline{\Omega})$ and $(\widetilde{\Omega})$ (which have been introduced by Wagner [15] and Vogt [14]) Meise and Vogt [5], [7] derived in the case $R = 1$ that $H(\mathbf{D}_a)$ is isomorphic to a power series space iff $H(\mathbf{D}_a)$ has the property $(\overline{\Omega})$, in this case equivalent to

$$\lim_{j\to\infty} -\frac{1}{\alpha_j}\log a_j = 0\,,$$

and that $H(\mathbf{D}_a)$ has the property $(\widetilde{\Omega})$ iff

$$\limsup_{j\to\infty} -\frac{1}{\alpha_j}\log a_j < \infty\,.$$

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Furthermore, they proved [6] that there exist a continuum of polydiscs $(\mathbf{D}_{a_r})_{r>0}$ such that the l.m.c. algebras $(H(\mathbf{D}_{a_r}))_{r>0}$ are pairwise nonisomorphic.

In contrast to the last statement we show in the present article that under the stability condition $\lim_{j\to\infty}(1/j)\log\alpha_j = 0$ the classes $(\overline{\Omega})$, $(\widetilde{\Omega})\setminus(\overline{\Omega})$ and $\neg(\widetilde{\Omega})$ are exactly the isomorphism classes of $\{H(\mathbf{D}_a) \mid a \in \Lambda_1(\alpha), a > 0\}$ in the category of locally convex spaces.

On the other hand, we show that in the case $\lim_{j\to\infty}(1/j)\log\alpha_j = 0$, $R = \infty$, $H(\mathbf{D}_a)$ and $H(\mathbf{D}_b)$ are isomorphic for all $0 < a$, $b \in \Lambda_\infty(\alpha)$.

The proofs of these results are based on the sequence space representation of the space $H(\mathbf{D}_a)$. The theorem of Hall and König [9] which has been used earlier by Mityagin [10] to investigate bases in nonnuclear Hilbert scales enables us to formulate a general criterion for two shift-stable, metrizable Köthe–Schwartz spaces to be isomorphic. We verify this criterion in the classes mentioned above using stability properties of the considered spaces and estimates of the number of lattice points in certain polytopes in $\mathbf{R}^n$.

I am indebted to Prof. Meise for inspiring me to work on these topics which form a part of my thesis.

1. Preliminaries.

In this section we introduce some notations and conventions used throughout the whole article. We also mention some results which we shall use in the subsequent sections without further references (see Jarchow [4], Pietsch [11] Schaefer [12]). Furthermore, we prove a criterion for two shift-stable, metrizable Köthe–Schwartz spaces to be isomorphic.

1.1. *Sequence spaces.* Let M be a countable set and let P be a set of nonnegative sequences with the following two properties:

(i) For all $j \in M$ there exists $p \in P$ with $p_j > 0$.
(ii) For all $p, q \in P$ there exists $\lambda > 0$, $r \in P$ with $p + q \le \lambda r$.

We define the locally convex space

$$\lambda(P) := \Big\{x \in \mathbf{C}^M \;\Big|\; \|x\|_p := \sum_{j\in M} |x_j| p_j < \infty\,,\ \forall p \in P\Big\}$$

endowed with the topology induced by the system of seminorms $(\|\cdot\|_p)_{p\in P}$. The nuclearity and the Schwartz property of $\lambda(P)$ can be characterized in terms of P:

1.1.1. THEOREM. (a) (Grothendieck–Pietsch criterion [11], 6.1.2). *$\lambda(P)$ is nuclear $\Leftrightarrow$ for all $p \in P$ there exists $q \in P$ with $p/q \in l^1$.*

(b) (Jarchow [4]). *$\lambda(P)$ is a Schwartz space $\Leftrightarrow$ for all $p \in P$ there exists $q \in P$ with $p/q \in c_0$.*

1.1.2. Remark. If $\lambda(P)$ is metrizable then $\lambda(P) = \lambda(A)$ with a so-called Köthe matrix $A = (a_{jp})_{(j,p)\in M\times\mathbf{N}}$, i.e.

(i)′ For all $j \in M$ there exists $p \in \mathbf{N}$ with $a_{jp} > 0$.
(ii)′ For all $j \in M$, $p \in \mathbf{N}$, $a_{jp} \le a_{j,p+1}$.

1.1.3. PROPOSITION. *If $\lambda(A)$ is nuclear then $\lambda(A)'_b = \lambda(P)$, $P = \{p \in \lambda(A) \mid p \ge 0\}$ where $\lambda(A)'_b$ denotes the strong dual of $\lambda(A)$.*

1.1.4. *Power series spaces.* Let α be an increasing, unbounded sequence of positive numbers and $0 < R \le \infty$. We define the power series space

$$\Lambda_R(\alpha) := \lambda(P), \quad P = \{(r^{\alpha_j})_{j\in\mathbf{N}} \mid 0 < r < R\}\,.$$

Then by a diagonal transformation $\Lambda_R(\alpha) \cong \Lambda_1(\alpha)$ for $0 < R < \infty$. $\Lambda_R(\alpha)$ is called of *finite* (*infinite*) *type* if $R < \infty$ ($R = \infty$).

1.1.5. DEFINITION (Vogt [14], Wagner [15]). Let E be a Fréchet space and let $(\|\cdot\|_p)_{p\in\mathbf{N}}$ be a fundamental system of seminorms on E. For an arbitrary seminorm $\|\cdot\|$ we define the dual seminorm:

$$\|\cdot\|^* : E' \to [0,\infty], \quad \|y\|^* := \sup\{|y(x)| \mid \|x\| \le 1\}\,.$$

E has the property

$$\left.\begin{array}{l}(\widetilde{\Omega})\ :\Leftrightarrow\ \forall p_1\ \exists d > 0, p_2\ \forall p_3\ \exists C > 0 :\\ (\overline{\Omega})\ :\Leftrightarrow\ \exists d > 0\ \forall p_1\ \exists p_2\ \forall p_3\ \exists C > 0 :\end{array}\right\}\quad (\|\cdot\|^*_{p_2})^{1+d} \le C\|\cdot\|^*_{p_3}(\|\cdot\|^*_{p_1})^d\,.$$

1.1.6. Remark. It is easy to see that $(\widetilde{\Omega})$ and $(\overline{\Omega})$ are inherited by quotients and that the implication $(\overline{\Omega}) \Rightarrow (\widetilde{\Omega})$ is true. For a Köthe space $\lambda(A)$ the properties $(\widetilde{\Omega})$ and $(\overline{\Omega})$ can be characterized in terms of A:

1.1.7. LEMMA (Vogt [14], Wagner [15]). *$\lambda(A)$ has the property*

$$\left.\begin{array}{l}(\widetilde{\Omega})\ \Leftrightarrow\ \forall p_1\ \exists d > 0, p_2\ \forall p_3\ \exists C > 0\ \forall j :\\ (\overline{\Omega})\ \Leftrightarrow\ \exists d > 0\ \forall p_1\ \exists p_2\ \forall p_3\ \exists C > 0\ \forall j :\end{array}\right\}\quad \frac{a_{jp_3}}{a_{jp_2}} \le C\Big(\frac{a_{jp_2}}{a_{jp_1}}\Big)^d\,.$$

The following definition is useful for constructing isomorphisms between Köthe spaces:

1.1.8. DEFINITION (Zakharyuta [16]). Let $\lambda(A)$ and $\lambda(B)$ be Köthe spaces with canonical bases $(e_j)_{j\in M}$ and $(f_j)_{j\in M}$. We define:

a) $\lambda(A) \overset{\text{q.d.}}{\hookrightarrow} \lambda(B)$ $:\Leftrightarrow$ There exists $\sigma : M \to M$ injective and a sequence $(d_m)_{m\in M}$ of positive numbers such that

$$T : \lambda(A) \to \lambda(B), \quad T(e_j) := d_j f_{\sigma(j)} \text{ for all } j \in M\,,$$

is a topological homeomorphism.

b) $\lambda(A) \overset{\text{q.d.}}{\cong} \lambda(B)$ $:\Leftrightarrow$ There exist $\sigma : M \to M$ bijective and a sequence $(d_m)_{m\in M}$ of a positive numbers such that

$$T : \lambda(A) \to \lambda(B), \quad T(e_j) := d_j f_{\sigma(j)} \text{ for all } j \in M,$$

is an isomorphism.

c) $\lambda(A)$ is called *quasi-diagonal* (q.d.) *shift-stable* if $\lambda(A) \overset{\text{q.d.}}{\cong} \lambda(A) \times \mathbb{C}$.

1.1.9. LEMMA (Zakharyuta [16]). *For two Köthe spaces $\lambda(A)$, $\lambda(B)$ the following statements are equivalent:*

(i) $\lambda(A) \overset{q.d.}{\hookrightarrow} \lambda(B)$ *and* $\lambda(B) \overset{q.d.}{\hookrightarrow} \lambda(A)$.

(ii) $\lambda(A) \overset{q.d.}{\cong} \lambda(B)$.

We shall make use of

1.1.10. THEOREM (Hall–König [9]). *Let $(A_i)_{i\in I}$ be a family of finite sets. Then the following statements are equivalent:*

(i) *There exists $\sigma : I \to \bigcup_{j\in J} A_j$ injective with $\sigma(j) \in A_j$ for all $j \in J$.*

(ii) *For all $M \in \mathcal{E}(I) := \{M \subset I \mid \#M < \infty\}$, $\#M \le \#\bigcup_{j\in M} A_j$.*

1.1.11. LEMMA. *Let $\lambda(A)$ and $\lambda(B)$ be metrizable Köthe–Schwartz spaces with $\lambda(A)$ q.d. shift-stable. Then the following statements are equivalent:*

(i) $\lambda(A) \overset{q.d.}{\hookrightarrow} \lambda(B)$.

(ii) *There exist strictly increasing sequences $(p_s)_{s\in\mathbb{N}}$, $(C_s)_{s\in\mathbb{N}}$ of natural numbers with*

$$\lim_{s\to\infty} \frac{1}{C_s} \log \frac{a_{mp_{s+7}}}{a_{mp_1}} = 0 \quad \textit{for all } m \in M$$

such that for $I := \{(\tau,t) \in \mathbb{N}^2 \mid \tau < t\} \times \mathbb{R}^2_{\ge 1}$ and for all $\mathcal{K} \in \mathcal{E}(\mathcal{E}(I))$

$$\# \bigcup_{I\in\mathcal{K}} \bigcap_{\nu\in I} A_\nu \le \# \bigcup_{I\in\mathcal{K}} \bigcap_{\nu\in I} B_\nu$$

where

$$A_{\tau,t,u_1,u_2} := \left\{ m \in M \,\middle|\, e^{C_t u_1} < \frac{a_{mp_{t+5}}}{a_{mp_{\tau+1}}},\ \frac{a_{mp_{t+4}}}{a_{mp_{\tau+2}}} \le e^{C_t u_2} \right\},$$

$$B_{\tau,t,u_1,u_2} := \left\{ m \in M \,\middle|\, e^{C_t u_1} < \frac{b_{mp_{t+6}}}{b_{mp_\tau}},\ \frac{b_{mp_{t+3}}}{b_{mp_{\tau+3}}} \le e^{C_t u_2} \right\}.$$

(iii) *There exist strictly increasing sequences $(p_s)_{s\in\mathbb{N}}$, $(C_s)_{s\in\mathbb{N}}$ of natural numbers with*

$$\lim_{s\to\infty} \frac{1}{C_s} \log \frac{a_{mp_{s+7}}}{a_{mp_1}} = 0 \quad \textit{for all } m \in M$$

such that for $I := \{(\tau,t) \in \mathbb{N}^2 \mid \tau < t\} \times \mathbb{N}$ and for all $\mathcal{K} \in \mathcal{E}(\mathcal{E}(I))$

$$\# \bigcup_{I\in\mathcal{K}} \bigcap_{\nu\in I} A_\nu \le \# \bigcup_{I\in\mathcal{K}} \bigcap_{\nu\in I} B_\nu$$

where

$$A_{\tau,t,u} := \left\{ m \in M \,\middle|\, e^{C_t u} < \frac{a_{mp_{t+5}}}{a_{mp_{\tau+1}}},\ \frac{a_{mp_{t+4}}}{a_{mp_{\tau+2}}} \le e^{C_t(u+1)} \right\},$$

$$B_{\tau,t,u} := \left\{ m \in M \,\middle|\, e^{C_t u} < \frac{b_{mp_{t+6}}}{b_{mp_\tau}},\ \frac{b_{mp_{t+3}}}{b_{mp_{\tau+3}}} \le e^{C_t(u+1)} \right\}.$$

Proof. (i)$\Rightarrow$(ii). Let $\lambda(A) \overset{q.d.}{\hookrightarrow} \lambda(B)$. Since $\lambda(A)$ and $\lambda(B)$ are Schwartz spaces there exist $\sigma : M \to M$ injective and sequences $q \in \mathbb{N}^{\mathbb{N}}$, $d \in \mathbb{R}^M_{>0}$ such that

$$\forall s \in \mathbb{N}\ \exists M_s \in \mathcal{E}(M)\ \forall m \in M\backslash M_s : \quad a_{ms} \le d_m b_{\sigma(m)q_s} \le a_{mq_{q_s}}.$$

We may assume $q_s > s^2$ and $M_{s+1} > M_s$ for all $s \in \mathbb{N}$ and $\bigcup_{s=1}^\infty M_s = M$. Choose $(p_s)_{s\in\mathbb{N}}$ and $(C_t)_{t\in\mathbb{N}}$ inductively: $p_1 := q_1$, $p_{s+1} := q_{p_s}$ and

$$C_1 > \max_{m\in M_{p_8}} \log \frac{a_{mp_8}}{a_{mp_1}}, \quad C_{t+1} > t \max\left(C_t, \max_{m\in M_{p_{t+8}}} \log \frac{a_{mp_{t+8}}}{a_{mp_1}} \right).$$

Then

$$\lim_{t\to\infty} \frac{1}{C_t} \log \frac{a_{mp_{t+7}}}{a_{mp_1}} = 0 \quad \text{for all } m \in M$$

and for $\nu = (\tau, t, u_1, u_2) \in I$, $m \in A_\nu$ we get

$$e^{u_1 C_t} < \frac{a_{mp_{t+5}}}{a_{mp_{\tau+1}}} \le \frac{b_{\sigma(m)p_{t+6}}}{b_{\sigma(m)p_\tau}}, \quad \frac{b_{\sigma(m)p_{t+3}}}{b_{\sigma(m)p_{\tau+3}}} \le \frac{a_{mp_{t+4}}}{a_{mp_{\tau+2}}} \le e^{u_2 C_t},$$

and therefore $\sigma(A_\nu) \subset B_\nu$. Thus we have for $\mathcal{K} \in \mathcal{E}(\mathcal{E}(I))$

$$\# \bigcup_{I\in\mathcal{K}} \bigcap_{\nu\in I} A_\nu = \# \bigcup_{I\in\mathcal{K}} \bigcap_{\nu\in I} \sigma(A_\nu) \le \# \bigcup_{I\in\mathcal{K}} \bigcap_{\nu\in I} B_\nu.$$

(ii)$\Rightarrow$(iii). Set $u = u_1 = u_2 - 1$.

(iii)$\Rightarrow$(i). Since

$$\lim_{s\to\infty} \frac{1}{C_s} \log \frac{a_{mp_{s+7}}}{a_{mp_1}} = 0 \quad \text{for all } m \in M$$

we have $\bigcap_{\nu\in L} A_\nu = \emptyset$ for $L \subset I$ infinite. Therefore $I_m := \{\nu \in I \mid m \in A_\nu\}$ is infinite for $m \in \widetilde{M} := \bigcup_{\nu\in I} A_\nu$. By hypothesis we have for any finite $L \subset \widetilde{M}$

$$\#L \le \# \bigcup_{m\in L} \bigcap_{\nu\in I_m} A_\nu \le \# \bigcup_{m\in L} \bigcap_{\nu\in I_m} B_\nu.$$

1.1.10 implies that there exists $\sigma : \widetilde{M} \to M$ injective with $\sigma(m) \in \bigcap_{\mu \in I_m} B_\mu$ for all $m \in \widetilde{M}$ and therefore

$$\sigma(A_\nu) \subset \bigcup_{m \in A_\nu} \bigcap_{\mu \in I_m} B_\mu \subset B_\nu \quad \text{for all } \nu \in I\,.$$

By definition of A_ν, B_ν and since $\bigcup_{u=1}^{\infty} A_{\tau,t,u} \supset \{m \in M \mid a_{mp_{t+5}}/a_{mp_{\tau+1}} > e^{C_t}\}$ we get for $\tau < t$, $m \in \widetilde{M}$, $a_{mp_{t+5}}/a_{mp_{\tau+1}} > e^{C_t}$,

$$\frac{a_{mp_{t+4}}}{a_{mp_{\tau+2}}} \le e^{C_t} \frac{b_{\sigma(m)p_{t+6}}}{b_{\sigma(m)p_\tau}}, \qquad \frac{b_{\sigma(m)p_{t+3}}}{b_{\sigma(m)p_{\tau+3}}} \le e^{C_t} \frac{a_{mp_{t+5}}}{a_{mp_{\tau+1}}}\,.$$

Then it follows that, with sufficiently large constants $D_t > 0$, for $m \in \widetilde{M}$,

$$\frac{a_{mp_{t+4}}}{a_{mp_{\tau+2}}} \le D_t \frac{b_{\sigma(m)p_{t+6}}}{b_{\sigma(m)p_\tau}}, \qquad \frac{b_{\sigma(m)p_{t+3}}}{b_{\sigma(m)p_{\tau+3}}} \le D_t \frac{a_{mp_{t+5}}}{a_{mp_{\tau+1}}}\,.$$

Hence

$$\frac{1}{D_t} \frac{a_{mp_{t+4}}}{b_{\sigma(m)p_{t+6}}} \le \frac{a_{mp_{\tau+2}}}{b_{\sigma(m)p_\tau}} \qquad \text{for } t > \tau\,,$$

$$\frac{a_{mp_{t+1}}}{b_{\sigma(m)p_{t+3}}} \le D_\tau \frac{a_{mp_{\tau+5}}}{b_{\sigma(m)p_{\tau+3}}} \qquad \text{for } t < \tau\,,$$

and therefore

$$\frac{1}{D_t} \frac{a_{mp_{t+1}}}{b_{\sigma(m)p_{t+6}}} \le D_\tau \frac{a_{mp_{\tau+5}}}{b_{\sigma(m)p_\tau}} \qquad \text{for all } \tau,\ t\,.$$

Denote by d_m the supremum of the left hand side over $t \in \mathbf{N}$. Then it follows that for all $t \in \mathbf{N}$, $m \in \widetilde{M}$

$$a_{mp_{t+1}} \le D_t d_m b_{\sigma(m)p_{t+6}} \le D_t D_{t+6} a_{mp_{t+11}}\,.$$

This implies $E := \overline{\mathrm{LH}(\{e_m \mid m \in \widetilde{M}\})}^{\lambda(A)} \overset{\text{q.d.}}{\hookrightarrow} \lambda(B)$. Since $M \backslash \widetilde{M}$ is finite and $\lambda(A)$ q.d. shift-stable we have $\lambda(A) \overset{\text{q.d.}}{\hookrightarrow} E \overset{\text{q.d.}}{\hookrightarrow} \lambda(B)$, which completes the proof.

1.1.12. Remark. The proof of 1.1.11 is based on an idea of Djakov [3] concerning the proof of the Crone–Robinson theorem on regular bases. A result similar to 1.1.11 has been proved by Chalov and Zakharyuta [2].

1.2. *Analytic functions on locally convex spaces.* Let E be a complex, locally convex space and let $\Omega \subset E$ be open. A function $f\colon \Omega \to \mathbf{C}$ is called *holomorphic* if it is continuous and for all $a \in \Omega$, $b \in E$ the mapping $z \mapsto f(a + zb)$ is holomorphic on its natural domain of definition. Then we define $H(\Omega) := \{f : \Omega \to \mathbf{C} \mid f \text{ is holomorphic }\}$ endowed with the compact-open topology.

1.2.1. Proposition (Meise–Vogt [6]). *Let $\Lambda_R(\alpha)$ be a nuclear power series space and $a \in \Lambda_R(\alpha)$, $a > 0$. Then*

$$\mathbf{D}_a := \{z \in \Lambda_R(\alpha)' \mid \sup_{j \in \mathbf{N}} |z_j| a_j < 1\}$$

is open in $\Lambda_R(\alpha)'_b$. $\mathbf{D}_a$ is called an open polydisc (with finite radii) in $\Lambda_R(\alpha)'_b$.

1.2.2. *Notation.* We define

$$\mathbf{M} := \{m \in \mathbf{N}_0^{\mathbf{N}} \mid m_j = 0 \text{ for almost all } j \in \mathbf{N}\}$$

and for $m \in \mathbf{M}$, $x \in \mathbf{C}^{\mathbf{N}}$, $(x \mid m) := \sum_{j=1}^{\infty} x_j m_j$.

Let $\Lambda_R(\alpha)$ be nuclear and $\mathbf{D}_a$ an open polydisc in $\Lambda_R(\alpha)'_b$. For $f \in H(\mathbf{D}_a)$ and $m = (m_1, \ldots, m_n, 0, \ldots) \in M$ we define for $0 < r_j < 1/a_j$

$$a_m(f) := \frac{1}{(2\pi i)^n} \int_{|z_1| = r_1} \cdots \int_{|z_n| = r_n} \frac{f(z_1, \ldots, z_n, 0, \ldots)}{z_1^{m_1+1} \ldots z_n^{m_n+1}}\, dz_n \ldots dz_1\,.$$

An easy modification of a theorem of Boland and Dineen [1] then gives

1.2.3. Lemma (Meise–Vogt [5]). *Let $\Lambda_R(\alpha)$ be a nuclear power series space and let $\mathbf{D}_\varrho \subset \Lambda_R(\alpha)'_b$ be an open polydisc, $\varrho > 0$. Then $T : H(\mathbf{D}_\varrho) \to \lambda(\exp(A \mid \mathbf{M}))$, $T(f) = (a_m(f))_{m \in \mathbf{M}}$, is an isomorphism where* $\exp(A \mid \mathbf{M}) := (\exp(a_p \mid m))_{(m,p) \in \mathbf{M} \times \mathbf{N}}$, $a_p := (\alpha_j r_{jp})_{j \in \mathbf{N}}$ *and*

a) *in the case $R = 1$,*

$$r_{jp} := \begin{cases} r_j - \dfrac{1}{1+p}, & j \le \nu_p\,, \\ -\dfrac{1}{p}, & j > \nu_p\,, \end{cases} \qquad r_j = -\frac{1}{\alpha_j} \log \varrho_j\,,$$

and ν_p is so large that $-1/p < r_j - 1/(1+p)$ for all $j > \nu_p$,

b) *in the case $R = \infty$,*

$$r_{jp} := \begin{cases} r_j - \dfrac{1}{1+p}, & j \le \nu_p\,, \\ p, & j > \nu_p\,, \end{cases} \qquad r_j = -\frac{1}{\alpha_j} \log \varrho_j\,,$$

and ν_p is so large that $p < r_j - 1/(1+p)$ for all $j > \nu_p$.

In the case $R = 1$ the properties $(\overline{\Omega})$ and $(\widetilde{\Omega})$ can be characterized in terms of ϱ:

1.2.4. Lemma (Meise–Vogt [8]). *Let $\Lambda_1(\alpha)$ be nuclear, $\varrho \in \Lambda_1(\alpha)$, $\varrho > 0$. Then*

a) *$H(\mathbf{D}_\varrho)$ has $(\overline{\Omega})$* $\Leftrightarrow$ $\displaystyle\lim_{j \to \infty} -\frac{1}{\alpha_j} \log \varrho_j = 0\,.$

b) $H(\mathbf{D}_\varrho)$ *has* $(\widetilde{\Omega}) \Leftrightarrow \limsup_{j\to\infty} -\frac{1}{\alpha_j}\log \varrho_j = 0$.

2. Isomorphism for spaces of holomorphic functions on infinite-dimensional polydiscs

2.1. THEOREM. *Let $\Lambda_1(\alpha)$ be a nuclear power series space of finite type with $\lim_{j\to\infty}(1/j)\log\alpha_j = 0$ and $\varrho,\sigma \in \Lambda_1(\alpha)$, $\varrho,\sigma > 0$. Then the following statements are equivalent*:

(i) $H(\mathbf{D}_\varrho) \cong H(\mathbf{D}_\sigma)$.

(i′) $H(\mathbf{D}_\varrho) \overset{q.d.}{\cong} H(\mathbf{D}_\sigma)$.

(ii) *The spaces $H(\mathbf{D}_\varrho)$, $H(\mathbf{D}_\sigma)$ both have one of the properties* (a) $(\overline{\Omega})$, (b) $(\widetilde{\Omega})$ *and* $\neg(\overline{\Omega})$ *or* (c) $\neg(\widetilde{\Omega})$.

(iii) *With $r_j := -(1/\alpha_j)\log\varrho_j$, $s_j := -(1/\alpha_j)\log\sigma_j$ one of the following conditions is fulfilled*:

(a′) $\lim_{j\to\infty} r_j = 0 = \lim_{j\to\infty} s_j$,

(b′) $0 < \limsup_{j\to\infty} r_j,\ \limsup_{j\to\infty} s_j < \infty$,

(c′) $\limsup_{j\to\infty} r_j = \infty = \limsup_{j\to\infty} s_j$.

2.2. Remark. (i) The implication 2.1(ii)(a) $\Rightarrow$ 2.1(i) has been proved by Meise and Vogt [5] without the restriction $\lim_{j\to\infty}(1/j)\log\alpha_j = 0$. Their proof is much simpler than ours.

(ii) The equivalences (a)$\Leftrightarrow$(a′), (b)$\Leftrightarrow$(b′) and (c)$\Leftrightarrow$(c′) in 2.1 follow from 1.2.4.

(iii) The condition 2.1(iii) is equivalent to: There exist sequences μ, ν of natural numbers with $\lim_{j\to\infty}\mu_j = \infty = \lim_{j\to\infty}\nu_j$ and a constant $C > 1$ such that for all $\varepsilon > 0$

$$\limsup_{j\to\infty} \frac{r_j}{s_{\mu_j}+\varepsilon} < C, \quad \limsup_{j\to\infty}\frac{s_j}{r_{\nu_j}+\varepsilon} < C.$$

2.3. THEOREM. *Let $\Lambda_\infty(\alpha)$ be a nuclear power series space of infinite type with $\lim_{j\to\infty}(1/j)\log\alpha_j = 0$ and $\varrho,\sigma\in\Lambda_\infty(\alpha)$, $\varrho,\sigma > 0$. Then $H(\mathbf{D}_\varrho) \overset{q.d.}{\cong} H(\mathbf{D}_\sigma)$.*

For the proofs of 2.1 and 2.3 we need some preparations concerning the estimation of the volume of certain polytopes in $\mathbf{R}^k$:

2.4. *Notation.* Let $A \in \mathbf{R}^{N\times\mathbf{N}}$ be a matrix with N rows and infinitely many columns with the following properties:

(i) There exists a positive row.

(ii) There exists $n > N$ such that for the columns A_j of A we have $Q_\infty := A_n = A_j$ for all $j \ge n$.

(iii) Let $(e_j)_{1\le j\le N}$ be the natural basis of $\mathbf{R}^N$ and

$$Q_j := \begin{cases} e_j, & 1\le j\le N,\\ A_{j-N}, & N+1\le j\le n+N,\\ Q_\infty, & \text{else.}\end{cases}$$

Then $(Q_{j_1},\ldots,Q_{j_N})$ is a basis of $\mathbf{R}^N$ for every choice of $1\le j_1,\ldots,j_N\le n+N$ pairwise distinct. Furthermore, we define $\mathcal{E}_{N,n} := \{\alpha\subset\{1,\ldots,N+n-1\} \mid \#\alpha = N-1\}$ and for $\alpha\in\mathcal{E}_{N,n}$, $j\notin\alpha$ we denote by $(Q^0_{\alpha,j,l})_{l\in\alpha\cup\{j\}}$ the basis dual to $(Q_l)_{l\in\alpha\cup\{j\}}$.

For $x\in\mathbf{R}$, $K\in\mathbf{N}$ set

$$H_0(x) := \begin{cases} 1, & x>0,\\ 1/2, & x=0,\\ 0, & x<0,\end{cases} \qquad H_K(x) := \begin{cases} \frac{1}{K!}x^K, & x>0,\\ 0, & x\le 0.\end{cases}$$

For $K\ge n$ we consider $V_K : \mathbf{R}^N \to \mathbf{R}$, $V_K(t) := \operatorname{vol}\{x\in\mathbf{R}^K_+ \mid Ax\le t\}$.

2.5. Remark. (i) $\operatorname{supp} V_n = \operatorname{supp} V_K = \sum_{j=1}^{N+n}\mathbf{R}_{\ge 0}Q_j =: \mathcal{K}$.

(ii) $\mathcal{K}^0 := \{y\in\mathbf{R}^N \mid \langle y, Q_j\rangle > 0 \text{ for all } j\in\mathbf{N}\} \ne \emptyset$.

(iii) V_K is continuous.

(iv) $V_K(t) \le V_K(t+\tau)$ for all $\tau\in\mathcal{K}, t\in\mathbf{R}^N$.

2.6. LEMMA. *With the notation of* 2.4,

$$\langle x,y\rangle^n = \sum_{\alpha\in\mathcal{E}_{N,n}}\prod_{j\notin\alpha}\langle x, Q^0_{\alpha,j,j}\rangle\langle Q_j, y\rangle \quad \textit{for all } x,y\in\mathbf{R}^n.$$

Proof. This follows from elementary linear algebra ([13]).

2.7. LEMMA. *Under the hypothesis of* 2.4 *we have for all $t\in\mathbf{R}^N$*

$$V_K(t) = \sum_{\alpha\in\mathcal{E}_{N,n}} c_\alpha H_K(\langle Q^0_{\alpha,\infty,\infty}, t\rangle)\prod_{j\in\alpha} H_0(\langle Q^0_{\alpha,\infty,j}, t\rangle)$$

with $c_\alpha := |\det(Q_{\alpha_1},\ldots,Q_{\alpha_{N-1}},Q_\infty)|^{-1}\prod_{j\notin\alpha}\langle Q^0_{\alpha,j,j}, Q_\infty\rangle$.

Proof. First we have for all $s\in\mathcal{K}^0 + i\mathbf{R}^N$, by 2.6,

$$(1)\quad \widehat{V}_K(s) := \int_{\mathbf{R}^N} V_K(t)e^{-\langle s,t\rangle}\,dt$$

$$= \int_{\mathbf{R}^K_+}\int_{\mathbf{R}^N}\prod_{j=1}^N \chi_{[(Ax)_j,\infty[}(t_j)e^{-\langle s,t\rangle}\,dt\,dx$$

$$= \int\limits_{\mathbf{R}_+^K} \prod_{j=1}^N \Big(\int\limits_{(Ax)_j}^{\infty} e^{-s_j t}\,dt\Big)\,dx = \prod_{j=1}^N \frac{1}{s_j} \int\limits_{\mathbf{R}_+^K} e^{-\langle Ax, s\rangle}\,dx$$

$$= \prod_{j=1}^N \frac{1}{s_j} \prod_{k=1}^K \frac{1}{(A^t s)_k} = \frac{1}{\langle Q_\infty, s\rangle^{K-n+1}} \prod_{j=1}^{N+n-1} \frac{1}{\langle Q_j, s\rangle}$$

$$= \sum_{\alpha\in\mathcal{E}_{N,n}} \prod_{j\notin\alpha} \langle Q_{\alpha,j,j}, Q_\infty\rangle \frac{1}{\langle Q_\infty, s\rangle^{K+1}} \prod_{j\in\alpha} \frac{1}{\langle Q_j, s\rangle}.$$

For $\sigma\in\mathcal{K}^0$ there exist constants $C > B > 0$ such that for all $\tau\in\mathbf{R}^N$, $1\le j\le N$,

$$|\langle Q_j, \sigma+i\tau\rangle| \le B \max_{l=N+1}^{2N} |\langle Q_l, \sigma+i\tau\rangle| \le C \prod_{l=N+1}^{2N} |\langle Q_l, \sigma+i\tau\rangle|$$

and therefore

$$\prod_{j=1}^N |\langle Q_j, \sigma+i\tau\rangle|^{1/N} \le \max_{j=1}^N |\langle Q_j, \sigma+i\tau\rangle| \le C \prod_{l=N+1}^{2N} |\langle Q_l, \sigma+i\tau\rangle|.$$

Since $n > N$ we get then with some constant $D > 0$, for all $\tau\in\mathbf{R}^N$,

$$|\widehat{V}_K(\sigma+i\tau)| \le D \prod_{j=1}^{2N} \frac{1}{|\langle Q_j, \sigma+i\tau\rangle|}$$

$$\le CD \prod_{j=1}^N \frac{1}{|\langle Q_j, \sigma+i\tau\rangle|} \prod_{j=1}^N \frac{1}{|\langle Q_j, \sigma+i\tau\rangle|^{1/N}}$$

$$\le CD \prod_{j=1}^N \frac{1}{(\sigma_j^2+\tau_j^2)^{(1+1/N)/2}},$$

hence

(2) $$\widehat{V}_K(\sigma+i(\cdot)) \in L^1(\mathbf{R}^N) \quad \text{for all } \sigma\in\mathcal{K}^0.$$

From this, (1), 2.5(ii), (iii) and the Fourier inversion formula we conclude that for all $t\in\mathbf{R}^N$

(3) $$V_K(t) = \frac{1}{(2\pi)^N} \int\limits_{\mathbf{R}^N} \widehat{V}_K(\sigma+i\tau) e^{\langle\sigma+i\tau, t\rangle}\,d\tau$$

$$= \frac{1}{(2\pi)^N} \int\limits_{\mathbf{R}^N} \sum_{\alpha\in\mathcal{E}_{N,n}} \prod_{j\in\alpha} \langle Q^0_{\alpha,j,j}, Q_\infty\rangle \frac{1}{\langle Q_\infty, \sigma+i\tau\rangle^{K+1}}$$

$$\times \prod_{j\in\alpha} \frac{1}{\langle Q_j, \sigma+i\tau\rangle} e^{\langle\sigma+i\tau, t\rangle}\,d\tau.$$

Now we define

$$\kappa(\tau) := \begin{cases} \prod_{j=1}^N (1-|\tau_j|), & \tau\in W := [-1,1]^N, \\ 0, & \tau\notin W, \end{cases}$$

and for $\alpha\in\mathcal{E}_{N,n}$, we set $Q_{\alpha,\infty} := (Q_{\alpha_1},\ldots,Q_{\alpha_{N-1}},Q_\infty)$ and $\mathcal{K}_{\alpha,\infty} := \sum_{j\in\alpha\cup\{\infty\}} \mathbf{R}_{\ge 0} Q_j$. From (3) we get for all $t\in\mathbf{R}^N$

(4) $$V_K(t) = \sum_{\alpha\in\mathcal{E}_{N,n}} L_\alpha(t)$$

where

$$L_\alpha(t) := \lim_{\Delta\to\infty} \frac{1}{(2\pi)^N}$$

$$\times \int\limits_{\mathbf{R}^N} \frac{1}{\langle Q_\infty, \sigma+i\tau\rangle^{K+1}} \prod_{j\in\alpha} \frac{1}{\langle Q_j, \sigma+i\tau\rangle} e^{\langle\sigma+i\tau, t\rangle} \kappa\circ Q^t_{\alpha,\infty}\Big(\frac{\tau}{\Delta}\Big)\,d\tau.$$

Then it follows that for all $t\in\mathbf{R}^N$

(5) $$L_\alpha(t) = |\det Q_{\alpha,\infty}|^{-1} H_K(\langle Q^0_{\alpha,\infty,\infty}, t\rangle) \prod_{j\in\alpha} H_0(\langle Q^0_{\alpha,\infty,j}, t\rangle).$$

Indeed, for $s\in\mathcal{K}^0 + i\mathbf{R}^N \subset \mathcal{K}^0_{\alpha,\infty} + i\mathbf{R}^N$ consider

$$\widehat{V}_\alpha(s) := \int\limits_{\mathbf{R}^N} V_\alpha(t) e^{-\langle s,t\rangle}\,dt = \frac{1}{\langle Q_\infty, s\rangle^{k+1}} \prod_{j\in\alpha} \frac{1}{\langle Q_j, s\rangle}.$$

Then we get for $t\in\mathbf{R}^N$, $\sigma\in\mathcal{K}^0$

$$L_{\alpha,\Delta}(t) := \frac{1}{(2\pi)^N} \int\limits_{\mathbf{R}^N} \int\limits_{\mathcal{K}_{\alpha,\infty}} V_\alpha(\xi) e^{-\langle\sigma+i\tau, \xi\rangle}\,d\xi\, e^{\langle\sigma+i\tau, t\rangle} \kappa\circ Q^t_{\alpha,\infty}\Big(\frac{\tau}{\Delta}\Big)\,d\tau$$

$$= \frac{1}{(2\pi)^N} \int\limits_{\mathcal{K}_{\alpha,\infty}} V_\alpha(\xi) e^{-\langle\sigma, \xi-t\rangle} \int\limits_{\Delta(Q^t_{\alpha,\infty})^{-1}W} \kappa\circ Q^t_{\alpha,\infty}\Big(\frac{\tau}{\Delta}\Big) e^{-i\langle\tau, \xi-t\rangle}\,d\tau\,d\xi$$

$$= \frac{1}{(2\pi)^N} \frac{\Delta^N}{|\det Q_{\alpha,\infty}|} \int\limits_{\mathcal{K}_{\alpha,\infty}} V_\alpha(\xi) e^{-\langle\sigma, \xi-t\rangle} \int\limits_W \kappa(\tau) e^{-i\Delta\langle\tau, Q^{-1}_{\alpha,\infty}(\xi-t)\rangle}\,d\tau\,d\xi$$

$$= \frac{1}{(2\pi)^N} \int\limits_{-\Delta Q^{-1}_{\alpha,\infty}(t)+\mathbf{R}^N_+} V_\alpha\Big(\frac{1}{\Delta} Q_{\alpha,\infty}\xi + t\Big) e^{-(1/\Delta)\langle\tau, Q^{-1}_{\alpha,\infty}\xi\rangle} \widehat{\kappa}(\xi)\,d\xi$$

where

$$\widehat{\kappa}(\xi) := \int\limits_W \kappa(\tau) e^{-i\langle\tau,\xi\rangle}\,d\tau = \prod_{j=1}^N \Big(2\frac{1-\cos\xi_j}{\xi_j^2}\Big).$$

We now consider

$$T := \prod_{j\in\alpha} T_j \times \mathbf{R}, \quad T_j := \begin{cases} \mathbf{R}, & \langle Q^0_{\alpha,\infty,j}, t\rangle > 0\,, \\ \mathbf{R}_+, & \langle Q^0_{\alpha,\infty,j}, t\rangle \le 0\,. \end{cases}$$

Since

$$\int_{\mathbf{R}} \frac{1-\cos\xi}{\xi^2}\,d\xi = 2\int_{\mathbf{R}_+} \frac{1-\cos\xi}{\xi^2}\,d\xi$$

and $\operatorname{supp} V_\alpha = \mathcal{K}_{\alpha,\infty}$ it follows that

$$L_\alpha(t) = \lim_{\Delta\to\infty} L_{\alpha,\Delta}(t) = |\det Q_{\alpha,\infty}|^{-1} H_K(\langle Q^0_{\alpha,\infty,\infty}, t\rangle)\frac{1}{(2\pi)^N}\int_T \widehat{\kappa}(\xi)\,d\xi$$

$$= \kappa(0)|\det Q_{\alpha,\infty}|^{-1} H_K(\langle Q^0_{\alpha,\infty,\infty}, t\rangle)\prod_{j\in\alpha} H_0(\langle Q_{\alpha,\infty,j}, t\rangle)\,.$$

Since $\kappa(0) = 1$ we get (5). Then (4) and (5) imply the lemma.

2.8. *Notation.* In the following we consider the functions $V_K, W_K : \mathbf{R}^N \to \mathbf{R}$ defined by

$$V_K(t) := \operatorname{vol}\{x \in \mathbf{R}^K_+ \mid Ax \le t\}, \quad W_K(t) := \operatorname{vol}\{x \in \mathbf{R}^K_+ \mid Bx \le t\}\,,$$

and vectors $(Q_j)_{1\le j\le n_1+N}$, $(R_j)_{1\le j\le n_2+N}$ with $n_1, n_2 > N$ satisfying the hypothesis of 2.4.

2.9. Lemma. *Let $n := \max(n_1, n_2)$, $R_\infty \in (\operatorname{supp} W_n)^\circ$, $Q_\infty \in R_\infty + (\operatorname{supp} V_n)^\circ$. Then the following statements are equivalent:*

(i) $\operatorname{supp} V_n \subset \operatorname{supp} W_n$.

(ii) *There exist $\varepsilon > 0$, $K_0 \in \mathbf{N}$ so that $V_K \le (1+\varepsilon)^{-K} W_K$ for all $K \ge K_0$. One can choose $\varepsilon = \frac{1}{2}\min_{\beta\in S'}\langle R^0_{\beta,\infty,\infty}, Q_\infty - R_\infty\rangle$ where*

$$S' := \Big\{\beta \in \mathcal{E}_{N,n_2} \;\Big|\; \sum_{l\in\beta} \mathbf{R}_{\ge 0} R_l \subset \partial \operatorname{supp} W_n\Big\}.$$

Proof. (ii)⇒(i) is trivial.

(i)⇒(ii). By 2.7 we have for $t \in \mathbf{R}^N$

$$V_K(t) = \sum_{\alpha\in\mathcal{E}_{N,n_1}} c_\alpha H_K(\langle Q^0_{\alpha,\infty,\infty}, t\rangle)\prod_{j\in\alpha} H_0(\langle Q^0_{\alpha,\infty,j}, t\rangle)\,,$$

$$W_K(t) = \sum_{\beta\in\mathcal{E}_{N,n_2}} d_\beta H_K(\langle R^0_{\beta,\infty,\infty}, t\rangle)\prod_{j\in\beta} H_0(\langle R^0_{\beta,\infty,j}, t\rangle)\,,$$

with constants c_α, d_β independent of K. We set $\mathcal{K} := \operatorname{supp} V_n$, $\mathcal{K}' := \operatorname{supp} W_n$ and for $\alpha \in \mathcal{E}_{N,n_1}$, $j \notin \alpha$

$$\mathcal{K}_\alpha := \sum_{l\in\alpha} \mathbf{R}_{\ge 0} Q_l, \quad \mathcal{K}_{\alpha,j} := \mathcal{K}_\alpha + \mathbf{R}_{\ge 0} Q_j\,.$$

Then for $t \in \mathcal{K}$ we have

$$K!V_K(t) \le \sum_{\alpha\in\mathcal{E}_{N,n_1}} |c_\alpha| \max_{\alpha\in\mathcal{E}_{N,n_1}, t\in\mathcal{K}_{\alpha,\infty}\setminus\mathcal{K}_\alpha} \langle Q^0_{\alpha,\infty,\infty}, t\rangle^K$$

and for $\beta \in S'$, $\alpha \in \mathcal{E}_{N,n_1}$,

$$R^0_{\beta,\infty,\infty} = \sum_{j\in\alpha\cup\{\infty\}} \langle R^0_{\beta,\infty,\infty}, Q_j\rangle Q^0_{\alpha,\infty,j}\,.$$

Since $\mathcal{K} \subset \mathcal{K}'$ by hypothesis, we conclude that for $t \in (\mathcal{K}'_{\beta,\infty}\setminus\mathcal{K}'_\beta) \cap (\mathcal{K}_{\alpha,\infty}\setminus\mathcal{K}_\alpha)$

$$\langle R^0_{\beta,\infty,\infty}, t\rangle \ge (1+2\varepsilon)\langle Q^0_{\alpha,\infty,\infty}, t\rangle$$

by the choice of $\varepsilon > 0$. Then it follows that for all $t \in \mathbf{R}^N$

$$(*) \qquad K!V_K(t) \le \sum_{\alpha\in\mathcal{E}_{N,n_1}} |c_\alpha|(1+2\varepsilon)^{-K} \max_{\beta\in S', t\in\mathcal{K}'_{\beta,\infty}\setminus\mathcal{K}'_\beta} \langle R^0_{\beta,\infty,\infty}, t\rangle^K\,.$$

For $\beta \in S'$, $t \in \mathcal{K}'_{\beta,\infty}$ we have $t - \langle R^0_{\beta,\infty,\infty}, t\rangle R_\infty \in \mathcal{K}'$ and by 2.5(iv)

$$W_K(t) \ge W_K(\langle R^0_{\beta,\infty,\infty}, t\rangle R_\infty) = \frac{1}{K!}\sum_{\alpha\in\mathcal{E}_{N,n_2}} d_\alpha \langle R^0_{\beta,\infty,\infty}, t\rangle^K \frac{1}{2^{N-1}}\,.$$

By hypothesis we have $R_\infty \in (\mathcal{K}')^\circ$ and therefore $\sum_{\alpha\in\mathcal{E}_{N,n_2}} d_\alpha = 2^{N-1} n! \times W_n(R_\infty) > 0$. Then $(*)$ implies

$$V_K(t) \le (1+2\varepsilon)^{-K} \sum_{\alpha\in\mathcal{E}_{N,n_1}} |c_\alpha| \max_{\beta\in S', t\in\mathcal{K}'_{\beta,\infty}\setminus\mathcal{K}'_\beta} \frac{1}{K!}\langle R^0_{\beta,\infty,\infty}, t\rangle^K$$

$$\le 2^{N-1}\frac{\sum_{\alpha\in\mathcal{E}_{N,n_1}} |c_\alpha|}{\sum_{\alpha\in\mathcal{E}_{N,n_2}} d_\alpha}(1+2\varepsilon)^{-K} W_K(t) \le (1+\varepsilon)^{-K} W_K(t)$$

where

$$K \ge \max\left(n, \frac{\log(2^{N-1}\sum_{\alpha\in\mathcal{E}_{N,n_1}} |c_\alpha| / \sum_{\alpha\in\mathcal{E}_{N,n_2}} d_\alpha)}{\log(1+\varepsilon/(1+\varepsilon))}\right) =: K_0\,.$$

2.10. *Notation.* Let $\varrho \in \Lambda_R(\alpha)$, $R = 1, \infty$ and let $(\nu_p)_{p\in\mathbf{N}}$, $(r_{j,p})_{j\in\mathbf{N}}$ be the sequences from 1.2.3. For a subsequence $(p_s)_{s\in\mathbf{N}}$ of $\mathbf{N}$ define $\nu_s := \nu_{p_s}$, and for $\tau, t \in \mathbf{R}^n$, $\tau < t$, define $R_j \in \mathbf{R}^{2n}$, $1 \le j \le N$, by

$$R_{j,2k} := r_{j,p_{t_k+5}} - r_{j,p_{\tau_k+1}}, \quad R_{j,2k-1} := r_{j,p_{t_k+4}} - r_{j,p_{\tau_k+2}}\,.$$

Moreover, set

$$C_R(p,\nu,n,\tau,t) := \sum_{k=1}^{2n} \mathbf{R}_{\ge 0}(-1)^{k+1} e_k + \sum_{j=1}^{N} \mathbf{R}_{\ge 0} R_j \subset \mathbf{R}^{2n}$$

where $(e_k)_{1\le k\le 2n}$ denotes the canonical basis in $\mathbf{R}^{2n}$.

For $\sigma \in \Lambda_R(\alpha)$ define $S_j \in \mathbf{R}^{2n}$, $1 \le j \le N$, by

$$S_{j,2k} := s_{j,p_{t_k+6}} - s_{j,p_{\tau_k}}, \qquad S_{j,2k-1} := s_{j,p_{t_k+3}} - s_{j,p_{\tau_k+3}},$$

and $C_S(p,\nu,n,\tau,t)$ respectively.

Here $N \in \mathbf{N}$ is such that for all $j \ge N$, $R_\infty := R_j$ and $S_\infty := S_j$ are independent of j. For $m \in \mathbf{M}$ set $a_{m,p} := \exp(\alpha r_p \mid m)$ and $b_{m,p} := \exp(\alpha s_p \mid m)$. Then we have:

2.11. Proposition. *Let $\lim_{j\to\infty}(1/j)\log\alpha_j = 0$ and let $(p_s)_{s\in\mathbf{N}}$, $(\nu_s)_{s\in\mathbf{N}}$ be sequences with $p_{s+1} > 3p_s$ such that $R_j \in (C_S(p,\nu,n,\tau,t))^\circ$ for all $\kappa \in \mathbf{N}$, $(\tau,t) \in \mathbf{N}^\kappa \times \mathbf{N}^\kappa$, $\tau < t$, $j \in \mathbf{N}$. Then $H(\mathbf{D}_\varrho) \overset{q.d.}{\hookrightarrow} H(\mathbf{D}_\sigma)$.*

Proof. We show that 1.1.11(iii) is fulfilled. For $\mu \in \mathbf{N}_{>1}$ we set

$$M_\mu := \left\{(\kappa,\tau,t) \;\middle|\; 1 \le \kappa \binom{n}{2},\ (\tau,t) \in \mathbf{N}^\kappa \times \mathbf{N}^\kappa,\ \max_{j=1}^{\kappa} t_j = \mu,\ \tau < t\right\}$$

and prove

$$(*) \quad \exists (C_\mu)_{\mu\in\mathbf{N}_{>1}}\ \forall \mu\ \exists D = D_\mu\ \forall (\kappa,\tau,t) \in M_\mu,\ (u,v) \in \mathbf{N}^\kappa \times \mathbf{N}^\kappa,\ v \le u+1:$$
$$\#\bigcap_{k=1}^{\kappa} A_{\tau_k,t_k,u_k,v_k} \le \frac{1}{D(|u|+|v|)^{1+2\kappa}}\,\#\bigcap_{k=1}^{\kappa} B_{\tau_k,t_k,u_k,v_k}.$$

One can choose

$$D_\mu = 2^{\mu+\binom{\mu}{2}} \max_{1\le\kappa\le\binom{\mu}{2}} \sum_{u\in\mathbf{N}^{2\kappa}} \frac{1}{|u|^{1+2\kappa}}$$

and C_μ such that

$$(1) \qquad \forall m \in \mathbf{M}: \quad \lim_{\mu\to\infty} \frac{1}{C_\mu} \log \frac{a_{m,p_{\mu+7}}}{a_{m,p_1}} = 0.$$

Let $\mu \in \mathbf{N}_{>1}$. By hypothesis we have

$$\forall \kappa \in \mathbf{N},\ (\tau,t) \in \mathbf{N}^\kappa \times \mathbf{N}^\kappa,\ \tau < t,\ j \in \mathbf{N}: \quad R_j \in (C_S(p,\nu,\kappa,\tau,t))^\circ.$$

Then it follows that there exist $1 > \gamma_\mu > 0$ such that

$$(2) \qquad \forall (\kappa,\tau,t) \in M_\mu: \quad (I_{2\kappa} - \gamma_\mu \operatorname{diag}((-1)^{k+1})_{1\le k\le 2\kappa})C_R \subset C_S.$$

We choose $0 < \delta_\mu < 1/\nu_{\mu+6}$ with

$$(3) \qquad \delta_\mu < \frac{1}{4}\varepsilon_\mu := \frac{1}{16} \min_{(\kappa,\tau,t)\in M_\mu} \min_{\beta\in S'} \langle S^0_{\beta,\infty,\infty}, R_\infty - S_\infty\rangle > 0$$

where S' is defined in 2.9. This is possible since $p_{s+1} > 3p_s$ by hypothesis

and in the case $\mathbf{D}_\varrho, \mathbf{D}_\sigma \subset \Lambda_1(\alpha)'_b$, by 2.10,

$$(R_\infty - S_\infty)_{2k-1} = \left(\frac{1}{p_{\tau_k+2}} - \frac{1}{p_{t_k+4}}\right) - \left(\frac{1}{p_{\tau_k+3}} - \frac{1}{p_{t_k+3}}\right) > 0,$$
$$(R_\infty - S_\infty)_{2k} = \left(\frac{1}{p_{\tau_k+1}} - \frac{1}{p_{t_k+5}}\right) - \left(\frac{1}{p_{\tau_k}} - \frac{1}{p_{t_k+6}}\right) > 0,$$

and therefore

$$(4) \qquad R_\infty - S_\infty \in C^\circ.$$

In the case $\mathbf{D}_\varrho, \mathbf{D}_\sigma \subset \Lambda_\infty(\alpha)'_b$, (4) follows in the same way.

Let δ_μ be so that

$$(5) \qquad \begin{aligned} &\frac{1+\varepsilon_\mu}{\varepsilon_\mu}\left(1 - \frac{(1+\delta_\mu)^2}{1+\varepsilon_\mu}(1+3\delta_\mu)\right) \ge 1 - \gamma_\mu, \\ 0 < &\frac{1+\varepsilon_\mu}{\varepsilon_\mu}\left(1 + \delta_\mu - \frac{(1-\delta_\mu)^2}{1+\varepsilon_\mu}\right) \le 1 + \gamma_\mu. \end{aligned}$$

From (2) it follows that for $(\kappa,\tau,t) \in M_\mu$, $x \in C_R$,

$$\sum_{k=1}^{\kappa} \left(\frac{1+\varepsilon_\mu}{\varepsilon_\mu}\left(1 - \frac{(1+\delta_\mu)^2}{1+\varepsilon_\mu}(1+3\delta_\mu)\right) x_{2k-1}e_{2k-1}\right.$$
$$\left. + \frac{1+\varepsilon_\mu}{\varepsilon_\mu}\left(1 + \delta_\mu - \frac{(1-\delta_\mu)^2}{1+\varepsilon_\mu}\right) x_{2k}e_{2k}\right)$$
$$\in (I_{2\kappa} - \gamma_\mu \operatorname{diag}((-1)^{k+1})_{1\le k\le 2\kappa})C_R \subset C_S.$$

Since C_S is positively homogeneous we have

$$(6) \qquad \sum_{k=1}^{\kappa} \left(\left(1 - \frac{(1+\delta_\mu)^2}{1+\varepsilon_\mu}(1+3\delta_\mu)\right) x_{2k-1}e_{2k-1}\right.$$
$$\left. + \left(1 + \delta_\mu - \frac{(1-\delta_\mu)^2}{1+\varepsilon_\mu}\right) x_{2k}e_{2k}\right) \in C_S.$$

We set

$$a_{\tau_k+1,t_k+5} := (\alpha_j R_{j,2k})_{j\in\mathbf{N}}, \qquad a_{\tau_k+2,t_k+4} := (\alpha_j R_{j,2k-1})_{j\in\mathbf{N}},$$
$$b_{\tau_k,t_k+6} := (\alpha_j S_{j,2k})_{j\in\mathbf{N}}, \qquad b_{\tau_k+3,t_k+3} := (\alpha_j S_{j,2k-1})_{j\in\mathbf{N}}$$

and

$$(7) \qquad E_\mu := \sup_{\substack{j\in\mathbf{N} \\ s<t<\mu+6 \\ \sigma<\tau<\mu+6}} \max\left(\frac{b_{s,t,j}}{b_{\sigma,\tau,j}}, \frac{a_{s,t,j}}{a_{\sigma,\tau,j}}, \frac{1}{\alpha_j} b_{s,t,j}, \frac{1}{\alpha_j} a_{s,t,j}\right).$$

We choose

(8) $\delta^A_{j,k}, \delta^B_{j,k} \in\ ]-\delta_\mu, \delta_\mu[$ such that for all $(\kappa,\tau,t) \in M_\mu$ and for

$$\widetilde{R}_{j,k} := (1+\delta^A_{j,k})R_{j,k}, \qquad \widetilde{S}_{j,k} := (1+\delta^B_{j,k})S_{j,k}$$

the following holds:

(i) $C_{\widetilde{R}} = C_R$, $C_{\widetilde{S}} = C_S$.

(ii) The first $\nu_{\mu+5}$ ($\nu_{\mu+6}$) columns of $\widetilde{A}$ ($\widetilde{B}$) are in general position.

(iii) $\widetilde{R}_\infty \in \widetilde{S}_\infty + C^\circ_{\widetilde{S}}$, $\widetilde{S}_\infty \in C^\circ_{\widetilde{S}}$, $\widetilde{R}_j \in C^\circ_{\widetilde{S}}$ for all $j \in \mathbf{N}$.

(iv) $\min_{\beta\in S'}\langle S^0_{\beta,\infty,\infty}, \widetilde{R}_\infty - \widetilde{S}_\infty\rangle \geq \varepsilon_\mu$.

This is possible by (4) and a simple perturbation argument. Then the hypotheses of 2.9 are fulfilled for $\widetilde{A}$, $\widetilde{B}$ modulo a diagonal transformation. We choose $K_\mu \in \mathbf{N}$ by 2.9(ii) and define for $v \in \mathbf{N}$

$$K(v) := K_\mu + [(1/\delta_\mu)\log(D_\mu v^{1+2\binom{\mu}{2}})]\,.$$

By hypothesis we have $\lim(1/j)\log\alpha_j = 0$. So there exists $F_\mu > 0$ such that

$$E_\mu \sum_{j=1}^{K(v)} \alpha_j \leq \frac{\delta_\mu}{16\binom{\mu}{2}E_\mu} v + F_\mu \quad \text{for all } v \in \mathbf{N}. \tag{9}$$

We choose

$$C_\mu > 8\frac{E_\mu}{\delta_\mu}F_\mu + \mu \max_{m\in\mathbf{N}_0^\mu,\,|m|\leq\mu} \log\frac{a_{m,p_{\mu+7}}}{a_{m,p_1}} \tag{10}$$

and show that $(C_\mu)_{\mu\in\mathbf{N}}$ satisfies (*).

Let $\mu \in \mathbf{N}_{>1}$, $(\kappa,\tau,t) \in M_\mu$, $(u,v) \in \mathbf{N}^{2\kappa}$, $v \leq u+1$. We consider the case

$$\bigcap_{k=1}^{\kappa} A_{\tau_k,t_k,u_k,v_k}$$
$$= \bigcap_{k=1}^{\kappa}\{m \mid C_{t_k}u_k < (a_{\tau_k+1,t_k+5} \mid m),\ (a_{\tau_k+2,t_k+4} \mid m) \leq C_{t_k}v_k\} \neq \emptyset$$

(otherwise (*) is trivial). Then it follows that

$$\max_{k=1}^{\kappa} C_t u_k \leq E_\mu \min_{k=1}^{\kappa} C_{t_k}v_k \tag{11}$$

by (6). For

$$K = K(u,v) = K_\mu + [(1/\delta_\mu)\log(D_\mu(|u|+|v|)^{1+2\kappa})] \tag{12}$$

we get by (9)–(11) for $\mu = \max_{k=1}^{\kappa} t_k$

$$E_\mu \sum_{j=1}^{K} \alpha_j \leq \frac{\delta_\mu}{16\binom{\mu}{2}E_\mu}(|u|+|v|) + F_\mu \leq \frac{\delta_\mu}{4E_\mu}\max_{k=1}^{\kappa} u_k + F_\mu \tag{13}$$
$$\leq \frac{\delta_\mu}{2E_\mu}\max_{k=1}^{\kappa} C_{t_k}u_k \leq \frac{\delta_\mu}{2}\min_{k=1}^{\kappa} C_{t_k}v_k \leq \delta_\mu \min_{k=1}^{\kappa} C_{t_k}u_k\,.$$

Define $\mathbf{M}_k := \{m \in \mathbf{M} \mid m_1 = \ldots = m_k = 0\}$. Then we have

$$\#\bigcap_{k=1}^{\kappa}\{m \mid C_{t_k}u_k < (a_{\tau_k+1,t_k+5} \mid m),\ (a_{\tau_k+2,t_k+4} \mid m) \leq C_{t_k}v_k\}$$
$$= \sum_{n\in\mathbf{M}_K} \#\bigcap_{k=1}^{\kappa}\{m \in \mathbf{N}_0^K \mid C_{t_k}u_k - (a_{\tau_k+1,t_k+5} \mid n) < (a_{\tau_k+1,t_k+5} \mid m),$$
$$(a_{\tau_k+2,t_k+4} \mid m) \leq C_{t_k}v_k - (a_{\tau_k+2,t_k+4} \mid n)\}$$
$$\leq \sum_{n\in\mathbf{M}_K} \operatorname{vol}\Big(\bigcap_{k=1}^{\kappa}\Big\{x \in \mathbf{R}^K_{\geq 0}\ \Big|\ C_{t_k}u_k - (a_{\tau_k+1,t_k+5} \mid n) < (a_{\tau_k+1,t_k+5} \mid x),$$
$$(a_{\tau_k+2,t_k+4} \mid x) \leq C_{t_k}v_k - (a_{\tau_k+2,t_k+4} \mid n) + \sum_{j=1}^{K} a_{\tau_k+2,t_k+4,j}\Big\}\Big)$$
$$\overset{(8)}{\leq} \sum_{n\in\mathbf{M}_K} \operatorname{vol}\Big(\bigcap_{k=1}^{\kappa}\Big\{x \in \mathbf{R}^K_{\geq 0}\ \Big|\ (1-\delta_\mu)(C_{t_k}u_k - (a_{\tau_k+1,t_k+5} \mid n))$$
$$< (\widetilde{a}_{\tau_k+1,t_k+5} \mid x),$$
$$(\widetilde{a}_{\tau_k+2,t_k+4} \mid x) \leq (1+\delta_\mu)(C_{t_k}v_k - (a_{\tau_k+2,t_k+4} \mid n)) + \sum_{j=1}^{K} a_{\tau_k+2,t_k+4,j}\Big\}\Big)$$
$$\overset{(3),2.9(\mathrm{ii})}{\leq} \sum_{n\in\mathbf{M}_K} \operatorname{vol}\Big(\bigcap_{k=1}^{\kappa}\Big\{x \in \mathbf{R}^K_{\geq 0}\ \Big|\ \frac{1-\delta_\mu}{1+\varepsilon_\mu}(C_{t_k}u_k - (a_{\tau_k+1,t_k+5} \mid n))$$
$$< (\widetilde{b}_{\tau_k,t_k+6} \mid x),$$
$$(\widetilde{b}_{\tau_k+3,t_k+3} \mid x) \leq \frac{1+\delta_\mu}{1+\varepsilon_\mu}\Big(C_{t_k}v_k - (a_{\tau_k+2,t_k+4} \mid n) + \sum_{j=1}^{K} a_{\tau_k+2,t_k+4,j}\Big)\Big\}\Big)$$
$$\overset{(13),(8)}{\leq} \frac{1}{D_\mu(|u|+|v|)^{1+2\kappa}}$$
$$\times \sum_{n\in\mathbf{M}_k} \operatorname{vol}\Big(\bigcap_{k=1}^{\kappa}\Big\{x \in \mathbf{R}^K_{\geq 0}\ \Big|\ \frac{(1-\delta_\mu)^2}{1+\varepsilon_\mu}(C_{t_k}u_k - (a_{\tau_k+1,t_k+5} \mid n)) < (b_{\tau_k,t_k+6} \mid x),$$
$$(b_{\tau_k+3,t_k+3} \mid x)$$
$$\leq \frac{(1+\delta_\mu)^2}{1+\varepsilon_\mu}e^{\delta_\mu}\Big(C_{t_k}v_k - (a_{\tau_k+2,t_k+4} \mid n) + \sum_{j=1}^{K} a_{\tau_k+2,t_k+4,j}\Big)\Big\}\Big) =: T\,.$$

By hypothesis we have $p_{s+1} > 3p_s$ and for $j > \nu_{\mu+6}$, $1 \leq k \leq \kappa$, in the case $\mathbf{D}_\varrho, \mathbf{D}_\sigma \subset \Lambda_1(\alpha)'_b$

$$\text{(14)}\quad a_{\tau_k+2,t_k+4,j} = \alpha_j\left(\frac{1}{p_{\tau_k+2}} - \frac{1}{p_{t_k+4}}\right) \le 2\alpha_j\left(\frac{1}{p_{\tau_k+3}} - \frac{1}{p_{t_k+3}}\right) = 2b_{\tau_k+3,t_k+3,j}\,,$$

$$2a_{\tau_k+1,t_k+5,j} = 2\alpha_j\left(\frac{1}{p_{\tau_k+1}} - \frac{1}{p_{t_k+5}}\right) \le \alpha_j\left(\frac{1}{p_{\tau_k}} - \frac{1}{p_{t_k+6}}\right) = b_{\tau_k,t_k+6}\,.$$

In the case $\mathbf{D}_\varrho, \mathbf{D}_\sigma \subset \Lambda_1(\alpha)'_b$, (14) follows in the same way. So we get, with $d' := 1/(D_\mu(|u|+|v|)^{1+2\kappa})$,

$$T \overset{(13),(12),(7)}{\le} d' \sum_{n\in\mathbf{M}} \mathrm{vol}\Big(\bigcap_{k=1}^{\kappa}\Big\{x\in\mathbf{R}^K_{\ge 0} \;\Big|\; \frac{(1-\delta_\mu)^2}{1+\varepsilon_\mu}C_{t_k}u_k < (b_{\tau_k,t_k+6} \mid x+n),$$
$$(b_{\tau_k+3,t_k+3} \mid x+n) \le \frac{(1+\delta_\mu)^2}{1+\varepsilon_\mu}(1+3\delta_\mu)C_{t_k}v_k\Big\}\Big)$$

$$\overset{(6),2.5(\mathrm{iv})}{\le} d' \sum_{n\in\mathbf{M}_k} \mathrm{vol}\Big(\bigcap_{k=1}^{\kappa}\{x\in\mathbf{R}^K_{\ge 0} \mid (1+\delta_\mu)C_{t_k}u_k < (b_{\tau_k,t_k+6} \mid x+n)\,,$$
$$(b_{\tau_k+3,t_k+3} \mid x+n) \le C_{t_k}v_k\}\Big)$$

$$\le d' \sum_{n\in\mathbf{M}_k} \#\bigcap_{k=1}^{\kappa}\Big\{m\in\mathbf{N}_0^K \mid (1+\delta_\mu)C_{t_k}u_k - \sum_{j=1}^{K} b_{\tau_k,t_k+6,j}$$
$$< (b_{\tau_k,t_k+6} \mid m+n), (b_{\tau_k+3,t_k+3} \mid m+n) \le C_{t_k}v_k\Big\}$$

$$\overset{(13),(7)}{\le} d'\,\#\bigcap_{k=1}^{\kappa}\{m\in\mathbf{M} \mid C_{t_k}u_k < (b_{\tau_k,t_k+6} \mid m),\ (b_{\tau_k+3,t_k+3} \mid m) \le C_{t_k}v_k\}$$

and $(*)$ is verified.

Now take I as in 1.1.11(iii) and $\mathcal{K}\in\mathcal{E}(\mathcal{E}(I))$ with $\max_{\nu=(\tau,t,u)\in J} t = \mu$ for all $J\in\mathcal{K}$. Then we have

$(**)$ $\exists I_\mu,\ \#I_\mu \le 2^{\binom{\mu}{2}}\ \forall j\in I_\mu\ \exists N = N_j\in\mathbf{N}, (\kappa_j,\tau_j,t_j)\in M_\mu\,,$
$\forall 1\le l\le N_j\ \exists(u_{l,j},v_{l,j})\in\mathbf{N}^{\kappa_j}\times\mathbf{N}^{\kappa_j},\ u_{l,j}\le v_{l,j}+1$
pairwise different :

$$\bigcup_{J\in\mathcal{K}}\bigcap_{\nu\in J}A_\nu = \bigcup_{J\in I_\mu}\bigcup_{l=1}^{N_j}\bigcap_{k=1}^{\kappa_j}A_{\tau_{j,k},t_{j,k},u_{l,j,k},v_{l,j,k}}\,.$$

To prove $(**)$ we consider on $\mathcal{K}$ the relation

$$J_1\sim J_2 \;:\Leftrightarrow\; \pi(J_1)=\pi(J_2),\qquad \pi = \begin{pmatrix}100\\010\end{pmatrix}.$$

Let $(T_j)_{j\in I_\mu}$ be an enumeration of $\mathcal{K}/\sim$. Then we have

$$\#I_\mu = \#\{\pi(J)\mid J\in\mathcal{K}\} \le \#\mathfrak{P}(\{(\tau,t)\in\mathbf{N}^2\mid \tau<t\le\mu\}) \le 2^{\binom{\mu}{2}}$$

where $\mathfrak{P}(M) := \{A\mid M\supset A\}$. For $J\in\mathcal{K}$ we set

$$\widetilde{J} := \{(\tau,t,\max_{(\tau,t,u)\in J}u,\ \min_{(\tau,t,u)\in J}(u+1))\mid (\tau,t)\in\pi(J)\}\,.$$

Then

$$\#\widetilde{J} = \#\pi(J) \le \#\{(\tau,t)\in\mathbf{N}^2\mid\tau<t\le\mu\} \le \binom{\mu}{2}$$

and $v\le u+1$ for $(\tau,t,u,v)\in\widetilde{J}$. For $j\in I_\mu$ let $(J_{j,l})_{1\le l\le N_j}$ be an enumeration of T_j. Then $\kappa_j := \#\widetilde{J}_{j,l} = \#\pi(J_{j,l}) \le \binom{\mu}{2}$ is independent of l. Let $(\tau_{j,k},t_{j,k},u_{l,j,k},v_{l,j,k})_{1\le k\le\kappa_j}$ be enumerations of $\widetilde{J}_{j,l}$. Then

$$\bigcup_{J\in\mathcal{K}}\bigcap_{\nu\in J}A_\nu = \bigcup_{J\in I_\mu}\bigcup_{l=1}^{N_j}\bigcap_{\nu\in J_{j,l}}A_\nu = \bigcup_{J\in I_\mu}\bigcup_{l=1}^{N_j}\bigcap_{\nu\in\widetilde{J}_{j,l}}A_\nu$$
$$= \bigcup_{J\in I_\mu}\bigcup_{l=1}^{N_j}\bigcap_{k=1}^{\kappa_j}A_{\tau_{j,k},t_{j,k},u_{l,j,k},v_{l,j,k}}\,.$$

So $(**)$ is verified. For $\bigcup_{J\in\mathcal{K}}\bigcap_{\nu\in J}B_\nu$ we get an analogous representation.

By $(*)$ and $(**)$ we get

$$(***)\quad \#\bigcup_{J\in\mathcal{K}}\bigcap_{\nu\in J}A_\nu \le 2^{\binom{\mu}{2}}\max_{j\in I_\mu}\sum_{l=1}^{N_j}\#\bigcap_{k=1}^{\kappa_j}A_{\tau_{j,k},t_{j,k},u_{l,j,k},v_{l,j,k}}$$
$$\le 2^{\binom{\mu}{2}}\max_{j\in I_\mu}\sum_{l=1}^{N_j}\frac{1}{D_\mu}\frac{1}{(|u_{l,j}|+|v_{l,j}|)^{1+2\kappa_j}}\#\bigcap_{k=1}^{\kappa_j}B_{\tau_{j,k},t_{j,k},u_{l,j,k},v_{l,j,k}}$$
$$\le 2^{\binom{\mu}{2}}\max_{j\in I_\mu}\Big(\sum_{l=1}^{N_j}\frac{1}{D_\mu}\frac{1}{(|u_{l,j}|+|v_{l,j}|)^{1+2\kappa_j}}\Big)\max_{l=1}^{N_j}\#\bigcap_{k=1}^{\kappa_j}B_{\tau_{j,k},t_{j,k},u_{l,j,k},v_{l,j,k}}$$
$$\le \frac{1}{2^\mu}\#\bigcup_{J\in I_\mu}\bigcup_{l=1}^{N_j}\bigcap_{k=1}^{\kappa_j}B_{\tau_{j,k},t_{j,k},u_{l,j,k},v_{l,j,k}} = \frac{1}{2^\mu}\#\bigcup_{J\in\mathcal{K}}\bigcap_{\nu\in J}B_\nu\,.$$

For an arbitrary $\mathcal{K}\in\mathcal{E}(\mathcal{E}(I))$ it follows from $(***)$ that

$$\#\bigcup_{J\in\mathcal{K}}\bigcap_{\nu\in J}A_\nu = \bigcup_{\mu=1}^{\infty}\ \bigcup_{J\in\mathcal{K},\max_{(\tau,t,u)\in J}t=\mu}\ \bigcap_{\nu\in J}B_\nu$$
$$\le \sum_{\mu=1}^{\infty}\frac{1}{2^\mu}\#\bigcup_{J\in\mathcal{K},\max_{(\tau,t,u)\in J}t=\mu}\ \bigcap_{\nu\in J}B_\nu \le \#\bigcup_{J\in\mathcal{K}}\bigcap_{\nu\in J}B_\nu\,,$$

which completes the proof.

2.12. LEMMA. *Let $\mathbf{D}_\varrho, \mathbf{D}_\sigma \subset \Lambda_1(\alpha)'_b$ be open polydiscs. With the notations of* 1.2.3 *let $(\mu_j)_{j\in\mathbf{N}}$ be a sequence of natural numbers with $\lim_{j\to\infty}\mu_j = \infty$ and let $C > 1$ be a constant such that $\limsup_{j\to\infty} r_j/(s_{\mu_j}+\varepsilon) < C$ for all $\varepsilon > 0$. Then, with the notations of* 2.10, *there exist sequences $(p_s)_{s\in\mathbf{N}}$, $(\nu_s)_{s\in\mathbf{N}}$ so that*

$$\forall n \in \mathbf{N},\ \tau, t \in \mathbf{N}^n,\ \tau < t: \qquad R_j \in (C_S(p,\nu,n,\tau,t))^\circ .$$

Proof. We choose inductively sequences $(p_l)_{l\in\mathbf{N}}$, $(\nu_l)_{l\in\mathbf{N}}$ such that for all $l \in \mathbf{N}$

(i) $p_{l+1} > 2Cp_l(1+p_l)$,

(ii) $\forall j > \nu_l: \qquad \dfrac{r_j}{s_{\mu_j} + \dfrac{1}{p_l(1+p_l)}} \le C$ and $-\dfrac{1}{2p_l(1+p_l)} < \min(r_j, s_j)$,

(iii) $\{j \mid \nu_{l-1} < j \le \nu_l\} \subset \{j \mid \nu_{l-2} < \mu_j \le \nu_{l+1}\}$,
$\{j \mid \nu_{l-1} < \mu_j \le \nu_l\} \subset \{j \mid \nu_{l-2} < j \le \nu_{l+1}\}$.

Let now $n \in \mathbf{N}$, $\tau, t \in \mathbf{N}^n$, $\tau < t$, be given. We show

$(*)$ $\quad \forall j \in \mathbf{N}, 1 \le k, \kappa \le n:$

$$T = T_{j,k,\kappa} := \frac{R_{j,2k} - S_{\infty,2k}}{S_{\mu_j,2k}} \cdot \frac{S_{\mu_j,2\kappa-1}}{R_{j,2\kappa-1} - S_{\infty,2\kappa-1}} < 1 .$$

We now consider 4 cases:

1. $\nu_{\tau_k+1} < j \le \nu_{t_k+5}$ and $\nu_{\tau_\kappa+3} < \mu_j \le \nu_{t_\kappa+3}$. Then it follows by (iii) that $\nu_{\tau_k} < \mu_j \le \nu_{t_k+6}$ and $\nu_{\tau_\kappa+2} < j \le \nu_{t_\kappa+4}$ and we get

(1) $\quad R_{j,2k} - S_{\infty,2k} \le r_j \quad$ by (i),

(2) $\quad S_{\mu_j,2k} > 0 \quad$ by (ii),

(3) $\quad 0 < S_{\mu_j,2\kappa-1} \le S_{\mu_j} + \dfrac{1}{p_{\tau_\kappa+3}} \quad$ by (ii),

(4) $\quad R_{j,2\kappa-1} - S_{\infty,2\kappa-1} > 0 \quad$ by (i) and (ii).

From (1)–(4) it follows in the case $\tau_\kappa + 3 > \tau_k$ that

$$T \le \frac{r_j}{r_j + \dfrac{1}{p_{\tau_\kappa+2}(1+p_{\tau_\kappa+2})}} \cdot \frac{S_{\mu_j} + \dfrac{1}{p_{\tau_\kappa+3}}}{S_{\mu_j} + \dfrac{1}{p_{\tau_\kappa}(1+p_{\tau_k})}} < 1 \quad \text{by (i)},$$

In the other cases:

2. $j \notin \{j \mid \nu_{\tau_k+1} < j \le \nu_{t_k+5}\}$ and $\nu_{\tau_\kappa+3} < \mu_j \le \nu_{t_\kappa+3}$,
3. $\nu_{\tau_k+1} < j \le \nu_{t_k+5}$ and $j \notin \{j \mid \nu_{\tau_\kappa+3} < \mu_j \le \nu_{t_\kappa+3}\}$,
4. $j \notin \{j \mid \nu_{\tau_k+1} < j \le \nu_{t_k+5}\}$ and $j \notin \{j \mid \nu_{\tau_\kappa+3} < \mu_j \le \nu_{t_\kappa+3}\}$,

the estimate $T < 1$ can be proved in an analogous way. (For more details see [13].) So $(*)$ is proved and therefore there exists $\gamma_j < 0$ so that

$$0 < \min_{\kappa=1}^{n} \frac{S_{\mu_j,2\kappa-1}}{R_{j,2\kappa-1} - S_{\infty,2\kappa-1}} > \gamma_j > \max_{k=1}^{n} \frac{R_{j,2k} - S_{\infty,2k}}{S_{\mu_j,2k}} .$$

Set $\lambda_{j,k} := (-1)^{k+1}(R_{j,k} - S_{\infty,k} - \gamma_j S_{\mu_j,k}) > 0$ for all $j, 1 \le k \le 2n$. Then for all $j \in \mathbf{N}$ we have

$$R_j = \sum_{k=1}^{2n} (-1)^{k+1}\lambda_{j,k} e_k + S_\infty + \gamma_j S_{\mu_j} \in (C_S(p,\nu,n,\tau,t))^\circ .$$

2.13. LEMMA. *Let $\mathbf{D}_\varrho, \mathbf{D}_\sigma \subset \Lambda_1(\alpha)'_b$ be open polydiscs. With the notations of* 1.2.3 *and* 2.10 *there exist sequences $(p_s)_{s\in\mathbf{N}}$, $(\nu_s)_{s\in\mathbf{N}}$ such that*

$$\forall n \in \mathbf{N},\ \tau, t \in \mathbf{N}^n,\ \tau < t: \qquad R_j \in (C_S(p,\nu,n,\tau,t))^\circ .$$

Proof. The procedure is similar to the proof of 2.12 (see [13]).

2.14. Proof of 2.1. (i)⇒(ii) is obvious. (ii)⇒(iii) follows from 2.2(ii). (iii)⇒(i′). By 2.12 and 2.2(iii) the hypotheses of 2.11 are fulfilled. So $H(\mathbf{D}_\varrho) \overset{\text{q.d.}}{\hookrightarrow} H(\mathbf{D}_\sigma)$ and by symmetry $H(\mathbf{D}_\sigma) \overset{\text{q.d.}}{\hookrightarrow} H(\mathbf{D}_\varrho)$. By 1.1.9 we have finished the proof.

(i′)⇒(i) is trivial.

2.15. Proof of 2.3. By 2.13 the hypotheses of 2.11 are fulfilled and so $H(\mathbf{D}_\varrho) \overset{\text{q.d.}}{\hookrightarrow} H(\mathbf{D}_\sigma)$ and by symmetry $H(\mathbf{D}_\sigma) \overset{\text{q.d.}}{\hookrightarrow} H(\mathbf{D}_\varrho)$. By 1.1.9, $H(\mathbf{D}_\varrho) \overset{\text{q.d.}}{\cong} H(\mathbf{D}_\sigma)$.

2.16. Remark. Using diametral dimension one can also give equivalent conditions for the isomorphism $H(\mathbf{D}_\varrho) \cong H(\mathbf{D}_\sigma)$ for polydiscs $\mathbf{D}_\varrho \subset \Lambda_1(\alpha)'_b \times \Lambda_\infty(\beta)'_b$ and $\mathbf{D}_\sigma \subset \Lambda_1(\widetilde{\alpha})'_b \times \Lambda_\infty(\widetilde{\beta})'_b$ and certain sequences $\varrho, \sigma, \alpha, \widetilde{\alpha}, \widetilde{\beta}$ (see [13]).

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Weighted weak type inequalities for certain maximal functions

by

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Abstract. We give an A_p type characterization for the pairs of weights (w,v) for which the maximal operator $Mf(y) = \sup \frac{1}{b-a}\int_a^b |f(x)|\,dx$, where the supremum is taken over all intervals $[a,b]$ such that $0 \le a \le y \le b/\psi(b-a)$, is of weak type (p,p) with weights (w,v). Here ψ is a nonincreasing function such that $\psi(0)=1$ and $\psi(\infty)=0$.

The Poisson integral for the Hermite expansion of a function f is given by

$$P_r f(y) = \int_{\mathbf{R}} P(r,y,z) f(z) e^{-z^2}\,dz \tag{1}$$

where

$$P(r,y,z) = \frac{1}{\sqrt{\pi(1-r^2)}} e^{-(r^2y^2-2ryz+r^2z^2)/(1-r^2)}.$$

C. Calderón [C] and B. Muckenhoupt [M1] proved that the maximal operator

$$P^* f(y) = \sup_{r\in(0,1)} |P_r f(y)|$$

is bounded in $L^p(e^{-x^2}dx)$ $(1<p\le\infty)$ and of weak type $(1,1)$ with respect to the Gaussian measure $e^{-x^2}dx$. We can write (1) in the form

$$P_r f(y) = \int_{\mathbf{R}} K(r,y,z)\, f(z)\,dz$$

where

$$K(r,y,z) = \frac{1}{\sqrt{\pi\,(1-r^2)}} e^{-((ry-z)/\sqrt{1-r^2})^2}.$$

If we take $\varepsilon = \sqrt{1-r^2}$ and $\chi_{(-1,1)}$ instead of e^{-t^2}, we are led to the

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