[3] F. Beckhoff, *Korovkin-Theory in Algebren*, Schriftenreihe Math. Inst. Univ. Münster, Ser. 2, Heft 45, 1987.

[4] —, *A counterexample in Korovkin theory*, Rend. Circ. Mat. Palermo (2) 37 (1988), 469–473.

[5] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Springer, Berlin 1979.

[6] J. Dixmier, *Von Neumann Algebras*, North-Holland Math. Library 27, 1981.

[7] N. Dunford and J. T. Schwartz, *Linear Operators II*, Interscience Publ., 1963.

[8] R. V. Kadison, *A generalized Schwarz inequality and algebraic invariants of operator algebras*, Ann. of Math. 56 (1952), 494–503.

[9] I. Kaplansky, *Groups with representations of bounded degree*, Canad. J. Math. 1 (1949), 105–112.

[10] B. V. Limaye and M. N. N. Namboodiri, *Korovkin approximation on $C^*$-algebras*, J. Approx. Theory 34 (1982), 237–246.

[11] —, —, *Weak Korovkin approximation by completely positive linear maps on $\beta(H)$*, ibid. 42 (1984), 201–211.

[12] —, —, *Weak approximation by positive maps on $C^*$-algebras*, to appear.

[13] L. H. Loomis, *An Introduction to Abstract Harmonic Analysis*, D. van Nostrand, New York 1953.

[14] M. Pannenberg, *Korovkin approximation in Waelbroeck algebras*, Math. Ann. 274 (1986), 423–437.

[15] W. M. Priestley, *A noncommutative Korovkin theorem*, J. Approx. Theory 16 (1976), 251–260.

[16] A. G. Robertson, *A Korovkin theorem for Schwarz maps on $C^*$-algebras*, Math. Z. 56 (1977), 205–207.

[17] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer, Berlin 1960.

[18] M. Takesaki, *Theory of Operator Algebras I*, Springer, New York 1979.

[19] B. Yood, *Hilbert algebras as topological algebras*, Ark. Mat. 12 (1974), 131–151.



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# Hölder continuity of proper holomorphic mappings*

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**Abstract.** We prove the Hölder continuity for proper holomorphic mappings onto certain piecewise smooth pseudoconvex domains with "good" plurisubharmonic peak functions at each point of their boundaries. We directly obtain a quite precise estimate for the exponent from an attraction property for analytic disks. Moreover, this way does not require any consideration of infinitesimal metric.

**1. Introduction.** The theorem of Carathéodory states that every biholomorphic map $F : D_1 \to D_2$ between bounded and simply connected domains in $\mathbf{C}$ extends to an homeomorphism $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$ if both domains satisfy the Schönflies condition at each point of their boundaries (see [9], p. 209).

For domains in $\mathbf{C}^n$ $(n > 1)$ the known generalizations require more precise assumptions. The basic result is due to Henkin [8]: if $D_1$ is bounded, defined by a plurisubharmonic function and if $D_2$ is bounded with $C^2$ strongly pseudoconvex boundary, then every proper holomorphic map $F : D_1 \to D_2$ extends to a Hölder continuous map $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$ with exponent 1/2. (See also [10] and [13]; [12] for piecewise smooth strongly pseudoconvex boundary; [5] for a local version of this theorem.)

This was generalized by Bedford–Fornæss and Diederich–Fornæss to the case where $D_1$ is bounded pseudoconvex with $C^2$ boundary and $D_2$ is bounded pseudoconvex with real-analytic boundary; they proved that every proper holomorphic map $F : D_1 \to D_2$ extends to a Hölder continuous map $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$ for some exponent $\varepsilon \in [0, 1]$ (see [2], [4]).

This paper is mainly motivated by the following observation: a proper holomorphic map is easily seen to be Hölder continuous if the image domain

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satisfies a simple attraction property for analytic disks and if the distance to the boundary behaves correctly under the map.

This leads us to introduce the following two properties:

• The boundary $\partial D$ of a domain $D \subset \mathbf{C}^n$ satisfies the *attraction property of order* $\alpha$ $(0 < \alpha < 1)$ if for each $r \in [0,1[$ there is a positive constant $C(r)$ such that the following estimate holds for any analytic disk $g : \Delta \to D$ and any $\eta \in \partial D$:

$$|u| \le r \;\Rightarrow\; |g(u) - \eta| \le C(r)|g(0) - \eta|^\alpha$$

($\Delta$ is the open unit disk in $\mathbf{C}$).

• A pair of domains $D_1$, $D_2$ satisfies the *property* $(D_1, D_2)_\beta$ $(0 < \beta < 1)$ if for every proper holomorphic map $F : D_1 \to D_2$ one has

$$\forall z \in D_1 : \quad d(F(z), \partial D_2) \le C d(z, \partial D_1)^\beta$$

for some positive constant $C$.

Our basic result is:

THEOREM 1. *Let $D_1$ and $D_2$ be bounded domains in $\mathbf{C}^n$ with piecewise smooth boundaries. Assume that $\partial D_2$ satisfies the attraction property of order $\alpha$ and that $D_1$, $D_2$ satisfy $(D_1, D_2)_\beta$ with $\alpha\beta < 1$. Then every proper holomorphic map $F : D_1 \to D_2$ extends to a Hölder continuous map $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$ with exponent $\alpha\beta$.*

(See [12], p. 206, for a precise definition of "piecewise smooth".)

Our first goal is to obtain a quite precise control of $\alpha\beta$ for domains with "good" plurisubharmonic peak functions at each point of their boundaries. This is done in Propositions 3.1 and 3.2.

In Theorems 2 and 3, we use recent results of Fornæss–Sibony ([6]) in order to apply Theorem 1 to intersections of domains in $\mathbf{C}^n$ (convex if $n > 2$) having finite types and lying in general position.

## 2. Results.

If $A(x)$ and $B(x)$ depend on a variable $x$, $A(x) \lesssim B(x)$ means that there is a constant $K$, $0 < K < \infty$, such that $A(x) \le KB(x)$ for all $x$.

Any finite intersection of bounded pseudoconvex domains in $\mathbf{C}^n$, with at least $C^2$ boundaries and lying in general position, will be called an *elementary-pseudoconvex* domain.

Remark 2.1. One can check that every elementary-pseudoconvex domain satisfies the following property, which we call the *cone property of order* $\gamma$, written $\mathcal{C}(\gamma)$ (actually this only follows from the "regularity" of the boundary).

$D \subset \mathbf{C}^n$ satisfies $\mathcal{C}(\gamma)$, $0 < \gamma < 1$, if there is a continuous map $N : \partial D \to \{\overrightarrow{u} \in \mathbf{C}^n : \|\overrightarrow{u}\| = 1\}$ and a constant $\lambda$, $0 < \lambda < \infty$, such that:



$\mathcal{C}_1$: the cone of vertex $p$, height $\lambda$, angle $\alpha\pi$ $(0 < \alpha < 1)$ and directed by $\overrightarrow{N(p)}$ is contained in $\overline{D}$.

$\mathcal{C}_2$: $\forall p, p' \in \partial D$, $\forall x, x' \in [0, \lambda] : p + x\overrightarrow{N(p)} = p' + x'\overrightarrow{N(p')} \Rightarrow p = p'$ and $x = x'$.

Remark 2.2. It directly follows from the Diederich–Fornæss exhaustion theorem ([3]) that every elementary-pseudoconvex domain $D$ admits a plurisubharmonic function $\varrho$ satisfying $0 < -\varrho(z) \lesssim d(z, \partial D)^a$ for all $z \in D$ and some $a \in ]0, 1]$. We shall say that $D$ is *hyperconvex of order* $a$.

We are now able to state precisely our main results.

THEOREM 2. *Let $D_1$ and $D_2$ be elementary-pseudoconvex domains in $\mathbf{C}^n$. Assume moreover that $D_2$ is a finite intersection of bounded domains in $\mathbf{C}^n$ (convex if $n > 2$) having types less than $2k$ and lying in general position. Then every proper holomorphic map $F : D_1 \to D_2$ extends to a Hölder continuous map $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$ with exponent $(a\gamma - 0)/2k$, where $a$ and $\gamma$ are such that $D_1$ is hyperconvex of order $a$ and $D_2$ satisfies the cone property of order $\gamma$.*

Here and below, $(b - 0)$ (resp. $(b + 0)$) denotes a constant which is strictly smaller (resp. larger) but arbitrarily close to $b$.

THEOREM 3. *Let $D_1$ (resp. $D_2$) be a finite intersection of bounded pseudoconvex domains in $\mathbf{C}^n$ (convex if $n > 2$) having types less than $2q$ (resp. $2k$) and lying in general position. Then every biholomorphism $F : D_1 \to D_2$ extends to a homeomorphism $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$ which satisfies*

$$\forall z, z' \in \overline{D}_1 : \quad |z - z'|^{(1+0)2q/\gamma_1} \lesssim |\widetilde{F}(z) - \widetilde{F}(z')| \lesssim |z - z'|^{(1-0)\gamma_2/2k}$$

*where $D_1$ and $D_2$ satisfy the cone property of order $\gamma_1$ and $\gamma_2$ respectively.*

## 3. Two technical propositions.

We first show that a domain admitting "good" plurisubharmonic peak functions satisfies the attraction property.

PROPOSITION 3.1. *Let $D$ be a bounded domain in $\mathbf{C}^n$ with neighbourhood $V$. Assume that for each $\eta \in \partial D$ there is a function $\varphi_\eta : V \to \mathbf{R}$, plurisubharmonic on $D$, peaking at $\eta$ and satisfying*

$\mathcal{P}_1$: $\forall z, z' \in V : \quad |\varphi_\eta(z) - \varphi_\eta(z')| \le A|z - z'|$,

$\mathcal{P}_2$: $\forall z \in D : \quad \varphi_\eta(z) \le 1 - B|z - \eta|^{2k}$ *and* $\varphi_\eta(\eta) = 1$,

*where $A$, $B$ and $k$ are strictly positive and independent of $\eta$. Then $\partial D$ satisfies the attraction property of order $(1 - 0)/2k$.*

Proof. Let $g : \Delta \to D$ be an analytic disk. Let $\eta$ be any point in $\partial D$ and $\varphi$ a plurisubharmonic peak function at $\eta$ which satisfies conditions $\mathcal{P}_1$

and $\mathcal{P}_2$. We shall assume without any loss of generality that $\eta = 0$. We write $\varepsilon$ for $|g(0) - \eta| = |g(0)|$.

For $r \in ]0,1[$ and $\lambda \in ]0,+\infty[$, we consider

$$M(r) = \sup_{\theta\in[0,2\pi]} |g(re^{i\theta})|^{2k},$$

$$\mu(r,\lambda) = \operatorname{mes}\{\theta \in [0,2\pi] : |g(re^{i\theta})|^{2k} \le (A/B)\lambda\varepsilon\},$$

$$\widetilde{\mu}(r,\lambda) = \operatorname{mes}\{\theta \in [0,2\pi] : \varphi \circ g(re^{i\theta}) \ge 1 - A\lambda\varepsilon\}.$$

Condition $\mathcal{P}_2$ implies that $\widetilde{\mu}(r,\lambda) \le \mu(r,\lambda)$. Therefore, since $\varphi \circ g$ is subharmonic and bounded by 1, we get

$$(1)\quad \varphi\circ g(0) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi\circ g(re^{i\theta})\,d\theta \le \frac{1}{2\pi}[\widetilde{\mu}(r,\lambda)+(1-A\lambda\varepsilon)(2\pi-\widetilde{\mu}(r,\lambda))].$$

Condition $\mathcal{P}_1$ implies that $\varphi \circ g(0) \ge 1 - A\varepsilon$ and therefore

$$(2)\qquad \mu(r,\lambda) \ge 2\pi(1-1/\lambda).$$

On the other hand, we have

$$(3)\qquad |g(u)|^{2k} \le \frac{2}{r_1-r_2}\cdot\frac{1}{2\pi}\int_0^{2\pi}|g(r_1e^{i\theta})|^{2k}\,d\theta \quad \text{for } |u| \le r_2 < r_1 < 1.$$

Hence

$$(4)\qquad M(r_2) \le \frac{2}{r_1-r_2}\left[2\pi\frac{A}{B}\lambda\varepsilon + (2\pi - \mu(r,\lambda))M(r_1)\right]$$

and by (2)

$$(5)\qquad M(r_2) \le \frac{C}{r_1-r_2}\left[\lambda\varepsilon + \frac{M(r_1)}{\lambda}\right] \quad \text{where } C = 2\max\{A/B, 1\}.$$

On choosing $\lambda = (M(r_1)/\varepsilon)^{1/2}$, (5) becomes

$$(6)\qquad M(r_2) \le \frac{C}{r_1-r_2}(M(r_1)\varepsilon)^{1/2}.$$

Now we take a decreasing sequence $(r_p)_{p\ge1}$, with $r_p < 1$ and $\lim r_p = r$. An obvious induction on (6) provides $M(r_{p+1}) \le C_p\varepsilon^{(2^p-1)/2^p}M(r_1)^{1/2^p}$ where $C_p$ only depends on $r_1,\ldots,r_{p+1}$. The proposition follows immediately. ■

The second proposition provides an estimate for the distance to the boundary under proper holomorphic mappings.

PROPOSITION 3.2. *Let $D_1$ and $D_2$ be bounded domains in $\mathbb{C}^n$. Assume that $D_1$ is hyperconvex of order $a$ and that $D_2$ satisfies the cone property of order $\gamma$. Then $(D_1, D_2)_{a\gamma}$ is satisfied.*



It is well known that this proposition remains true for $\gamma = 1$ if $D_2$ has at least $C^2$ boundary. This is actually an easy consequence of the classical Hopf lemma (see [11], p. 177).

In our case, Proposition 3.2 will follow from a suitable version of this lemma for domains satisfying the cone property:

LEMMA 3.1. *Let $D$ be a bounded domain in $\mathbb{C}^n$ which satisfies the cone property of order $\gamma$. Assume $\varphi : \overline{D} \to ]-\infty, 0]$ is a plurisubharmonic continuous function on $D$ which vanishes on $\partial D$. Then $\varphi(z) \lesssim -d(z,\partial D)^{1/\gamma}$ for all $z \in D$.*

Proof. Let $T$ be the triangle with vertices $b_1 = 0$, $b_2 = \lambda(1+i\tan\gamma\pi/2)$ and $b_3 = \lambda(1-i\tan\gamma\pi/2)$. Let $f : \overline{\Delta} \to \overline{T}$ be a conformal map. (One can take

$$f(u) = C\int_0^u (a_1-t)^{\gamma-1}(a_2-t)^{-(\gamma+1)/2}(a_3-t)^{-(\gamma+1)/2}\,dt + C'$$

where $f(a_j) = b_j$ for $j = 1,2,3$.)

For each $p \in \partial D$ one defines $\Psi_p : \overline{T} \to ]-\infty, 0]$ by $\Psi_p(u) = \varphi(p+u\overrightarrow{N(p)})$. This is possible because of condition $\mathcal{C}_1$. By applying the Poisson integral formula to $\Psi_p$ and using obvious estimates for $f$ one easily gets

$$(1)\qquad \varphi(p+x\overrightarrow{N(p)}) \lesssim x^{1/\gamma}\left[\frac{1}{2\pi}\int_0^{2\pi}\varphi(p+f(e^{it})\overrightarrow{N(p)})\,dt\right] = M(p)x^{1/\gamma}$$

for $x > 0$. Since $\varphi$ is continuous on $\overline{D}$, $M$ is continuous on $\partial D$. So $M \lesssim -1$ and (1) becomes

$$(2)\qquad \forall p \in \partial D:\quad \varphi(p+x\overrightarrow{N(p)}) \lesssim -x^{1/\gamma} \quad \text{for } x \in [0,\lambda].$$

Finally, for each $z \in D$ sufficiently close to $\partial D$ condition $\mathcal{C}_2$ provides a unique $x \in [0,\lambda]$ and a unique $p \in \partial D$ such that $z = p + x\overrightarrow{N(p)}$. Then $d(z,\partial D) \ge (\tan\gamma\pi/2)x$ and the lemma directly follows from (2). ■

We now sketch the proof of Proposition 3.2. Let $\varrho_1$ be a plurisubharmonic function on $D_1$ which satisfies $0 < -\varrho_1(w) \lesssim d(w,\partial D_1)^a$. Assume $F : D_1 \to D_2$ is a proper holomorphic map. By applying Lemma 3.1 to the plurisubharmonic function $S(z) = \sup\{\varrho_1(w) : F(w) = z\}$, one gets $S(z) \lesssim -d(z,\partial D_2)^{1/\gamma}$. On the other hand, if $F(w) = z$ one has $\varrho_1(w) \gtrsim -d(w,\partial D_1)^a$. Comparing these inequalities one finds $d(F(w),\partial D_2) \lesssim d(w,\partial D_1)^{a\gamma}$ and the proposition is proved (see [1], p. 140, for more details). ■

**4. Proofs of Theorems 1–3.** We first prove Theorem 1.

Let $D_1$, $D_2$ be domains in $\mathbf{C}^n$ with piecewise smooth boundaries. Assume that $\partial D_2$ satisfies the attraction property of order $\alpha$ and that $(D_1, D_2)_\beta$ is fulfilled. Let $F : D_1 \to D_2$ be a proper holomorphic map. Let $z_0 \in D_1$. We shall write $\varepsilon$ for $d(z_0, \partial D_1)$ and we choose $\eta \in \partial D_2$ such that $|F(z_0) - \eta| = d(F(z_0), \partial D_2)$.

Now we define an analytic disk $g : \Delta \to D_2$ by $g(u) = F(z_0 + \frac{1}{2}\varepsilon u \overrightarrow{x})$ where $\overrightarrow{x}$ is a fixed unit vector in $\mathbf{C}^n$.

It follows from $(D_1, D_2)_\beta$ that

$$d(F(z_0), \partial D_2) \lesssim \varepsilon^\beta . \tag{1}$$

Since $\partial D_2$ satisfies the attraction property of order $\alpha$ one gets

$$|u| \le 1/2 \ \Rightarrow\ |g(u) - \eta| \lesssim |g(0) - \eta|^\alpha \lesssim \varepsilon^{\alpha\beta} . \tag{2}$$

We have $\alpha\beta < 1$ and so

$$|u| \le 1/2 \ \Rightarrow\ |g(u) - g(0)| \le |g(u) - \eta| + |\eta - g(0)| \lesssim \varepsilon^{\alpha\beta} . \tag{3}$$

Hence by Cauchy's inequality

$$|g'(0)| \lesssim \frac{1}{\varepsilon}\varepsilon^{\alpha\beta} . \tag{4}$$

Denote the holomorphic tangent map of $F$ by $F'_{z_0}$. We have $F'_{z_0} \cdot \overrightarrow{x} = g'(0)$ and therefore

$$|||F'_{z_0}||| = \sup_{\|\overrightarrow{u}\|=1} \|F'_{z_0} \cdot \overrightarrow{u}\| \lesssim d(z_0, \partial D_1)^{\alpha\beta - 1} \quad \text{for all } z \in D_1 . \tag{5}$$

Finally, the Hölder continuity of $F$ with exponent $\alpha\beta$ is obtained from (5) by a classical integration argument for which we refer to [8] or [7], p. 62. Theorem 1 is proved. ∎

We now explain how Theorems 2 and 3 are obtained from Theorem 1.

Proof of Theorem 2. Assume that $D_1$ and $D_2$ are elementary-pseudoconvex domains, $D_1$ is hyperconvex of order $a$ and $D_2$ satisfies the cone property of order $\gamma$. Then, by Proposition 3.2, $(D_1, D_2)_{a\gamma}$ is fulfilled.

On the other hand, it follows from a result of Fornæss and Sibony ([6]) that if $D_2$ is an intersection of domains of finite type (convex if $n > 2$) then the assumptions of Proposition 3.1 are satisfied for some integer $k$. Therefore Theorem 2 is directly obtained by applying Proposition 3.1 and Theorem 1.

(In the case $n > 2$ the peak functions of Fornæss–Sibony satisfy slightly different conditions than the conditions $\mathcal{P}_1$ and $\mathcal{P}_2$ described in Proposition 3.1 (see [6], p. 651). However, one easily sees that this does not modify our result.) ∎



Proof of Theorem 3. Under the assumptions of Theorem 3 one can apply Theorem 2 to $F$ and $F^{-1}$. So $F$ extends to an homeomorphism $\widetilde{F} : \overline{D}_1 \to \overline{D}_2$. Then it is possible to use a suitable version of Proposition 3.2 based on local plurisubharmonic exhausting functions instead of global ones, and to take the constant $a$ arbitrarily close to 1. This leads to the estimates of Theorem 3. ∎

## References

[1] H. Alexander, *Proper holomorphic mappings in* $\mathbf{C}^n$, Indiana Univ. Math. J. 26 (1977), 137–146.

[2] E. Bedford and J. E. Fornæss, *Biholomorphic maps of weakly pseudoconvex domains*, Duke Math. J. 45 (1978), 711–749.

[3] K. Diederich and J. E. Fornæss, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. 39 (1977), 129–141.

[4] —, —, *Proper holomorphic maps onto pseudoconvex domains with real-analytic boundary*, Ann. of Math. (2) 110 (1979), 575–592.

[5] F. Forstneric and J.-P. Rosay, *Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings*, Math. Ann. 279 (1987), 239–252.

[6] J. E. Fornæss and N. Sibony, *Construction of p.s.h. functions on weakly pseudoconvex domains*, Duke Math. J. 58 (1989), 633–655.

[7] J. E. Fornæss and B. Stensønes, *Lectures on Counterexamples in Several Complex Variables*, Math. Notes 33, Princeton Univ. Press, 1987.

[8] G. M. Henkin, *An analytic polyhedron is not holomorphically equivalent to a strictly pseudoconvex domain*, Soviet. Math. Dokl. 14 (1973), 858–862.

[9] M. Hervé, *Les fonctions analytiques*, Presses Univ. de France, 1982.

[10] G. A. Margulis, *Boundary correspondence under biholomorphic mappings of multivariate domains*, in: Abstracts of the All-Union Conf. on the Theory of Functions of Several Complex Variables, Kharkov 1971, 137–138.

[11] S. I. Pinchuk, *Holomorphic maps in* $\mathbf{C}^n$ *and the problem of holomorphic equivalence*, in: Several Complex Variables, Encyclopaedia Math. Sci. 19, Springer, 1989, 173–201.

[12] R. M. Range, *On the topological extension to the boundary of biholomorphic maps in* $\mathbf{C}^n$, Trans. Amer. Math. Soc. 216 (1976), 203–216.

[13] N. Vormoor, *Topologische Fortsetzung biholomorpher Funktionen auf dem Rande bei beschränkten streng-pseudokonvexen Gebieten im* $\mathbf{C}^n$ *mit* $C^\infty$*-Rand*, Math. Ann. 204 (1973), 239–261.

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