STUDIA MATHEMATICA 100 (2) (1991)

Translation invariant projections in Sobolev spaces on tori in the L^1 and uniform norms*

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Abstract. The idempotent multipliers on Sobolev spaces on the torus in the L^1 and uniform norms are characterized in terms of the coset ring of the dual group of the torus. This result is deduced from a more general theorem concerning certain translation invariant subspaces of vector-valued function spaces on tori.

Introduction. The classical result of Cohen [C] (cf. also [G-McG], p. 2) on idempotent measures can be restated as a characterization of translation invariant projections acting either on the space of all continuous scalar-valued functions or on the space of absolutely integrable functions (with respect to the Haar measure) on a locally compact abelian group. It says that the family of the supports of the multipliers of translation invariant projections coincides with the coset ring of the dual group, i.e. with the boolean ring generated by the cosets of all closed subgroups of the dual group. The purpose of the present paper is to show that for finite-dimensional tori the same characterization holds for translation invariant projections acting on the spaces of k times continuously differentiable functions and on the corresponding Sobolev spaces in L^1 norm (Corollaries 2 and 3). In fact, the same characterization holds for every Sobolev spaces determined by a smoothness (= a nonempty set of partial derivatives) with elliptic fundamental polynomial (Theorem 3). Our results in particular imply that Sobolev spaces with elliptic fundamental polynomials have no infinite-dimensional translation invariant complemented subspaces isomorphic to Hilbert spaces (cf. [P-W] for details). Our results for Sobolev spaces are consequences of more general results which characterize (in terms of the coset ring of the dual group of the d-dimensional torus) translation invariant projections on certain translation invariant subspaces of the spaces $C(\mathrm{T}^d, E)$ and $L^1(\mathrm{T}^d, E)$ of

1991 *Mathematics Subject Classification*: 42B15, 46E35, 46E40.

* This is a part of the author's Ph.D. thesis written under the supervision of Prof. A. Pełczyński.

functions on the d-dimensional torus $\mathbf{T}^d$ with values in a finite-dimensional complex Hilbert space E. The considered subspaces X have the property that for every character κ on $\mathbf{T}^d$ the subspace $X_\kappa = \{e \in E : e\kappa \in X\}$ is one-dimensional. It is well known that one can identify Sobolev spaces determined by smoothnesses with such translation invariant function spaces (cf. [K-P], [P-S], [P]). Assuming "regular behaviour" at infinity of the function $\kappa \to X_\kappa$ (called a bundle) we show that the translation invariant projections on X are characterized by the coset ring of the dual group of $\mathbf{T}^d$.

The paper consists of 4 sections. Sections 1 contains preliminaries. We introduce there the concepts of stable and asymptotically symmetric bundles which are the regularity conditions we need. The case of translation invariant subspaces of $L^1(\mathbf{T}^d, E)$ is treated in Section 2 and the case of translation invariant subspaces of $C(\mathbf{T}^d, E)$ in Section 3. The main results are Theorems 1 and 2. Section 4 is devoted to applications to Sobolev spaces.

Acknowledgement. The author wishes to express his sincere gratitude to Professor A. Pełczyński for inspiration, stimulating discussions and relevant remarks at all stages of work on this paper.

1. Preliminaries. The symbols $\langle\ ,\ \rangle$ and $|\ |$ denote the standard scalar product and Hilbert norm in $\mathbf{R}^d$; $x^{(j)}$ denotes the jth coordinate of an $x \in \mathbf{R}^d$. A linear manifold in $\mathbf{R}^d$ of codimension one is called a $(d-1)$-*dimensional hyperplane*, shortly a *hyperplane*. A hyperplane is called *rational* if it is perpendicular to a nonzero vector with integral coordinates. $\mathbf{Z}^d$ is the sublattice of points in $\mathbf{R}^d$ with integral coordinates. The ball centred at $x \in \mathbf{R}^d$ with radius r is denoted by $B(x,r)$. $\mathbf{T}^d$ denotes the d-dimensional torus. The dual group of $\mathbf{T}^d$ can be identified with $\mathbf{Z}^d$. The *coset ring* of $\mathbf{Z}^d$ is the smallest ring of subsets of $\mathbf{Z}^d$ which contains all cosets of all subgroups of $\mathbf{Z}^d$.

A set $A \subseteq \mathbf{Z}^d$ is called *essentially periodic* with *essential period* $\varrho \in \mathbf{Z}^d$, $\varrho^{(j)} \neq 0$ for $j = 1, \ldots, d$, and *exceptional family* $H_1, \ldots, H_k$ of $(d-1)$-dimensional hyperplanes if there exists $B \subseteq \mathbf{Z}^d$ such that

$$A \div B \subseteq \bigcup_{j=1}^{k} H_j$$

where B is a periodic set of period $\varrho = (\varrho^{(j)})_{j=1}^d$, i.e.

$$B + (0, \ldots, 0, \varrho^{(j)}, 0, \ldots, 0) = B \quad \text{for } j = 1, \ldots, d.$$

FACT 1. *If A belongs to the coset ring of $\mathbf{Z}^d$ then A is essentially periodic.*

For the proof observe that each coset of each subgroup of $\mathbf{Z}^d$ is an essentially periodic set and the essentially periodic sets form a ring. ∎

COROLLARY 1. *If A belongs to the coset ring of $\mathbf{Z}^d$ and A is contained in some halfspace of $\mathbf{R}^d$, say $A \subseteq \{x \in \mathbf{R}^d : \langle x, a\rangle \geq t\}$ for some $a \in \mathbf{R}^d$ and $t \in \mathbf{R}$, then there exist $M > 0$ and $\alpha \in \mathbf{Z}^d$ such that*

$$A \subseteq \{\gamma \in \mathbf{Z}^d : |\langle \gamma, \alpha\rangle| < M\}.$$

For the proof use Fact 1, induction with respect to d and the following observation: if A is in the coset ring of $\mathbf{Z}^d$ and H is a rational hyperplane then $A \cap H$ is a translate of some set from the coset ring of a subgroup of $\mathbf{Z}^d$ isometric to $\mathbf{Z}^{d-1}$. ∎

Let E denote an arbitrary finite-dimensional complex Hilbert space with the norm $|\cdot|_E$. Then $B(E)$ stands for the ring of linear operators on E and $L^1(\mathbf{T}^d, E)$ denotes the space of equivalence classes of E-valued functions absolutely summable with respect to the Haar measure of $\mathbf{T}^d$ with the norm

$$\|f\| = \int_{\mathbf{T}^d} |f(t)|\, dt.$$

By $G(E,1)$ we denote the Grassmannian of one-dimensional subspaces of E and by $d(\ ,\ )$ the usual metric on $G(E,1)$, i.e. $d(X,Y) =$ the Hausdorff distance of the sets $X \cap B_E(0,1)$ and $Y \cap B_E(0,1)$. A *one-dimensional bundle* is a function $\psi : \mathbf{Z}^d \to G(E,1)$. By $L^1_\psi = L^1_\psi(\mathbf{T}^d, E)$ we denote the closed linear subspace of $L^1(\mathbf{T}^d, E)$ generated by the set $\{xe^{2\pi i\langle\gamma,t\rangle} : \gamma \in \mathbf{Z}^d,\ x \in \psi(\gamma)\}$. By $L^1_{|F}(\mathbf{T}^d, E)$ we denote the subspace of $L^1(\mathbf{T}^d, E)$ consisting of those functions whose Fourier transform vanishes outside a subset F of $\mathbf{Z}^d$; we put $L^1_{\psi|F} = L^1_\psi \cap L^1_{|F}(\mathbf{T}^d, E)$. If $\dim E = 1$ we write $L^1(\mathbf{T}^d)$ instead of $L^1(\mathbf{T}^d, E)$. By $C(\mathbf{T}^d, E)$ we denote the space of all E-valued continuous functions with the norm

$$\|f\|_\infty = \sup_{t \in \mathbf{T}^d} |f(t)|_E.$$

Given a bundle $\psi : \mathbf{Z}^d \to G(E,1)$ the symbol C_ψ denotes the closed linear subspace of $C(\mathbf{T}^d, E)$ generated by the set $\{xe^{2\pi i\langle\gamma,t\rangle} : \gamma \in \mathbf{Z}^d,\ x \in \psi(\gamma)\}$.

The concepts of translation invariant operators and the corresponding multipliers acting on translation invariant function spaces on a fixed group have the usual meaning (cf. [P-S], [K-P]). In particular, if $T : L^1(\mathbf{T}^d, E) \to L^1(\mathbf{T}^d, E)$ is a translation invariant operator, then the corresponding multiplier $\widehat{T}$ is a function from $\mathbf{Z}^d$ into $B(E)$ defined by $\widehat{T}(\gamma) =$ the restriction of T to the space $\{xe^{2\pi i\langle\gamma,t\rangle} : x \in E\}$. We write $\operatorname{supp} \widehat{T} = \{\gamma \in \mathbf{Z}^d : \widehat{T}(\gamma) \neq 0\}$. If $T : L^1_\psi \to L^1_\psi$ is translation invariant then

$$T(\psi(\gamma)e^{2\pi i\langle\gamma,t\rangle}) \subseteq \psi(\gamma)e^{2\pi i\langle\gamma,t\rangle}.$$

Thus $\widehat{T}(\gamma) = T_{|\psi(\gamma)e^{2\pi i\langle\gamma,t\rangle}}$ can be identified with a linear operator on the one-dimensional space $\psi(\gamma)$, hence it can be identified with a complex number. Thus $\widehat{T}$ can be identified with a complex-valued function on $\mathbf{Z}^d$. If $P : L^1_\psi(\mathbf{T}^d, E) \to L^1_\psi(\mathbf{T}^d, E)$ is a translation invariant projection, then $\widehat{P} : \mathbf{Z}^d \to \{0,1\}$.

In the present paper we consider a class of bundles ψ which are stable and asymptotically symmetric.

DEFINITION. A set $F \subset \mathbf{Z}^d$ is called *ε-stable* for the bundle ψ if $d(\psi(\gamma_1), \psi(\gamma_2)) < \varepsilon$ for any $\gamma_1, \gamma_2 \in F$.

A bundle $\psi : \mathbf{Z}^d \to G(E,1)$ is called *stable* if for every $m > 0$ and $\varepsilon > 0$ there exists $M > 0$ such that $|\gamma| > M$ implies that the ball $B(\gamma, m)$ is ε-stable.

A bundle ψ is *asymptotically symmetric* if for every $\varepsilon > 0$ there exists $M > 0$ such that if $|\gamma| > M$ then the set $\{\gamma, -\gamma\}$ is ε-stable.

By the Sobolev space $W^1_S(\mathbf{T}^d)$ ($C_S(\mathbf{T}^d)$) we mean, as usual, the completion of the set of all trigonometric polynomials equipped with the norm

$$\|f\|_{W^1_S} = \int_{\mathbf{T}^d} \Big(\sum_{D\in S} |Df(t)|^2 \Big)^{1/2} dt$$

$$\Big(\|f\|_{C_S} = \sup_{t\in\mathbf{T}^d} \Big(\sum_{D\in S} |Df(t)|^2 \Big)^{1/2} \text{ respectively}\Big)$$

where S denotes a (d-dimensional) *smoothness*, i.e. an arbitrary finite set of partial derivatives (in d variables) containing the identity operator. For $D \in S$ we denote by $\widehat{D}$ the *symbol* of D, i.e. if $D = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}$ then $\widehat{D}(\xi) = (i\xi^{(1)})^{\alpha_1} \ldots (i\xi^{(d)})^{\alpha_d}$. Given a smoothness S the *fundamental polynomial* Q_S is

$$Q_S(\xi) = \sum_{D\in S} |\widehat{D}(\xi)|^2 .$$

A smoothness S is called *elliptic* provided there exists $C > 0$ such that

$$|Q_S(\xi)| \ge C|\xi|^{\deg Q_S} \quad \text{for } \xi \in \mathbf{R}^d .$$

With a d-dimensional smoothness S we associate the bundle $\psi_S = \psi : \mathbf{Z}^d \to G(E,1)$ defined as follows. E is the complex Hilbert space $\mathbf{C}^{\operatorname{card} S}$ and $\psi_S(\gamma) = \operatorname{span}\{x_\gamma\}$ where $x_\gamma = (\widehat{D}(\gamma)/(Q_S(\gamma))^{1/2})_{D\in S} \in E$ for $\gamma \in \mathbf{Z}^d$. The Sobolev space $W^1_S(\mathbf{T}^d)$ is "canonically" isometrically isomorphic to $L^1_\psi(\mathbf{T}^d, E)$ (cf. [K-P]) via an operator $H : W^1_S \to L^1_\psi$ defined by

$$(Hf)\hat{\ }(\gamma) = \widehat{f}(\gamma)(Q_S(\gamma))^{1/2} x_\gamma \quad \text{for } \gamma \in \mathbf{Z}^d .$$

Clearly H is a translation invariant isometric isomorphism. Indeed, for any trigonometric polynomial f one has

$$\begin{aligned}
\|f\|_{W^1_S} &= \int_{\mathbf{T}^d} \Big(\sum_{D\in S} |Df(t)|^2 \Big)^{1/2} dt \\
&= \int_{\mathbf{T}^d} \Big(\sum_{D\in S} \Big| D\Big(\sum_{\gamma\in\mathbf{Z}^d} \widehat{f}(\gamma) e^{2\pi i\langle\gamma,t\rangle} \Big) \Big|^2 \Big)^{1/2} dt \\
&= \int_{\mathbf{T}^d} \Big(\sum_{D\in S} \Big| \sum_{\gamma\in\mathbf{Z}^d} \widehat{D}(\gamma) \widehat{f}(\gamma) e^{2\pi i\langle\gamma,t\rangle} \Big|^2 \Big)^{1/2} dt \\
&= \int_{\mathbf{T}^d} \Big| \sum_{\gamma\in\mathbf{Z}^d} \widehat{f}(\gamma)(Q_S(\gamma))^{1/2} x_\gamma e^{2\pi i\langle\gamma,t\rangle} \Big|_E dt = \|Hf\|_{L^1_\psi} .
\end{aligned}$$

Notice that a similar argument shows the existence of a translation invariant isometric isomorphism from $C_S(\mathbf{T}^d)$ onto $C_\psi(\mathbf{T}^d, E)$.

A smoothness S is called *stable* (resp. *asymptotically symmetric*) if ψ_S is stable (resp. asymptotically symmetric).

2. Characterization of translation invariant projections on $L^1_\psi(\mathbf{T}^d, E)$ for stable and asymptotically symmetric bundles.

Our main result is

THEOREM 1. *If ψ is a stable and asymptotically symmetric bundle and $P : L^1_\psi \to L^1_\psi$ is a translation invariant projection then* supp $\widehat{P}$ *belongs to the coset ring of $\mathbf{Z}^d$.*

In order to prove Theorem 1 we need several lemmas.

LEMMA 1. *If $F \subset \mathbf{Z}^d$ is an n-element set which is $1/(3n)$-stable for the bundle ψ then there exists a translation invariant isomorphism $H : L^1_{|F} \to L^1_{\psi|F}$ with $\|H\| \cdot \|H^{-1}\| \le 2$.*

Proof. Since F is $1/(3n)$-stable for ψ, there exist $y \in E$ with $|y|_E = 1$ and a system of vectors $(x_\gamma)_{\gamma\in F}$ with $x_\gamma \in \psi(\gamma)$ satisfying $|x_\gamma - y|_E \le 1/(3n)$. Let

$$H(e^{2\pi i\langle\gamma,t\rangle}) = x_\gamma e^{2\pi i\langle\gamma,t\rangle} \quad \text{for } \gamma \in F .$$

Then for arbitrary scalars $(a_\gamma)_{\gamma\in F}$ we have

$$\begin{aligned}
\Big\| H\Big(\sum_{\gamma\in F} a_\gamma e^{2\pi i\langle\gamma,t\rangle} \Big) \Big\| &= \Big\| \sum_{\gamma\in F} a_\gamma x_\gamma e^{2\pi i\langle\gamma,t\rangle} \Big\| \\
&= \Big\| \sum_{\gamma\in F} a_\gamma y e^{2\pi i\langle\gamma,t\rangle} + \sum_{\gamma\in F} a_\gamma (x_\gamma - y) e^{2\pi i\langle\gamma,t\rangle} \Big\| \\
&\le \Big\| \sum_{\gamma\in F} a_\gamma e^{2\pi i\langle\gamma,t\rangle} \Big\|_1 + \sum_{\gamma\in F} |a_\gamma| |x_\gamma - y|_E
\end{aligned}$$

$$\leq \Big\| \sum_{\gamma\in F} a_\gamma e^{2\pi i\langle\gamma,t\rangle}\Big\|_1 + n\cdot\frac{1}{3n}\max_{\gamma\in F}|a_\gamma|$$
$$\leq \frac{4}{3}\Big\| \sum_{\gamma\in F} a_\gamma e^{2\pi i\langle\gamma,t\rangle}\Big\|_1$$

because $|a_\gamma| \leq \|\sum_{\gamma\in F} a_\gamma e^{2\pi i\langle\gamma,t\rangle}\|_1$ for $\gamma\in F$. Similarly

$$\Big\|H\Big(\sum_{\gamma\in F} a_\gamma e^{2\pi i\langle\gamma,t\rangle}\Big)\Big\| = \Big\|\sum_{\gamma\in F} a_\gamma y e^{2\pi i\langle\gamma,t\rangle} + \sum_{\gamma\in F} a_\gamma(x_\gamma - y)e^{2\pi i\langle\gamma,t\rangle}\Big\|$$
$$\geq \Big\| \sum_{\gamma\in F} a_\gamma e^{2\pi i\langle\gamma,t\rangle}\Big\|_1 - \sum_{\gamma\in F}|a_\gamma||x_\gamma - y|_E$$
$$\geq \Big\| \sum_{\gamma\in F} a_\gamma e^{2\pi i\langle\gamma,t\rangle}\Big\|_1 - n\cdot\frac{1}{3n}\max_{\gamma\in F}|a_\gamma|$$
$$\geq \frac{2}{3}\Big\| \sum_{\gamma\in F} a_\gamma e^{2\pi i\langle\gamma,t\rangle}\Big\|_1 .$$

Thus $\|H\|\cdot\|H^{-1}\| \leq 2$. ■

LEMMA 2. *Let ψ be a stable and asymptotically symmetric bundle and let $P : L^1_\psi \to L^1_\psi$ be a translation invariant projection. Then each unbounded sequence $(\alpha'_n)_{n=1}^\infty \subseteq \mathbf{Z}^d$ contains an unbounded subsequence $(\alpha_n)_{n=1}^\infty$ such that $\lim_{n\to\infty}\widehat{P}(\gamma+\alpha_n)$ exists for each $\gamma\in\mathbf{Z}^d$ and the formula*

$$(1)\qquad \widehat{R}(\gamma) = \lim_{n\to\infty}\widehat{P}(\gamma+\alpha_n)\quad \textit{for } \gamma\in\mathbf{Z}^d$$

determines a translation invariant projection $R : L^1(\mathbf{T}^d)\to L^1(\mathbf{T}^d)$.

Proof. Given a cube

$$Q = Q(\alpha,h) = \{\gamma\in\mathbf{Z}^d : |\gamma^{(i)} - \alpha^{(i)}| \leq h \text{ for } i = 1,\ldots,d\}$$

we put

$$f_Q(t) = e^{2\pi i\langle\alpha,t\rangle}\prod_{j=1}^d \sum_{|k^{(j)}|\leq h}\Big(1 - \frac{|k^{(j)}|}{n+1}\Big)e^{2\pi i k^{(j)}t^{(j)}}$$

for $t = (t^{(j)})_{j=1}^d \in \mathbf{R}^d$. Let $F_q : L^1(\mathbf{T}^d)\to L^1_{|Q}(\mathbf{T}^d)$ be the operator of convolution by f_Q. Since f_Q is a translate of a d-dimensional Fejér kernel, $\|F_Q\| = 1$. Let $H_Q : L^1_{|Q}(\mathbf{T}^d)\to L^1_{\psi|Q}$ be any translation invariant isomorphism, i.e. H_Q is defined by a sequence $(x_\gamma)_{\gamma\in Q}$ with $0\neq x_\gamma\in\psi(\gamma)$ via the formula $H_Q(e^{2\pi i\langle\gamma,t\rangle}) = x_\gamma e^{2\pi i\langle\gamma,t\rangle}$. Having chosen H_Q we define S_Q by the commutative diagram

$$\begin{array}{ccccc}
L^1(\mathbf{T}^d) & \xrightarrow{F_Q} & L^1_{|Q}(\mathbf{T}^d) & \xrightarrow{H_Q} & L^1_{\psi|Q}\\
\big\downarrow{\scriptstyle S_Q} & & & & \big\downarrow{\scriptstyle P_Q}\\
L^1(\mathbf{T}^d) & \xleftarrow{J_Q} & L^1_{|Q}(\mathbf{T}^d) & \xleftarrow{H_Q^{-1}} & L^1_{\psi|Q}
\end{array}$$

i.e. we put

$$(2)\qquad S_Q = J_Q\circ H_Q^{-1}\circ P_Q\circ H_Q\circ F_Q$$

where $P_Q : L^1_{\psi|Q}\to L^1_{\psi|Q}$ is the restriction of P and $J_Q : L^1_{|Q}(\mathbf{T}^d)\to L^1(\mathbf{T}^d)$ is the inclusion. Finally, we put

$$(3)\qquad R_Q = [-\alpha]\circ S_Q\circ[\alpha]$$

where, for $\gamma\in\mathbf{Z}^d$, $[\gamma]$ denotes the operator of multiplication by the function $t\mapsto e^{2\pi i\langle\gamma,t\rangle}$. Clearly R_Q and S_Q are translation invariant operators acting on $L^1(\mathbf{T}^d)$. Moreover, we have

$$(4)\qquad \widehat{R}_Q(\gamma) = \widehat{S}_Q(\gamma+\alpha) = \widehat{F}_Q(\gamma+\alpha)\widehat{P}_Q(\gamma+\alpha) = \widehat{F}_{Q-\alpha}(\gamma)\widehat{P}(\gamma+\alpha)$$

for $\gamma\in\mathbf{Z}^d$ and

$$(5)\qquad \|R_Q\| = \|S_Q\| \leq \|P\|\cdot\|H_Q\|\cdot\|H_Q^{-1}\|.$$

Now assume that we are given an unbounded sequence $(\alpha'_n)_{n=1}^\infty\subseteq\mathbf{Z}^d$. Then using the stability of ψ we can extract from it an unbounded subsequence $(\alpha_n)_{n=1}^\infty$ such that each cube $Q_n = Q(\alpha_n, n)$ for $n = 1,2,\ldots$ (which has $(2n+1)^d$ elements) is $1/(3(2n+1)^d)$-stable for the bundle ψ. Hence, by Lemma 1, there exists an isomorphism $H_n = H_{Q_n}$ with

$$(6)\qquad \|H_n\|\cdot\|H_n^{-1}\| \leq 2.$$

Let $S_n = S_{Q_n}$ and $R_n = R_{Q_n}$ be defined by (2) and (3) for the above choice of H_n $(n = 1,2,\ldots)$. Note that $|\widehat{R}_Q(\gamma)| \leq 1$ for all $\gamma\in\mathbf{Z}^d$ and all cubes $Q\subseteq\mathbf{Z}^d$. Thus, passing if necessary again to a subsequence, we can also assume

$$(7)\qquad \lim_{n\to\infty}\widehat{R}_n(\gamma)\quad\text{exists for } \gamma\in\mathbf{Z}^d.$$

It follows from (5) and (6) that

$$(8)\qquad \|R_n\| \leq 2\|P\|\quad \text{for } n = 1,2,\ldots$$

Thus (7) and (8) yield the existence of a unique translation invariant operator $R : L^1(\mathbf{T}^d)\to L^1(\mathbf{T}^d)$ such that $\widehat{R}(\gamma) = \lim\widehat{R}_n(\gamma)$ and moreover $\|R\|\leq 2\|P\|$.

Put $F_n = F_{Q_n-\alpha_n}$, $f_n = f_{Q_n-\alpha_n}$ for $n = 1,2,\ldots$ The cubes $Q_n - \alpha_n$ satisfy: $Q_1-\alpha_1\subseteq Q_2-\alpha_2\subseteq\ldots$; $\lim_{n\to\infty}(Q_n-\alpha_n) = \mathbf{Z}^d$. Thus, by the well known property of the Fejér kernels,

$$\lim_{n\to\infty} \widehat{F}_n(\gamma) = \lim_{n\to\infty} \widehat{f}_n(\gamma) = 1 \quad \text{for } \gamma \in \mathbf{Z}^d .$$

Hence, by (4),

$$\widehat{R}(\gamma) = \lim_{n\to\infty} \widehat{R}_n(\gamma) = \lim_{n\to\infty} \widehat{P}(\gamma + \alpha_n) \quad \text{for } \gamma \in \mathbf{Z}^d .$$

This proves (1). Clearly (1) implies that $\widehat{R}(\gamma)$ equals either 0 or 1 for all $\gamma \in \mathbf{Z}^d$ because $\widehat{P}$ being a multiplier induced by a projection has the same property. Thus $\widehat{R}(\gamma) = (R(\gamma))^2$ for all $\gamma \in \mathbf{Z}^d$. Hence R is a translation invariant projection. ■

We shall also need the following technical

LEMMA 3. *Let $Q : L^1_\psi \to L^1_\psi$ be a translation invariant projection ($\psi : \mathbf{Z}^d \to G(E,1)$ stable). Assume that Q satisfies either*

(i) *there exist $M_0 > 0$, a sequence $(x_k)_{k=1}^\infty \subseteq \mathbf{R}^d$ with $|x_k| = 1$ for $k = 1,2,\ldots$ and a sequence of balls $(B(\alpha_k, r_k))_{k=1}^\infty$ with $(\alpha_k)_{k=1}^\infty \subseteq \mathbf{Z}^d$ and $\lim r_k = \infty$ such that for $k = 1,2,\ldots$*

$$\alpha_k \in \operatorname{supp} \widehat{Q} \cap B(\alpha_k, r_k) \subset \{z : \langle z - \alpha_k, x_k\rangle \le M_0\},$$

or

(ii) *there exist sequences of balls $(B(a_k, s_k))_{k=1}^\infty$ with $\lim s_k = \infty$, $(a_k)_{k=1}^\infty \subset \mathbf{R}^d$ and $(\alpha_k)_{k=1}^\infty \subset \mathbf{Z}^d$ with $|\alpha_k - a_k| = s_k$ for $k = 1,2,\ldots$ such that for $k = 1,2,\ldots$*

$$\widehat{Q}(\alpha_k) = 1 \quad \textit{and} \quad \operatorname{supp} \widehat{Q} \cap B(a_k, s_k) = \emptyset .$$

Then there exist $M > 0$, $x \in \mathbf{Z}^d$ and a subsequence $(\beta_k)_{k=1}^\infty$ of $(\alpha_k)_{k=1}^\infty$ such that for $k = 1,2,\ldots$

$$\beta_k \in \operatorname{supp} \widehat{Q} \cap B(\beta_k, k) \subset \{z : |\langle z - \beta_k, x\rangle| \le M\}.$$

Proof. (i) Passing to a subsequence if necessary, we can assume that $\lim x_k = y$ and (by Lemma 2) that there exists a translation invariant projection $R : L^1(\mathbf{T}^d) \to L^1(\mathbf{T}^d)$ satisfying

$$\widehat{R}(\gamma) = \lim_{n\to\infty} \widehat{Q}(\gamma + \alpha_k) \quad \text{for } \gamma \in \mathbf{Z}^d .$$

If $\langle \gamma, y\rangle > M_0$ then $\langle \gamma, x_k\rangle = \langle(\alpha_k + \gamma) - \alpha_k, x_k\rangle > M_0$ for large k. Hence, by (i), $\widehat{Q}(\alpha_k + \gamma) = 0$. Thus if $\langle \gamma, y\rangle > M_0$ then $\widehat{R}(\gamma) = 0$. Therefore, by Corollary 1, there exist $M > 0$ and $x \in \mathbf{Z}^d$ such that

$$\widehat{R}(\gamma) = 0 \quad \text{for } \gamma \in \mathbf{Z}^d \text{ satisfying } |\langle \gamma, x\rangle| > M . \tag{9}$$

Since for all $\gamma \in \mathbf{Z}^d$ the sequence $(Q(\gamma + \alpha_k))_{k=1}^\infty$ is integer-valued and tends to $\widehat{R}(\gamma)$, there is $k_\gamma > 0$ such that $\widehat{Q}(\gamma + \alpha_k) = \widehat{R}(\gamma)$ for every

$k > k_\gamma$. Putting $k_m = \max\{k_\gamma : \gamma \in B(0,m)\}$ we infer that if $k \ge k_\gamma$ and $\gamma \in B(0,m)$ then $\widehat{Q}(\gamma + \alpha_k) = \widehat{R}(\gamma)$. Thus, by (9),

$$\operatorname{supp} \widehat{Q} \cap B(\alpha_k, m) \subset \{z : |\langle z - \alpha_k, x\rangle| \le M\} \quad \text{for } k \ge k_m .$$

On the other hand, by (i), $\widehat{P}(\alpha_k) = 1$. We put $\beta_m = \alpha_{k_m}$ for $m = 1,2,\ldots$

(ii) Let $x_k = (a_k - \alpha_k)/|a_k - \alpha_k|$ and $r_k = \sqrt{2s_k}$ for $k = 1,2,\ldots$ Then

$$B(\alpha_k, r_k) \subset \{z : \langle z - \alpha_k, x_k\rangle \le 1\} \subseteq B(a_n, s_n).$$

Hence, by (i), we get (ii) with $M_0 = 1$. ■

Proof of Theorem 1. We use induction with respect to the dimension d. The case $d = 0$ is trivial. Assume the validity of the assertion of Theorem 1 for all integers d' with $0 \le d' \le d-1$ and for all stable and asymptotically symmetric bundles parametrized by $\mathbf{Z}^{d'}$.

First observe that the inductive hypothesis implies:

(*) *For every stable and asymptotically symmetric bundle ψ parametrized by $\mathbf{Z}^d$, for every translation invariant projection $P : L^1_\psi \to L^1_\psi$ and for every $(d-1)$-dimensional hyperplane H of $\mathbf{R}^d$ the set* $\operatorname{supp} \widehat{P} \cap H$ *belongs to the coset ring of $\mathbf{Z}^d$.*

Indeed, for every $(d-1)$-dimensional hyperplane $H \subset \mathbf{R}^d$ for which $H \cap \mathbf{Z}^d$ is nonempty, there exist $\alpha \in \mathbf{Z}^d$ and linearly independent vectors $\beta_1, \ldots, \beta_{d'} \in \mathbf{Z}^d$ for some integer d' with $0 \le d' \le d-1$ such that

$$H \cap \mathbf{Z}^d = \{\alpha + n_1\beta_1 + \ldots + n_{d'}\beta_{d'} : n_j \in \mathbf{Z} \text{ for } 1 \le j \le d'\}.$$

Consider the bundle $\psi' : \mathbf{Z}^{d'} \to G(E,1)$ defined by

$$\psi'(n_1, \ldots, n_{d'}) = \psi(\alpha + n_1\beta_1 + \ldots + n_{d'}\beta_{d'}).$$

One can easily verify that ψ' is asymptotically symmetric and stable. Define an isomorphism $G : L^1_{\psi'} \to L^1_{\psi|H\cap\mathbf{Z}^d}$ as follows. Put

$$G(xe^{2\pi i\langle(n_1,\ldots,n_{d'}),s\rangle}) = xe^{2\pi i\langle(\alpha+n_1\beta_1+\ldots+n_{d'},\beta_{d'}),t\rangle}$$

and extend G linearly to E-valued trigonometric polynomials whose coefficients belong to the bundle ψ'. For any such polynomial, say f, we have

$$\|Gf\|_{L^1_{\psi|H\cap\mathbf{Z}^d}} = C\|f\|_{L^1_{\psi'}}$$

where C = square root of the Gram determinant of the vectors $\beta_1, \ldots, \beta_{d'}$. Thus G uniquely extends to an isomorphism from $L^1_{\psi'}$ onto $L^1_{\psi|H\cap\mathbf{Z}^d}$. Define $P_G = G^{-1} \circ P \circ G$. Then P_G is a translation invariant projection on $L^1_{\psi'}$. Hence, by the inductive hypothesis, $\operatorname{supp} \widehat{P}_G$ belongs to the coset ring of $\mathbf{Z}^d$.

Let $P : L^1_\psi \to L^1_\psi$ be a translation invariant projection for some stable and asymptotically symmetric bundle $\psi : \mathbf{Z}^d \to G(E,1)$. Assume to the contrary

(A) supp $\widehat{P}$ *does not belong to the coset ring of* $\mathbf{Z}^d$.

We shall show in several steps that (A) will lead to a contradiction.

Step 1. (A)⇒(B) *where*

(B) *there exists a translation invariant projection* $Q : L^1_\psi \to L^1_\psi$ *such that* supp $\widehat{Q}$ *does not belong to the coset ring of* $\mathbf{Z}^d$ *and for some sequence of balls* B_n *with unbounded sequence of radii*, supp $\widehat{Q} \cap B_n = \emptyset$ *for* $n = 1, 2, \ldots$

Proof. Given P satisfying (A) and any unbounded sequence $(\alpha'_n) \subseteq \mathbf{Z}^d$ we construct a translation invariant projection $R : L^1(\mathbf{T}^d) \to L^1(\mathbf{T}^d)$ and a subsequence $(\alpha_n)_{n=1}^\infty$ of $(\alpha'_n)_{n=1}^\infty$ as in Lemma 2. Then supp $\widehat{R}$ belongs to the coset ring of $\mathbf{Z}^d$ (by the idempotent measure theorem; cf. [G-McG], p. 2). Hence, in view of Fact 1, supp $\widehat{R}$ is essentially periodic with an essential period ϱ and exceptional rational hyperplanes $H_1, \ldots, H_k$. Replacing $(\alpha_n)_{n=1}^\infty$ by a subsequence if necessary, one can assume that there exists $\alpha \in \mathbf{Z}^d$ such that for $n = 1, 2, \ldots$

$$\alpha_n \equiv \alpha \mod \varrho,$$

i.e. $\alpha_n^{(j)} \equiv \alpha^{(j)} \mod \varrho^{(j)}$ for $j = 1, \ldots, d$.

Put $R_\alpha = [\alpha] \circ R \circ [-\alpha]$. Then $\widehat{R}_\alpha(\gamma) = \widehat{R}(\gamma - \alpha)$ for $\gamma \in \mathbf{Z}^d$. Let $R^\psi_\alpha = [\alpha] \circ R^\psi \circ [-\alpha] : L^1_\psi \to L^1_\psi$ where R^ψ is the translation invariant projection with supp $\widehat{R}^\psi =$ supp $\widehat{R}$ (R^ψ is the restriction to L^1_ψ of the projection $S : L^1(\mathbf{T}^d, E) \to L^1(\mathbf{T}^d, E)$ such that $\widehat{S}(\gamma) = \mathrm{Id}_E$ for $\gamma \in$ supp $\widehat{R}$ and $\widehat{S}(\gamma) = 0$ otherwise). Put

$$P^{(1)} = R^\psi_\alpha - P \circ R^\psi_\alpha, \quad P^{(2)} = P - P \circ R^\psi_\alpha.$$

Then $P^{(1)}$ and $P^{(2)}$ are translation invariant projections of L^1_ψ (because P and R^ψ_α commute). Clearly

$$P = P^{(1)} - P^{(2)} + R^\psi_\alpha.$$

Thus either supp $\widehat{P}^{(1)}$ or supp $\widehat{P}^{(2)}$ does not belong to the coset ring of $\mathbf{Z}^d$ because supp $\widehat{P}$ does not belong, while supp $\widehat{R}^\psi_\alpha =$ supp $\widehat{R}_\alpha$ does. Assume that supp $\widehat{P}^{(1)}$ belongs to the coset ring of $\mathbf{Z}^d$ (the argument for $P^{(2)}$ is similar). Arguing as in the proof of Lemma 3 we infer that for every $m = 1, 2, \ldots$ there exists $k_m > 0$ such that if $n \ge k_m$ then

$$\widehat{P}(\gamma + \alpha_n) = \widehat{R}(\gamma) \quad \text{for } \gamma \in B(0, m).$$

Hence for $n \ge k_m$

$$\begin{aligned}\widehat{P}^{(1)}(\gamma + \alpha_n) &= (R^\psi_\alpha - P \circ R^\psi_\alpha)^\wedge(\gamma + \alpha_n) \\ &= \widehat{R}(\gamma + \alpha_n - \alpha) - \widehat{P}(\gamma + \alpha_n)\widehat{R}(\gamma + \alpha_n - \alpha) \\ &= \widehat{R}(\gamma + \alpha_n - \alpha) - \widehat{R}(\gamma)\widehat{R}(\gamma + \alpha_n - \alpha).\end{aligned}$$

Now fix $n > k_m$. Since $\gamma + \alpha_n - \alpha \equiv \gamma \mod \varrho$ and ϱ is an essential period for supp $\widehat{R}$, we infer that $\widehat{P}^{(1)}(\gamma + \alpha_n) = 0$ whenever

$$\gamma \in A = B(0, m) \setminus \Big(\bigcup_{j=1}^k H_j \cup \bigcup_{j=1}^k (H_j - \alpha_n + \alpha) \Big).$$

Since A is the difference of a ball of radius m and the union of $2k$ hyperplanes, a simple argument involving comparison of volumes shows that A contains a ball of radius Cm where C depends on k and d only. ■

It follows from the observation (∗) that one can assume without loss of generality

(∗∗) supp $\widehat{Q}$ *is not contained in a union of finitely many* $(d-1)$*-dimensional hyperplanes.*

Step 2. *If* Q *satisfies* (∗∗) *then* (B)⇒(C) *where*

(C) *for* $n = 1, 2, \ldots$, *there exist a ball* $C_n = B(a_n, n)$ *with* $a_n \in \mathbf{R}^d$ *and a point* $\alpha_n \in \mathbf{Z}^d$ *with* $|\alpha_n - a_n| = n$ *such that* supp $\widehat{Q} \cap C_n = \emptyset$ *and* $\widehat{Q}(\alpha_n) = 1$. *Moreover*, $\langle y_k, \alpha_n - \alpha_i \rangle \ge n$ *for* $i, k < n$ *where* $(y_k)_{k=1}^\infty$ *is any enumeration of the set* $\{\gamma/|\gamma| : \gamma \in \mathbf{Z}^d \setminus \{0\}\}$.

Proof. Take for C_1 the ball of maximal radius with the same centre as B_1 and contained in $\mathbf{R}^d \setminus$ supp $\widehat{Q}$; take for α_1 any point in the (obviously nonempty) intersection supp $\widehat{Q} \cap$ closure of C_1.

Fix $n \ge 1$. Assume that we have constructed C_m and α_m for $1 \le m < n$. Denote by $\mathcal{R}_n$ the family of hyperplanes

$$\{y : \langle y, y_m \rangle = \langle \alpha_i, y_m \rangle\} \quad \text{for } i, m < n.$$

We have to show that

(10) *there exist a ball* $C = B(a, n)$ *and* $\alpha \in \mathbf{Z}^d$ *such that* $|a - \alpha| = n$, supp $\widehat{Q} \cap C = \emptyset$, $\widehat{Q}(\alpha) = 1$ *and* $\mathrm{dist}(\alpha, \bigcup \mathcal{R}_n) > n$.

Let $\mathcal{A}$ be the family of all components of $\mathbf{R}^d \setminus \bigcup \mathcal{R}_n$ which are not contained in any strip determined by two parallel rational hyperplanes.

For any nonempty $A \subseteq \mathbf{R}^d$ and $r > 0$ we put

$$A^r = \{x \in A : B(x, r) \subseteq A\}.$$

First we prove the implication (11)⇒(10) where

(11) *there exist* $A \in \mathcal{A}$, $b \in A^{2n}$ *and* $\delta \in A^{2n} \cap \mathbf{Z}^d$ *such that* supp $\widehat{Q} \cap B(b, n) = \emptyset$ *and* $\widehat{Q}(\delta) = 1$.

Indeed, put $B_s = B(s\delta + (1-s)b, n)$ for $0 \le s \le 1$. Clearly supp $\widehat{Q} \cap B_0 = \emptyset$ and supp $\widehat{Q} \cap B_1 \ne \emptyset$. Put $s_0 = \mathrm{supp}\{s : \mathrm{supp}\, \widehat{Q} \cap B_s = \emptyset\}$. Since B_{s_0} is open, $\operatorname{supp}\widehat{Q}\cap B_{s_0}=\emptyset$. Since $\operatorname{supp}\widehat{Q}$ is a discrete subset of $\mathbf{R}^d$, there exists an $\alpha\in\operatorname{supp}\widehat{Q}$ such that $|\alpha-(s_0\delta+(1-s_0)b)|=n$.

Proof of the implication (B)$\Rightarrow$(11). Given $A_j\in\mathcal{A}$ ($j=1,2$) we write $A_1\sim A_2$ provided there exist a versor x normal to one of the hyperplanes of $\mathcal{R}_n$, $c>0$ and a sequence of balls $(B(a_k',r_k))_{k=1}^{\infty}$ such that $\lim r_k=\infty$ and

$$B(a_k',r_k)\cap\{y:\langle y-a_k',x\rangle>c\}\subseteq A_1\,,$$
$$B(a_k',r_k)\cap\{y:\langle y-a_k',x\rangle<-c\}\subseteq A_2\,.$$

Note

(12) *Any two members of $\mathcal{A}$, say A and B, can be joined by a chain $A_1,\ldots,A_m$ of members of $\mathcal{A}$ such that $A_j\sim A_{j+1}$ for $j=1,\ldots,m-1$ and $A_1=A$, $A_m=B$.*

(13) *$\mathbf{Z}^d\setminus\bigcup\mathcal{A}$ is contained in a finite union of rational hyperplanes.*

The proof of (12) is given in the appendix. We leave to the reader the routine proof of (13).

Observe that if $A_1\sim A_2$ then there exists another sequence of balls $(D_k)_{k=1}^{\infty}$ such that $D_k=B(a_k,k)$ and

$$D_k\cap\{y:\langle y-a_k,x\rangle>c\}\subseteq A_1^{2n}\,,\qquad D_k\cap\{y:\langle y-a_k,x\rangle<-c\}\subseteq A_2^{2n}\,. \tag{14}$$

It follows from (B) that there exists a ball C of radius n such that $\operatorname{supp}\widehat{Q}\cap C\neq\emptyset$ and $C\subseteq A_1^{2n}$ for some $A_1\in\mathcal{A}$. Now if (11) were false, then $\widehat{Q}(\gamma)=0$ for every $\gamma\in A_1^{2n}\cap\mathbf{Z}^d$. Pick $A_2\in\mathcal{A}$ so that $A_1\sim A_2$. Then, by (14), we have $\widehat{Q}(\gamma)=0$ for $\gamma\in\mathbf{Z}^d\cap D_k\cap\{y:\langle y-a_k,x\rangle>c\}$ ($k=1,2,\ldots$). Now, by Lemma 3(i) (applied to the constant sequence $x_k=x$ and $M_0=c$) we infer that $\widehat{Q}(\gamma)=0$ for some $M>0$ and $\gamma\in\mathbf{Z}^d\cap D_k\cap\{y:|\langle y-a_k,x\rangle|>M\}$ for $k=1,2,\ldots$ For large k the set $D_k\cap\{y:\langle y-a_k,x\rangle>-M\}$ contains a ball of radius n which is contained in A_2^{2n}. Thus, assuming that (11) is false, we get $\widehat{Q}(\gamma)=0$ for $\gamma\in A_2^{2n}\cap\mathbf{Z}^d$. Hence, by (12), $\operatorname{supp}\widehat{Q}\subseteq\mathbf{Z}^d\setminus\bigcup\mathcal{A}$. This, by (13), contradicts (**). Thus we have shown that (B) and (**) imply (10), which completes the induction.

Step 3. *If Q satisfies (**) then* (C)$\Rightarrow$(D) *where*

(D) *there exist a sequence $(\alpha_n)_{n=1}^{\infty}\subseteq\mathbf{Z}^d$, a vector $x\in\mathbf{R}^d$ and $M>0$ such that for $n=1,2,\ldots$*

$$\widehat{Q}(\alpha_n)=1\,, \tag{15}$$

$$\operatorname{supp}\widehat{Q}\cap B(\alpha_n,n)\cap\{y:|\langle y-a_k',x\rangle|>M\}=\emptyset\,, \tag{16}$$

(17) *the sequence $(\langle\alpha_n,x\rangle)_{n=1}^{\infty}$ is unbounded.*

Proof. Apply Lemma 3(ii) for $s_n=n$ ($n=1,2,\ldots$). Since $x\in\mathbf{Z}^d$, $x/|x|=y_k$ for some $k\in\mathbf{N}$. Thus, by the "moreover" part of (C) we obtain (17). ∎

Step 4. *The assumption* (D) *leads to a contradiction.*

Proof. Let $n=3^m$. Passing to a subsequence of $(\alpha_n)_{n=1}^{\infty}$ if necessary, we may assume that after appropriate renumbering of coordinates

$$\alpha_k^{(1)}\to\infty\quad\text{as } k\to\infty\,, \tag{18}$$

$$\bigcup_{k=1}^{\infty}(B(\alpha_k,k)\cup B(-\alpha_k,k))\quad\text{is } 1/(3n)\text{-stable for } \psi\,. \tag{19}$$

Next we pick inductively an m-element subsequence $(\gamma_j)_{j=1}^{m}$ of $(\alpha_k)_{k=1}^{\infty}$ and balls $(B(\gamma_j,n_j))_{j=1}^{\infty}$ such that

$$\gamma_j^{(1)}>0\qquad(j=1,\ldots,m)\,, \tag{20}$$

$$n_j>\sum_{k=1}^{j-1}|\gamma_k|+M\qquad(j=1,\ldots,m)\,, \tag{21}$$

$$|\langle x,\gamma_j\rangle|>\sum_{k=1}^{j-1}|\gamma_k|+M\qquad(j=1,\ldots,m) \tag{22}$$

where x and M are the same as in (16) and (17).

Next we consider the "Riesz product"

$$g(t)=\prod_{j=1}^{m}\Big(1+\frac{e^{2\pi i\langle\gamma_j,t\rangle}+e^{2\pi i\langle-\gamma_j,t\rangle}}{2}\Big)=\prod_{j=1}^{m}(1+\cos\langle\gamma_j,t\rangle)\,.$$

Using (21) and the standard properties of Riesz products (cf. [G-McG]) we infer that

(23) all the vectors $\sum_{j=1}^{m}\sigma_j\gamma_j$ are distinct where σ_j is either 0 or 1 or -1 for $j=1,2,\ldots$

Hence

$$\|g\|_1=\int_{\mathbf{T}^d}|g(t)|\,dt=\int_{\mathbf{T}^d}g(t)\,dt=\widehat{g}(0)=1\,, \tag{24}$$

$$\widehat{g}(\gamma_j)=1/2\quad\text{for } j=1,2,\ldots \tag{25}$$

Furthermore, (21) implies

$$\operatorname{supp}\widehat{g}\subseteq\{0\}\cup\bigcup_{j=1}^{\infty}(B(\gamma_j,n_j)\cup B(-\gamma_j,n_j))\,. \tag{26}$$

Now, using (22) we get for $j = 1, \ldots, m$

$$\operatorname{supp}\widehat{g} \cap B(\gamma_j, n_j) \cap \{y : |\langle y - \gamma_j, x\rangle| < M\} = \{\gamma_j\}. \tag{27}$$

Define the trigonometric polynomial $f : \mathbf{T}^d \to \mathbf{C}$ by

$$\widehat{f}(\gamma) = \begin{cases} \widehat{Q}(\gamma)\widehat{g}(\gamma) & \text{for } \gamma \in \mathbf{Z}^d \setminus \{0\}, \\ 0 & \text{for } \gamma = 0. \end{cases} \tag{28}$$

It follows from (26), (27), (15) and (16) that

$$\operatorname{supp}\widehat{f} \cap B(\gamma_j, n_j) = \{\gamma_j\} \quad \text{for } j = 1, \ldots, m. \tag{29}$$

Now we show that

$$\|f\|_1 \ge C \log m. \tag{30}$$

Indeed, since $\operatorname{supp}\widehat{f} \subseteq \operatorname{supp}\widehat{g}$, it follows from (25), (26) and (29) that

$$f(t) = \frac{1}{2}\sum_{j=1}^{m} e^{2\pi i\langle\gamma_j, t\rangle} + W(t)$$

where the trigonometric polynomial W satisfies: if $\gamma \in \operatorname{supp}\widehat{W}$ then $\gamma^{(1)} \le 0$. Hence, taking into account (20) for $t = (t^{(1)}, \ldots, t^{(d)})$ we have

$$f(t) = \sum_{j=1}^{m} a_j(t^{(2)}, \ldots, t^{(d)}) e^{2\pi i \gamma_j^{(1)} t^{(1)}} + \sum_{\{\gamma \in \operatorname{supp}\widehat{f} : \gamma^{(1)} \le 0\}} b_\gamma(t^{(2)}, \ldots, t^{(d)}) e^{2\pi i \gamma^{(1)} t^{(1)}}$$

where $|a_j(t^{(2)}, \ldots, t^{(d)})| = 1/2$. Thus, using the McGehee–Pigno–Smith inequality (cf. [McG-P-S]) we get, for some $C > 0$,

$$\begin{aligned}\int_{\mathbf{T}^d} |f(t)|\, dt &= \int_{\mathbf{T}} \cdots \int_{\mathbf{T}} dt^{(2)} \ldots dt^{(d)} \int_{\mathbf{T}} |f(t^{(1)}, \ldots, t^{(d)})|\, dt^{(1)} \\ &\ge \left(\frac{1}{2\pi}\right)^d \int_{\mathbf{T}} \cdots \int_{\mathbf{T}} \sum_{j=1}^{m} \frac{|a_j(t^{(2)}, \ldots, t^{(d)})|}{j}\, dt^{(2)} \ldots dt^{(d)} \\ &\ge C \log m.\end{aligned}$$

The set $\operatorname{supp}\widehat{g}$ has at most 3^m elements and it is $1/(3n)$-stable for ψ (this follows from (19) and (26)). It follows from Lemma 1 that there exists a translation invariant isomorphism, say H, from $L^1_{|\operatorname{supp}\widehat{g}}$ to $L^1_{\psi|\operatorname{supp}\widehat{g}}$ with $\|H\| \cdot \|H^{-1}\| \le 2$. It follows from the comparison of the Fourier coefficients of f and g that $Q \circ H(g - 1) = Hf$. Thus, by (30),

$$\begin{aligned}C \log m &\le \|f\|_1 \le \|H^{-1}\| \cdot \|Hf\| \\ &\le \|H^{-1}\| \cdot \|Q\| \cdot \|H\| \cdot \|g - 1\| \le 2\|Q\| \cdot \|H^{-1}\| \cdot \|H\|.\end{aligned}$$

Hence $\|Q\| \ge \frac{1}{4} C \log m$, which is impossible for large m. ∎

Remark. In Step 4 of the above proof, the McGehee–Pigno–Smith theorem can be replaced by an argument involving the Kolmogorov theorem on the weak type of the Riesz projection.

3. Translation invariant projections on $C_\psi(\mathbf{T}^d, E)$ spaces. If ψ is a stable and asymptotically symmetric bundle then the assertion of Theorem 1 is also valid if L^1_ψ is replaced by $C_\psi(\mathbf{T}^d, E)$. For, observe that if X is a translation invariant subspace of $C(\mathbf{T}^d, E)$ and $T : X \to X$ is a translation invariant operator then T has a unique extension to a translation invariant operator from X_1 to X_1, where X_1 is the closure of X in $L^1(\mathbf{T}^d, E)$ (cf. [P-W]). However, the assumption of asymptotical symmetry is superfluous.

THEOREM 2. *If ψ is a stable bundle and $P : C_\psi \to C_\psi$ is a translation invariant projection then $\operatorname{supp}\widehat{P}$ belongs to the coset ring of $\mathbf{Z}^d$.*

Proof. First observe that all lemmas of Section 2 and Steps 1–3 are true if we replace L^1_ψ by C_ψ and $\|\ \|_1$ by $\|\ \|_\infty$. Then for sequences $(\gamma_j)_{j=1}^m$ and $(B(\gamma_j, n_j))_{j=1}^m$ satisfying (20)–(22) and for any sequence $(v_j)_{j=1}^m \subset \mathbf{C}$ we consider the following "Rudin–Shapiro construction". Define

$$g_1(t) = v_1 e^{2\pi i\langle\gamma_1, t\rangle}, \quad h_1(t) \equiv 1, \tag{31}$$

and, by induction, for $k = 2, 3, \ldots, m$

$$\begin{aligned} g_k(t) &= g_{k-1}(t) + v_k e^{2\pi i\langle\gamma_k, t\rangle} h_{k-1}(t), \\ h_k(t) &= h_{k-1}(t) - v_k e^{-2\pi i\langle\gamma_k, t\rangle} g_{k-1}(t). \end{aligned} \tag{32}$$

Put now $g = g_m$. Using (20)–(22) and standard properties of this construction (cf. [W]) we obtain

$$\|g\|_\infty \le C\Big(\sum_{j=1}^{m} |v_j|^2\Big)^{1/2} \quad \text{for some } C > 0, \tag{33}$$

$$\widehat{g}(\gamma_1) = v_1 \quad \text{for } k = 1, \ldots, m, \tag{34}$$

$$\operatorname{supp}\widehat{g} \subseteq \bigcup_{j=1}^{m} (B(\gamma_j, n_j) \setminus \{y : |\langle y - \gamma_j, x\rangle| < M\}) \cup \{\gamma_j\}. \tag{35}$$

Defining now $f : \mathbf{Z}^d \to \mathbf{C}$ via (28) we see from (34) and (35) that $\operatorname{supp}\widehat{f} = \{\gamma_1, \ldots, \gamma_m\}$ and $\widehat{f}(\gamma_j) = v_j$ for $j = 1, \ldots, m$. Hence, by lacunarity of $(\gamma_j)_{j=1}^m$, for some constant $C' > 0$ we get

$$\|f\|_\infty \ge C' \sum_{j=1}^{m} |v_j|. \tag{36}$$

By (33) and (36), we may assume that taking a suitable sequence $(v_j)_{j=1}^m$, for some $C, C' > 0$ we obtain

$$\|g\|_\infty \le C\sqrt{m}, \tag{37}$$

$$\|f\|_\infty \ge C'm. \tag{38}$$

By (19) and Lemma 1, there exists a translation invariant isomorphism

$$H : C_{|\operatorname{supp}\hat{g}} \to C_{\psi|\operatorname{supp}\hat{g}}$$

with $\|H\| \cdot \|H^{-1}\| \le 2$. Comparing Fourier coefficients we obtain $Q \circ Hg = Hf$. Hence, by (37) and (38),

$$\begin{aligned} C'm \le \|f\|_\infty &\le \|H^{-1}\| \cdot \|Hf\| \le \|H^{-1}\| \cdot \|Q\| \cdot \|H\| \cdot \|g\|_\infty \\ &\le \|Q\| \cdot \|H^{-1}\| \cdot \|H\| \cdot C\sqrt{m}. \end{aligned}$$

Thus $\|Q\| \ge 2(C'/C)\sqrt{m}$, which is impossible for large m. ∎

4. Application to Sobolev spaces

THEOREM 3. *If the smoothness S is elliptic then for each translation invariant projection $P : W_S^1(\mathbf{T}^d) \to W_S^1(\mathbf{T}^d)$ the set $\operatorname{supp} \widehat{P}$ belongs to the coset ring of $\mathbf{Z}^d$.*

Proof. By Theorem 1 it is enough to prove that the bundle ψ_S corresponding to the smoothness S is stable and asymptotically symmetric. Assume that $\deg Q_S = m$ and

$$|Q_S(\xi)| \ge C|\xi|^m \quad \text{for } \xi \in \mathbf{R}^d.$$

Stability. Expanding the polynomial Q_S at a point $\xi \in \mathbf{R}^d$ in a Taylor series with respect to $h \in \mathbf{R}^d$ we get

$$Q_S(\xi + h) = Q_S(\xi) + \sum P_\alpha(\xi) h^\alpha$$

where P_α are polynomials with $\deg P_\alpha < m$ and the sum on the right hand side extends over all nonzero $\alpha \in \mathbf{Z}^d$ with $\alpha^{(j)} \ge 0$, for $j = 1, \ldots, d$, and $\sum_{j=1}^d \alpha^{(j)} < m$. In particular, $\lim_{|\xi|\to\infty} |P_\alpha(\xi)|/|\xi|^m = 0$. Thus the ellipticity of Q_S yields

$$\lim_{|\xi|\to\infty} Q_S(\xi + h)/Q_S(\xi) = 1.$$

Hence for $\xi \in \mathbf{Z}^d$ and every fixed $h \in \mathbf{Z}^d$,

$$\begin{aligned} &\lim_{|\xi|\to\infty} (d(\psi(\xi+h), \psi(\xi)))^2 \\ &= \lim_{|\xi|\to\infty} \sum_{D\in S} \left| \frac{\widehat{D}(\xi+h)}{(Q_S(\xi+h))^{1/2}} - \frac{\widehat{D}(\xi)}{(Q_S(\xi))^{1/2}} \right|^2 \end{aligned}$$

$$\begin{aligned} &= \lim_{|\xi|\to\infty} \sum_{D\in S} \left| \frac{\widehat{D}(\xi+h)}{(Q_S(\xi))^{1/2}} \left(\frac{Q_S(\xi)}{Q_S(\xi+h)} \right)^{1/2} - \frac{\widehat{D}(\xi)}{(Q_S(\xi))^{1/2}} \right|^2 \\ &= \lim_{|\xi|\to\infty} \sum_{D\in S} \left| \frac{\widehat{D}(\xi+h) - \widehat{D}(\xi)}{(Q_S(\xi))^{1/2}} \right|^2 \le \lim_{|\xi|\to\infty} \sum_{D\in S} \left| \frac{\widehat{D}(\xi+h) - \widehat{D}(\xi)}{C|\xi|^m} \right|^2 = 0 \end{aligned}$$

(because $\deg(D(\xi+h) - D(\xi))^2 < m$). This is equivalent to stability of ψ.

Asymptotical symmetry. Consider two cases: m odd and m even. For m even and $\gamma \in \mathbf{Z}^d$ remembering that $Q_S(\gamma) = Q_S(-\gamma)$ we have

$$\psi_S(\gamma) = \operatorname{span}\left\{ \left(\frac{\widehat{D}(\gamma)}{(Q_S(\gamma))^{1/2}} \right)_{D\in S} \right\},$$

$$\psi_S(-\gamma) = \operatorname{span}\left\{ \left(\frac{\widehat{D}(-\gamma)}{(Q_S(\gamma))^{1/2}} \right)_{D\in S} \right\}.$$

Hence we obtain

$$\begin{aligned} (d(\psi_S(\gamma), \psi_S(-\gamma)))^2 &= \sum_{D\in S} \frac{|\widehat{D}(\gamma) - \widehat{D}(-\gamma)|^2}{Q_S(\gamma)} = \sum_{\deg D<m} \frac{|\widehat{D}(\gamma) - \widehat{D}(-\gamma)|^2}{Q_S(\gamma)} \\ &\le \sum_{|\alpha|<m} \frac{2|\gamma^\alpha|}{Q_S(\gamma)} \quad \text{for } |\gamma| \to \infty. \end{aligned}$$

Thus $\lim_{|\gamma|\to\infty} d(\psi_S(\gamma), \psi_S(-\gamma)) = 0$.

For m odd the argument is similar. The only difference is that to estimate $d(\psi_S(\gamma), \psi_S(-\gamma))$ we use the vectors

$$\left(\frac{\widehat{D}(\gamma)}{(Q_S(\gamma))^{1/2}} \right)_{D\in S} \in \psi_S(\gamma) \quad \text{and} \quad \left(\frac{-\widehat{D}(-\gamma)}{(Q_S(\gamma))^{1/2}} \right)_{D\in S} \in \psi_S(-\gamma). \ ∎$$

In particular, for the classical isotropic Sobolev spaces of k times differentiable functions we obtain

COROLLARY 2. *Let $k = 1, 2, \ldots$ and let $P : W_k^1(\mathbf{T}^d) \to W_k^1(\mathbf{T}^d)$ (resp. $P : C^k(\mathbf{T}^d) \to C^k(\mathbf{T}^d)$) be a translation invariant projection. Then $\operatorname{supp} \widehat{P}$ belongs to the coset ring of $\mathbf{Z}^d$.*

For the proof apply Theorem 3 and observe that the fundamental polynomial $Q_k(\xi) = \sum_{|\alpha|\le k} |\xi^\alpha|^2$ is elliptic. ∎

Remark 1. Theorems 1–3 and Corollaries 1 and 2 can be stated as characterizations of the translation invariant projections in terms of the coset ring. Namely, in each case, for every member of the coset ring, its characteristic function coincides with the multiplier of some projection in a corresponding Banach space. The proof is done by repeating the argument in Step 1 of the proof of Theorem 1.

Remark 2. A necessary and sufficient condition for the stability of a d-dimensional smoothness S with $d \geq 2$ is the following. For every nonempty set $A \subset \{1,\ldots,d\}$ and every $\alpha \in \mathbf{Z}^d$ such that $\partial^\alpha \in S$ and $\alpha^{(k)} \neq 0$ for at least one $k \in A$, the orthogonal projection of α on the subspace $H_A = \{x \in \mathbf{R}^d : x^{(k)} = 0 \text{ for } k \in A\} \subseteq \mathbf{R}^d$ belongs to the interior of the convex hull of the set $\{\gamma \in \mathbf{Z}^d : \partial^\gamma \in S \text{ and } \gamma \in H_A\} \subseteq \mathbf{R}^d \cap H_A$.

Remark 3. A necessary condition for the asymptotical symmetry of the smoothness S is the following. The extremal points of $\operatorname{conv}\{\gamma \in \mathbf{Z}^d : \partial^\gamma \in S\} \setminus \{0\}$ are either all even, or all odd ($\gamma \in \mathbf{Z}^d$ is *even* if $\gamma^{(1)} + \ldots + \gamma^{(d)}$ is even and it is odd otherwise).

Remark 4. It follows from Remarks 2 and 3 that there exist smoothnesses which are: stable and not asymptotically symmetric; asymptotically symmetric and not stable; and neither stable nor asymptotically symmetric.

5. Appendix

Proof of (12). We will prove (12) for a family $\mathcal{A}$ defined by any family of hyperplanes $\mathcal{R}$ in $\mathbf{R}^d$. We use induction with respect to the number of elements of $\mathcal{R}$. The statement is trivial if $\mathcal{R}$ is a one-element set. Fix a family $\mathcal{R}$ with $k > 1$ elements and assume (12) for all families with less than k elements. There are two possibilities:

1° Among the hyperplanes $H_1, \ldots, H_k$ in $\mathcal{R}$ there are two parallel, say $H_{k-1} \| H_k$. Put $\mathcal{R}' = \mathcal{R} \setminus \{H_k\}$ and let $\mathcal{A}'$ be the family of all components of $\mathbf{R}^d \setminus \bigcup \mathcal{R}'$ which are not contained in any strip determined by two parallel rational hyperplanes. Then there exists a 1-1 map $\Phi : \mathcal{A} \to \mathcal{A}'$ such that $A \subseteq \Phi(A)$ for $A \in \mathcal{A}$ and $A \sim B$ if and only if $\Phi(A) \sim \Phi(B)$. By the inductive hypothesis for any $A, B \in \mathcal{A}$ there exists a chain $A_1, \ldots, A_s$ such that $A_1 = \Phi(A)$, $A_s = \Phi(B)$ and $A_j \sim A_{j+1}$ for $j = 1, \ldots, s-1$. Then the chain $\Phi^{-1}(A_1), \ldots, \Phi^{-1}(A_s)$ joins A and B.

2° No two hyperplanes in $\mathcal{R}$ are parallel. Define $\mathcal{R}'$ and $\mathcal{A}'$ like in case 1°, and define $\Phi : \mathcal{A} \to \mathcal{A}'$ by

$$\Phi(A) = \text{ the unique } X \in \mathcal{A}' \text{ with } A \subseteq X.$$

The function Φ has the following properties:

(i) $A \subseteq \Phi(A)$ for $A \in \mathcal{A}$,

(ii) $\Phi^{-1}(X) \subseteq \mathcal{A}$ consists of one or two elements for every $X \in \mathcal{A}'$,

(iii) if $X \sim Y$ for $X, Y \in \mathcal{A}'$ then there exist $A \in \Phi^{-1}(X)$ and $B \in \Phi^{-1}(Y)$ such that $A \sim B$,

(iv) if $A, B \in \Phi^{-1}(X)$ and $A \neq B$ then $A \sim B$.

To prove (iii) and (iv) it is enough to observe that, in case 2°, $A \sim B$ if and only if the intersection of the closures $\bar{A} \cap \overline{B}$ contains some $(d-1)$-dimensional halfcone. By the inductive hypothesis for any $A, B \in \mathcal{A}$ there exists a chain $X_1, \ldots, X_s \in \mathcal{A}'$ such that $X_1 = \Phi(A)$, $X_2 = \Phi(B)$ and $X_i \sim X_{i+1}$ for $i = 1, \ldots, s-1$. Observe that for any $i < s$, by (iii), there exist $A_i, B_i \in \mathcal{A}$ such that $A_i \subseteq X_i$, $B_i \subseteq X_{i+1}$ and $A_i \sim B_i$; by (iv), either $A_i = B_{i-1}$ or $A_i \sim B_{i-1}$. Thus we have (12). ■

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Received November 22, 1990 (2747)