ANNALES
POLONICI MATHEMATICI
LXXI.2 (1999)

A remark on the Lion–Rolin Preparation Theorem for LA-functions

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Abstract. A correct formulation of the Lion–Rolin Preparation Theorem for logarithmic-subanalytic functions (LA-functions) is given.

In [2] Lion and Rolin give an explicit description of functions on $\mathbb{R}^n$ ($n \in \mathbb{Z}$, $n > 0$), called by them *LE-functions*, defined as finite compositions of globally subanalytic functions with logarithmic and with exponential functions. This enables them to obtain the fundamental results of van den Dries, Macintyre and Marker [1] without making use of model theory. One important step in their study is their Preparation Theorem for LA-functions. To quote this theorem we first recall some basic definitions from [2].

If $\mathcal{F}$ is any family of real functions on $\mathbb{R}^n$, a subset E of $\mathbb{R}^n$ is called an $\mathcal{F}$*-set* if

$$E = \bigcup_{i \in I} \bigcap_{j \in J} E_{ij}$$

where I and J are finite and for each $(i,j) \in I \times J$, $E_{ij} = \{\phi_{ij} > 0\}$ or $E_{ij} = \{\phi_{ij} = 0\}$ or $E_{ij} = \{\phi_{ij} < 0\}$, with some $\phi_{ij} \in \mathcal{F}$.

A subset C of $\mathbb{R}^{n+1} = \{(x,y) \mid x \in \mathbb{R}^n,\ y \in \mathbb{R}\}$ is called an $\mathcal{F}$*-cylinder* if either

$$C = \{(x,y) \mid x \in B,\ y = \phi(x)\}$$

where B is an $\mathcal{F}$-set in $\mathbb{R}^n$ and $\phi \in \mathcal{F}$, or

$$C = \{(x,y) \mid x \in B,\ \phi(x) < y < \psi(x)\},$$

where B is an $\mathcal{F}$-set in $\mathbb{R}^n$, $\phi, \psi \in \mathcal{F} \cup \{-\infty\}^B \cup \{\infty\}^B$ and $\phi(x) < \psi(x)$ for each $x \in B$.

Let $f : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. We say that

1991 *Mathematics Subject Classification*: Primary 32B20; Secondary 33B10.
Key words and phrases: subanalytic function, logarithmic-subanalytic function, reducible.

(1) f is an *LA-function of type* 0 if f can be represented as

$$f(x,y)=F(a_1(x),\ldots,a_m(x),y),$$

where $m\in\mathbb{Z}$, $m>0$, F is a globally subanalytic function on $\mathbb{R}^{m+1}$, and a_j are LE-functions on $\mathbb{R}^n$;

(2) f is an *LA-function of type* r, $r\ge 1$, if f can be represented as

$$f(x,y)=F(f_1(x,y),\ldots,f_m(x,y),\log|f_{m+1}(x,y)|,\ldots,\log|f_{m+l}(x,y)|),$$

where F is globally subanalytic, $m,l\in\mathbb{Z}$, $m,l>0$, and f_i are LA-functions of type $r-1$.

An LA-function $f:\mathbb{R}^{n+1}\to\mathbb{R}$ of type r is called *reducible* if there exists a finite covering $\mathcal{C}$ of $\mathbb{R}^{n+1}$ by LE-cylinders such that, on each $C\in\mathcal{C}$, f can be represented in the form

$$f(x,y)=|y_0|^{\alpha_0}\ldots|y_r|^{\alpha_r}A(x)U(x,y_0,\ldots,y_r),$$

where $y_0=y-\theta_0(x)$, $y_1=\log|y_0|-\theta_1(x)$, $\ldots$, $y_r=\log|y_{r-1}|-\theta_r(x)$, with some LE-functions θ_j such that $|y_j|\le M|\theta_j(x)|$ on C, A is an LE-function, $\alpha_j\in\mathbb{Q}$ $(j=0,\ldots,r)$, and

$$U(x,y_0,\ldots,y_r)=V(\psi(x,y)),$$

where

$$\psi(x,y)=\left(\phi_1(x),\ldots,\psi_s(x),\frac{|y_0|^{1/p_0}}{a_0(x)},\ldots,\frac{|y_r|^{1/p_r}}{a_r(x)},\ldots,\frac{b_0(x)}{|y_0|^{1/p_0}},\ldots,\frac{b_r(x)}{|y_r|^{1/p_r}}\right),$$

with some LE-functions ϕ_i, a_j, b_j, positive integers p_j, and an analytic non-vanishing function V of constant sign in a neighbourhood of the compact set $\overline{\psi(C)}$ in $(\mathbb{P}^1)^{s+2r+2}$, where $\mathbb{P}^1$ denotes the real projective line.

Lion and Rolin formulate the following Preparation Theorem for LA-functions [2, Théorème 2]: *Every LA-function is reducible.*

Our goal here is to observe that this formulation requires some correction. To see this, consider the function $f:\mathbb{R}\to\mathbb{R}$ defined as follows:

$$f(y)=\begin{cases} y^y=e^{y\log y} & \text{if } y\in(0,1),\\ 0 & \text{if } y\notin(0,1).\end{cases}$$

Of course, f is an LA-function of type 1. However, it is not reducible in the above sense. If it were, we would have the following equality in some interval $(0,\varepsilon)$ $(\varepsilon>0)$:

$$f(y)=|y_0|^{\alpha_0}|y_1|^{\alpha_1}V(|y_0|^{1/p_0},|y_1|^{1/p_1},|y_0|^{-1/p_0},|y_1|^{-1/p_1}),$$

where $y_0=y-\theta_0$, $y_1=\log|y_0|-\theta_1$, p_0, p_1 are positive integers, $\alpha_0,\alpha_1\in\mathbb{Q}$ and V is an analytic positive function in a neighbourhood of $\overline{\psi(0,\varepsilon)}$ in $(\mathbb{P}^1)^4$, where

$$\psi(y)=(|y_0|^{1/p_0},|y_1|^{1/p_1},|y_0|^{-1/p_0},|y_1|^{-1/p_1}).$$

If $\theta_0 \neq 0$, then f would be subanalytic near 0, which is not the case, so suppose that $\theta_0 = 0$. Then

$$f(y) = |y|^{\alpha_0}|y_1|^{\alpha_1}\widetilde{V}(|y|^{1/p_0}, |y_1|^{-1/p_1}),$$

where $\widetilde{V}$ is analytic in a neighbourhood of $(0,0)$ and $\widetilde{V}(0,0) > 0$.

If $\alpha_0 \neq 0$ or $\alpha_1 \neq 0$, the right-hand side would tend to 0 or ∞ as y tends to 0; a contradiction. Hence,

$$f(y) = \widetilde{V}(|y|^{1/p_0}, |y_1|^{-1/p_1}) \quad \text{and} \quad y\log y = (\log \widetilde{V})(|y|^{1/p_0}, |y_1|^{-1/p_1})$$

for $y \in (0, \varepsilon')$. Expanding $\log \widetilde{V}$ with respect to the first variable, we have

$$\begin{aligned} y\log y &= \sum_{\nu=k}^{\infty} C_\nu(|y_1|^{-1/p_1})|y|^{\nu/p_0} \\ &= C_k(|y_1|^{-1/p_1})|y|^{k/p_0} \\ &\quad \times \left[1 + \frac{|y|^{1/p_0}}{C_k(|y_1|^{-1/p_1})} \sum_{\nu>k}^{\infty} C_\nu(|y_1|^{-1/p_1})|y|^{(\nu-k-1)/p_0}\right]. \end{aligned}$$

Hence $y^{1-k/p_0}(\log y)[C_k(|y_1|^{-1/p_1})]^{-1}$ tends to 1 as y tends to 0; this is asymptotically equivalent to

$$y^{1-k/p_0}(\log y)|y_1|^{l/p_1} \quad \text{for some } l > 0,$$

so tends to 0 or ∞.

This example indicates that in order to obtain a correct formulation of the theorem one should allow ψ in the definition of reducibility in a more general form, viz.

$$\psi(x,y) = (a_1(x)|y_0|^{\beta_{10}} \ldots |y_r|^{\beta_{1r}}, \ldots, a_s(x)|y_0|^{\beta_{s0}} \ldots |y_r|^{\beta_{sr}}),$$

where $\beta_{ij} \in \mathbb{Q}$.

The proof after this modification is that of [2].

References

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Reçu par la Rédaction le 24.9.1998