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Continuous linear extension operators on spaces of holomorphic functions on closed subgroups of a complex Lie group

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Abstract. We show that the restriction operator of the space of holomorphic functions on a complex Lie group to an analytic subset V has a continuous linear right inverse if it is surjective and if V is a finite branched cover over a connected closed subgroup Γ of G. Moreover, we show that if Γ and G are complex Lie groups and $V \subset \Gamma \times G$ is an analytic set such that the canonical projection $\pi_1 : V \to \Gamma$ is finite and proper, then $R_V : O(\Gamma \times G) \to \operatorname{Im} R_V \subset O(V)$ has a right inverse.

Introduction. Let M be a complex space. We denote by $O(M)$ the Fréchet space of analytic functions on M equipped with the topology of uniform convergence on compacta. If V is a closed subvariety of M the question of whether one can find a continuous linear extension operator from $O(V)$ into $O(M)$ was studied by various authors (see [2], [10], [12]). For example if V is a closed subvariety of $\mathbb{C}^n$ a continuous linear extension operator exists if V is an algebraic variety of $\mathbb{C}^n$ [2]. Moreover, in [8] Vogt has given an important condition for existence of a right inverse of a continuous linear surjection between nuclear Fréchet spaces.

In this note we take up the question of existence of continuous extension operators from subvarieties of $\mathbb{C}^n$, in the category of analytic subsets in a complex Lie group, by using the splitting theorem of Vogt. Namely, we prove the following two theorems.

Theorem 1. *Let Γ be a connected closed subgroup of a complex Lie group G and V an analytic set in G such that V is a branched cover over Γ and the restriction map $R_V : O(G) \to O(V)$ is surjective. Then R_V has a right inverse.*

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THEOREM 2. *Let Γ and G be complex Lie groups and $V \subset \Gamma \times G$ an analytic set such that the canonical projection $\pi_1 : V \to \Gamma$ is finite and proper. Then $R_V : O(\Gamma \times G) \to \operatorname{Im} R_V \subset O(V)$ has a right inverse.*

We now recall some definitions and relevant properties. Let E be a Fréchet space with a fundamental system $\{\|\cdot\|_k\}$ of seminorms. We say that E has the property

- (DN) if there exists p such that $\forall q$, $\exists k$, $\exists C > 0$:
$$\|\cdot\|_q^2 \le C\|\cdot\|_k\,\|\cdot\|_p,$$
- (Ω) if $\forall p$, $\exists q$, $\forall k$, $\exists C, d > 0$:
$$\|\cdot\|_q^{*1+d} \le C\|\cdot\|_k^*\,\|\cdot\|_p^{*d},$$
- $(\overline{\Omega})$ if $\exists d > 0$, $\forall p$, $\exists q$, $\forall k$, $\exists C > 0$:
$$\|\cdot\|_q^{*1+d} \le C\|\cdot\|_k^*\,\|\cdot\|_p^{*d},$$

where for each p we define $\|x^*\|_p^* = \sup\{x^*(x) : \|x\|_p \le 1\}$ for $x^* \in E^*$, the dual space of E.

The properties (DN), (Ω), $(\overline{\Omega})$ and many other properties were introduced and investigated by Vogt. It is known [8] that a Fréchet space $F \in (DN)$ (respectively $F \in (\Omega)$) if and only if F is isomorphic to a subspace (respectively a quotient space) of the space s of rapidly decreasing sequences of complex numbers. In [8], Vogt has proved that a continuous linear map R from a nuclear Fréchet space E onto a nuclear Fréchet space F has a right inverse if $F \in (DN)$ and $\operatorname{Ker} R \in (\Omega)$.

By the above splitting theorem of Vogt, to prove Theorems 1 and 2, it suffices to show that
$$O(V), \operatorname{Im} R_V \in (DN) \quad \text{and} \quad \operatorname{Ker} R_V \in (\Omega).$$
The proofs of these relations are given in Sections 1 and 3 respectively.

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1. Proof of Theorem 1

LEMMA 1.1. *Let θ be a finite proper holomorphic map from a complex space X onto a complex manifold Y. Then $O(X) \in (DN)$ if and only if $O(Y) \in (DN)$.*

Proof. Since $O(Y)$ is a subspace of $O(X)$, the necessity is trivial.

Now, we prove the sufficiency. It is known [10] that a Fréchet space $F \in (DN)$ if and only if every continuous linear map T from $\Lambda_1(\alpha)$ into F is bounded on a neighbourhood of zero in $\Lambda_1(\alpha)$, where α is any exponent sequence and

$$\Lambda_1(\alpha) = \Big\{(\xi_j) \subset \mathbb{C}^n : \sum_{j=1}^{\infty} |\xi_j| r^{\alpha_j} < \infty \text{ for } 0 < r < 1\Big\}.$$

Assume that $O(Y) \in (DN)$. We must prove $O(X) \in (DN)$. By the above mentioned result of Vogt it suffices to show that every continuous linear map T from $\Lambda_1(\alpha)$ into $O(X)$ is bounded on a neighbourhood of zero in $\Lambda_1(\alpha)$.

By the integrality lemma [3] it follows that there exists p such that

$$f^p + a_{p-1}(f) f^{p-1} + \ldots + a_0(f) = 0$$

for every $f \in O(X)$, where $a_{p-1}(f), \ldots, a_0(f) \in O(Y)$ are given by

$$a_{p-1}(f)\ (y) = \sum_{\theta(x)=y} f(x),$$

$$\ldots$$

$$a_0(f)\ (y) = \prod_{\theta(x)=y} f(x).$$

Clearly $a_{p-1}(f), \ldots, a_0(f)$ are continuous polynomials in f with values in $O(Y)$. Hence $a_{p-1}T, \ldots, a_0 T$ are also continuous polynomials on $\Lambda_1(\alpha)$. Since $\underbrace{\Lambda_1(\alpha) \hat{\otimes}_\pi \ldots \hat{\otimes}_\pi \Lambda_1(\alpha)}_{(p-1)\text{ times}}, \ldots, \Lambda_1(\alpha) \in (\overline{\Omega})$, by the theorem of Vogt $a_{p-1}T, \ldots, a_0 T$ and hence T are bounded on a neighbourhood of zero in $\Lambda_1(\alpha)$. ■

LEMMA 1.2. $O(V) \in (DN)$.

P r o o f. As V is a branched cover over Γ, by Lemma 1.1 it suffices to show that $O(\Gamma) \in (DN)$.

Put $\Gamma_e = \{z \in \Gamma : f(z) = f(e) \text{ for every } f \in O(\Gamma)\}$. It is well known [6] that Γ_e is abelian and normal. Moreover $\dim O(\Gamma_e) = 1$ and Γ/Γ_e is Stein. This yields that $O(\Gamma) \cong O(\Gamma/\Gamma_e)$ and hence we may assume that Γ is Stein.

We now prove that $O(\Gamma) \in (DN)$. By the theorem of Zaharyuta [12] it suffices to check that every plurisubharmonic function φ on Γ with $\sup_\Gamma \varphi < \infty$ is constant.

Consider the exponential map $\exp\colon T_e\Gamma \to \Gamma$. Take a neighbourhood U of zero in $T_e\Gamma$ such that

$$\exp : U \cong \exp U = V \quad \text{and} \quad V = V^{-1}.$$

Given $b \in \Gamma$ and $a \in V$, let $z_a \in U$ for which $\exp z_a = a^{-1}$ and $\sigma(\lambda) = b(\exp \lambda z_a)a$ for every $\lambda \in \mathbb{C}$. Since $\varphi\sigma = \text{const}$, we have

$$(*) \qquad \varphi(ba) = \varphi\sigma(0) = \varphi\sigma(1) = \varphi(b) \qquad \text{for every } b \in \Gamma \text{ and every } a \in V.$$

Let b be an arbitrary point in Γ. By the connectedness of Γ we can find $a_1 = e, a_2, \ldots, a_n \in V$ such that $b = a_1 a_2 \ldots a_n$. By $(*)$ we have

$$\varphi(b) = \varphi(a_1 a_2 \ldots a_n) = \varphi(a_1 a_2 \ldots a_{n-1}) = \ldots = \varphi(a_1) = \varphi(e).$$

Thus $\varphi = \text{const}$ and $O(\Gamma) \in (DN)$. ∎

LEMMA 1.3. *Let X be a Stein space. Then $H^0(X, \mathcal{S}) \in (\Omega)$ for every coherent sheaf $\mathcal{S}$ on X.*

Proof. Let $\{K_p\}$ be an increasing exhaustion sequence of compact sets in X. By the Cartan Theorem A, for each $x \in X$ there exist a neighbourhood U_x of x and $\sigma_{1x}, \ldots, \sigma_{mx} \in H^0(X, \mathcal{S})$ which generate $\mathcal{S}_y$ for every $y \in U_x$.

By the compactness of K_p there exists a sequence $\{\sigma_n\} \subset H^0(X, \mathcal{S})$ such that $\{\sigma_{nx}\}$ generate $\mathcal{S}_x$ for every $x \in X$.

Since $H^0(X, \mathcal{S})$ is Fréchet we may assume that $\{\sigma_n\}$ is bounded in $H^0(X, \mathcal{S})$. Consider the Banach coherent sheaf $O_X^{\ell^1}$ of germs of holomorphic functions on X with values in ℓ^1 and the morphism η from $O_X^{\ell^1}$ into $\mathcal{S}$ given by

$$\eta(f)(x) = \sum_{n \ge 1} \sigma_n(x) f_n(x) \quad \text{for } f = \{f_n\} \in O_X^{\ell^1}.$$

By the choice of σ_n we infer that η is surjective. By a theorem of Leiterer [5], $\operatorname{Ker} \eta$ is a Banach coherent sheaf and hence $H^1(X, \operatorname{Ker} \eta) = 0$ (see [5]). It follows that the map $\widehat{\eta} : H^0(X, O_X^{\ell^1}) \cong O(X, \ell^1) \to H^0(X, \mathcal{S})$ is surjective.

On the other hand, since $O(X, \ell^1) \cong O(X) \hat{\otimes}_\pi \ell^1 \in (\Omega)$ when $O(X) \in (\Omega)$, it remains to check that $O(X) \in (\Omega)$. For each n, let X_n denote the union of irreducible branches of X of dimension $\le n$. We have

$$O(X) \cong \lim\operatorname{proj} O(X_n)$$

and the restriction maps $R_n : O(X) \to O(X_n)$ are surjective. Hence $O(X) \in (\Omega)$ if $O(X_n) \in (\Omega)$ for $n \ge 1$. For each $n \ge 1$, choose a proper injection $\theta : X_n \to \mathbb{C}^{2n+1}$. Since $O(\mathbb{C}^{2n+1}) \in (\Omega)$ we have

$$O(X_n) \cong H^0(\mathbb{C}^{2n+1}, (\theta_n) * O_{X_n}) \in (\Omega). \ \blacksquare$$

REMARK 1.4. While this paper was in preparation, we were not aware of the results of D. Vogt [11] and A. Aytuna [1] who had proved Lemma 1.3 earlier. We thank the referee for pointing out these papers.

LEMMA 1.5. $\operatorname{Ker} R_V = J(V) = \{f \in O(G) : f|_V = 0\} \in (\Omega)$.

Proof. Let η denote the canonical map from G onto G/G_e and let

$$\widehat{V} = \{\overline{z} \in G/G_e : f(\overline{z}) = 0 \text{ for every } f \in J(V)\}.$$

Then $J(V) = J(\widehat{V})$ and as G/G_e is Stein we have

$$J(\widehat{V}) = H^0(G/G_e, J_{\widehat{V}})$$

where $J_{\widehat{V}}$ denotes the coherent ideal sheaf defined by $\widehat{V}$. By Lemma 1.3, this yields that $J(\widehat{V}) \in (\Omega)$ and hence $J(V) \in (\Omega)$. ∎

Now Theorem 1 is deduced immediately from Lemmas 1.2 and 1.5. ∎

2. It is known [7] that every non-compact connected complex Lie group G with $\dim O(G) = 1$ contains a closed subgroup Γ for which R_Γ is not surjective.

Thus the following question arises naturally. When is the restriction map R_V in Theorem 1 surjective?

The following proposition gives an answer.

PROPOSITION 2.1. *Let Γ be a connected closed subgroup of a complex Lie group G such that $G_e \subset \Gamma$. Then $R_\Gamma : O(G) \to O(\Gamma)$ is surjective.*

Proof. By [6] there exists a closed subgroup K of G such that for some n the groups G and $K \times \mathbb{C}^n$ are isomorphic as complex Lie groups.

Moreover, there exists a closed Stein subgroup S_0 of K such that for the centre Z of K, the map

$$\varrho_0 : Z \times S_0 \to K, \qquad (x, y) \mapsto xy,$$

is a finite covering homomorphism.

By the result of [6], $G_e \subset Z$ and $Z \cong G_e \times \mathbb{C}^{*\nu} \times \mathbb{C}^\mu$ for some non-negative integers ν and μ.

Putting $S = \mathbb{C}^{*\nu} \times \mathbb{C}^\mu \times S_0 \times \mathbb{C}^n$ we get a finite covering homomorphism $\varrho : G_e \times S \to G$ of degree n, given by

$$\varrho(x_0, x_1, x_2, x_3, x_4) = (\varrho_0((x_0, x_1, x_2), x_3), x_4).$$

It is easy to see that

$$\varrho^{-1}(\Gamma) = G_e \times (\Gamma \cap S).$$

Since S is Stein, the restriction map $\widetilde{R} : O(S) \to O(\Gamma \cap S)$ and hence also the restriction map $R : O(G_e \times S) \to O(G_e \times \Gamma \cap S)$ is surjective.

Now given $g \in O(\Gamma)$, define $f \in O(G)$ by

$$f(y) = \frac{1}{n} \sum_{\varrho(x,z)=y} \widehat{g}(x, z) \quad \text{with} \quad \widehat{g} \in O(G_e \times S),\ \widehat{g}|_{G_e \times \Gamma \cap S} = g\varrho.$$

Then $f|_\Gamma = g$. ∎

3. Proof of Theorem 2. Let $SO(V)$ denote the spectrum of the Fréchet algebra $O(V)$ equipped with the weak topology. Since $\pi_1 : V \to \Gamma$ is a branched covering map and $SO(\Gamma) \cong SO(\Gamma/\Gamma_e) \cong \Gamma/\Gamma_e$ it follows that π_1 induces a branched covering map $\widetilde{\pi}_1 : SO(V) \to \Gamma/\Gamma_e$.

Then $SO(V)$ is a complex space and $O(V) \cong O(SO(V))$.

Now since $\Gamma/\Gamma_e \times G/G_e$ is Stein, there exists a commutative diagram of holomorphic maps

$$\begin{tikzcd}
 & V \arrow[rr] \arrow[dl, "\delta"'] \arrow[dr, "\eta"] & & \Gamma \times G \arrow[dr, "\eta"] & \\
SO(V) \arrow[rr, "\beta"] \arrow[d, "\widetilde{\pi}_1"'] & & \operatorname{Im}\beta \arrow[rr] \arrow[dll, "\pi_1"] & & \Gamma/\Gamma_e \times G/G_e \\
\Gamma/\Gamma_e & & & &
\end{tikzcd}$$

where δ and η are canonical maps.

Then it is easy to see that β is proper and hence $\operatorname{Im}\beta$ is an analytic set in $\Gamma/\Gamma_e \times G/G_e$. Moreover, $O(\operatorname{Im}\beta) \cong \operatorname{Im} R_V$. By Lemma 1.1, $\operatorname{Im} R_V \in (DN)$ and by Lemma 1.5, $\operatorname{Ker} R_V \in (\Omega)$. Hence Vogt's splitting theorem implies that $R_V : O(\Gamma \times G) \to \operatorname{Im} R_V$ has a right inverse. ■

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