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Markov operators on the space of vector measures; coloured fractals

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Abstract. We consider the family $\mathcal{M}$ of measures with values in a reflexive Banach space. In $\mathcal{M}$ we introduce the notion of a Markov operator and using an extension of the Fortet–Mourier norm we show some criteria of the asymptotic stability. Asymptotically stable Markov operators can be used to construct coloured fractals.

Introduction. The theory of Markov operators started in 1906 when A. A. Markov showed that asymptotic properties of some stochastic processes can be studied by using stochastic matrices [8]. Such matrices define positive, linear operators on $\mathbb{R}^n$. Markov's ideas were generalized in many directions. In particular, W. Feller developed the theory of Markov operators acting on Borel measures defined on some topological spaces and E. Hopf proposed to study Markov operators on L^1 spaces (see [6]). Another important idea is to study Markov operators on an arbitrary measurable space. This approach, some historical remarks and a vast literature can be found in the book of E. Nummelin [9].

In all these generalizations a *Markov operator* is a linear operator P which satisfies the condition of nonnegativity

$$P\mu \geq 0 \quad \text{for } \mu \geq 0 \tag{0.1}$$

and the normalization property

$$P\mu(X) = 1 \quad \text{for } \mu(X) = 1,\ \mu \in \mathcal{M}_{\mathbb{R}}, \tag{0.2}$$

where $\mathcal{M}_{\mathbb{R}}$ denotes the class of real-valued measures.

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It is easy to verify that conditions (0.1) and (0.2) imply the inequality

$$|P\mu|(X) \le |\mu|(X), \tag{0.3}$$

where $|\mu|$ denotes the total variation of μ. On the other hand, for real-valued measures conditions (0.2) and (0.3) imply (0.1). The starting point of our generalization of Markov operators is the pair of conditions (0.2), (0.3).

The purpose of this paper is to develop the theory of Markov operators acting on the space $\mathcal{M}_E$ of vector measures defined on Borel subsets of a compact metric space X. These measures take values in a reflexive Banach space E. In the definition of a Markov operator on $\mathcal{M}_E$ we replace conditions (0.2), (0.3) by similar ones. Namely in (0.2) the number 1 is replaced by an arbitrary element $e \in E$ and in (0.3) the total variation by the semivariation or variation. These two norms lead to two different definitions of Markov operators. However, the main results are similar.

The Fortet–Mourier norm originally defined for real-valued measures [11] can be extended to the space $\mathcal{M}_E$. It allows us to obtain sufficient conditions for the asymptotic stability of Markov operators of both types.

The reason for studying Markov operators on vector measures is not purely theoretical. We believe that a sequence or flow of vector measures is an excellent tool for describing the evolution of complicated objects. We illustrate this possibility by proving a convergence theorem for Iterated Function Systems (see [1]) acting on vector measures. Such systems can be used to construct coloured fractals.

The paper is organized as follows. In Section 1 we introduce the Fortet–Mourier norm $\| \|_{\mathcal{F}}$ in the space $\mathcal{M}_E$, denoted in the sequel simply by $\mathcal{M}$, and we prove the completeness of some subsets of $(\mathcal{M}, \| \|_{\mathcal{F}})$. In Section 2 we study the properties of the space adjoint to $(\mathcal{M}, \| \|_{\mathcal{F}})$. The last section contains the definitions of two types of Markov operators and some criteria of the asymptotic stability.

1. Vector measures with the Fortet–Mourier norm. Let (X, ϱ) be a compact metric space and $(E, \| \|)$ a separable reflexive (real) Banach space. These assumptions will not be repeated in the sequel.

By $C(X)$ we denote the space of continuous functions $f : X \to \mathbb{R}$ with the supremum norm $\|f\|_\infty$. Let $\mathcal{B}_X$ denote the family of Borel subsets of X. By $\mathcal{M}$ we denote the space of all σ-additive measures $\mu : \mathcal{B}_X \to E$. It is well known that in the condition of σ-additivity

$$\mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} \mu(B_n),$$

where $B_n \in \mathcal{B}_X$ for $n \in \mathbb{N}$, and $B_i \cap B_j = \emptyset$ for $i \neq j$, the requirements of the weak and strong convergence of the series are equivalent. Let

$$\mathrm{Lip}_1(X) = \{f \in C(X) : |f(x) - f(z)| \le \varrho(x,z) \text{ for } x,z \in X\}.$$

In $\mathcal{M}$ we introduce the *Fortet–Mourier norm* by the formula

$$\|\mu\|_{\mathcal{F}} = \sup\Big\{\Big\|\int_X f\,d\mu\Big\| : f \in \mathrm{Lip}_1(X),\ \|f\|_\infty \le 1\Big\}. \tag{1.1}$$

For every functional $\lambda \in E^*$ and measure $\mu \in \mathcal{M}$ the set function $\nu = \lambda\mu$ is a real-valued measure for which

$$\|\nu\|_{\mathcal{F}} = \sup\Big\{\Big|\int_X f\,d\nu\Big| : f \in \mathrm{Lip}_1(X),\ \|f\|_\infty \le 1\Big\}$$

is the classical Fortet–Mourier norm. Evidently

$$\begin{aligned}\|\mu\|_{\mathcal{F}} &= \sup\Big\{\Big|\lambda\int_X f\,d\mu\Big| : f \in \mathrm{Lip}_1(X),\ \|f\|_\infty \le 1;\ \lambda \in E^*,\ \|\lambda\| \le 1\Big\}\\ &= \sup\{\|\lambda\mu\|_{\mathcal{F}} : \lambda \in E^*,\ \|\lambda\| \le 1\}\end{aligned}$$

for $\mu \in \mathcal{M}$. From this equality it follows that (1.1) defines a norm in $\mathcal{M}$. The *semivariation* of $\mu \in \mathcal{M}$ is defined by

$$\|\mu\|(B) = \sup\{|\lambda\mu|(B) : \lambda \in E^*,\ \|\lambda\| \le 1\} \quad \text{for } B \in \mathcal{B}_X, \tag{1.2}$$

where $|\lambda\mu|$ is the total variation of the real-valued measure $\lambda\mu$.

For $K > 0$ fixed, we are going to study properties of the set

$$\mathcal{M}_K = \{\mu \in \mathcal{M} : \|\mu\|(X) \le K\}. \tag{1.3}$$

Theorem 1.1. *For every sequence (μ_n) in $\mathcal{M}_K$ there exists $\mu \in \mathcal{M}_K$ and a strictly increasing sequence (m_n) of positive integers such that*

$$\lim_{n\to\infty} \lambda\int_X f\,d\mu_{m_n} = \lambda\int_X f\,d\mu \quad \textit{for } f \in C(X) \textit{ and } \lambda \in E^*. \tag{1.4}$$

In order to prove Theorem 1.1 we need the following lemma.

Lemma 1.1. *For every sequence (ν_n) of real-valued measures on $\mathcal{B}_X$ with $|\nu_n|(X) \le K$ for $n \in \mathbb{N}$ there exists a real-valued measure ν on $\mathcal{B}_X$ and a strictly increasing sequence (m_n) of positive integers such that*

$$|\nu|(X) \le \lim_{n\to\infty} |\nu_{m_n}|(X) \le K$$

and

$$\lim_{n\to\infty} \int_X f\,d\nu_{m_n} = \int_X f\,d\nu \quad \textit{for } f \in C(X).$$

Proof. By the Jordan decomposition we may restrict ourselves to the case of nonnegative measures satisfying $\nu_n(X) \le K$. If there exists a subsequence of $(\nu_n(X))$ converging to zero then the statement is immediate with $\nu = 0$. If not, we may choose a sequence (m_n) of positive integers such that the sequence $(\nu_{m_n}(X))$ of numbers converges to an $\alpha > 0$ and the sequence

$(\nu_{m_n}/\nu_{m_n}(X))$ of probabilistic measures is weakly convergent (Prokhorov's theorem, see [10; Theorems 6.1 and 6.4]) to a probability measure $\overline{\nu}$. In this case the sequence (ν_{m_n}) of measures converges weakly to $\alpha\overline{\nu}$.

Proof of Theorem 1.1. We divide the proof into two steps. Let (μ_n) be a sequence in $\mathcal{M}_K$.

STEP I. We now prove that there exists a strictly increasing sequence (m_n) of positive integers with the following property.

(P) For every functional $\lambda \in E^*$ there exists a real-valued measure ν_λ on $\mathcal{B}_X$ such that

$$|\nu_\lambda|(X) \le K\|\lambda\| \tag{1.5}$$

and

$$\lim_{n\to\infty} \int_X f\, d(\lambda\mu_{m_n}) = \int_X f\, d\nu_\lambda \quad \text{for } f \in C(X). \tag{1.6}$$

Fix $\lambda \in E^*$. According to (1.2) and (1.3) we have

$$|\lambda\mu_n|(X) \le \|\lambda\| \cdot \|\mu_n\|(X) \le K\|\lambda\| \quad \text{for } n \in \mathbb{N}, \tag{1.7}$$

and it follows from Lemma 1.1 that there exists a real-valued measure ν_λ on $\mathcal{B}_X$ satisfying (1.5), and a strictly increasing sequence $(m_n(\lambda))$ of positive integers such that

$$\lim_{n\to\infty} \int_X f\, d(\lambda\mu_{m_n(\lambda)}) = \int_X f\, d\nu_\lambda \quad \text{for } f \in C(X).$$

Since E is separable and reflexive, E^* is separable. Let (λ_k) be a dense sequence in E^*. Using Cantor's diagonal method we infer that for every $k \in \mathbb{N}$ there exists a real-valued measure ν_k on $\mathcal{B}_X$ satisfying

$$|\nu_k|(X) \le K\|\lambda_k\|, \tag{1.8}$$

and a strictly increasing sequence (m_n) of positive integers such that

$$\lim_{n\to\infty} \int_X f\, d(\lambda_k\mu_{m_n}) = \int_X f\, d\nu_k \quad \text{for } f \in C(X). \tag{1.9}$$

Now we are ready to show that condition (P) holds.

Fix $\lambda \in E^*$ and let (k_n) be a sequence of positive integers satisfying

$$\lim_{n\to\infty} \|\lambda_{k_n} - \lambda\| = 0. \tag{1.10}$$

From (1.8) and Lemma 1.1 it follows that there exists a real-valued measure ν_λ on $\mathcal{B}_X$ and a strictly increasing sequence (p_n) of positive integers such that

$$|\nu_\lambda|(X) \le \lim_{n\to\infty} |\nu_{k_{p_n}}|(X) \le \lim_{n\to\infty} K\|\lambda_{k_{p_n}}\| = K\|\lambda\|$$

and

$$\lim_{n\to\infty}\int_X f\,d\nu_{k_{p_n}}=\int_X f\,d\nu_\lambda\quad\text{ for } f\in C(X). \tag{1.11}$$

Further, the inequalities

$$\begin{aligned}\Big|\int_X f\,d(\lambda\mu_{m_n})-\int_X f\,d\nu_\lambda\Big| &\le \Big|\int_X f\,d(\lambda\mu_{m_n})-\int_X f\,d(\lambda_{k_{p_q}}\mu_{m_n})\Big|\\ &\quad+\Big|\int_X f\,d(\lambda_{k_{p_q}}\mu_{m_n})-\int_X f\,d\nu_{k_{p_q}}\Big|\\ &\quad+\Big|\int_X f\,d\nu_{k_{p_q}}-\int_X f\,d\nu_\lambda\Big|\end{aligned}$$

and (cf. (1.7))

$$\begin{aligned}\Big|\int_X f\,d(\lambda\mu_{m_n})-\int_X f\,d(\lambda_{k_{p_q}}\mu_{m_n})\Big| &\le \|f\|_\infty|(\lambda-\lambda_{k_{p_q}})\mu_{m_n}|(X)\\ &\le K\|\lambda-\lambda_{k_{p_q}}\|\cdot\|f\|_\infty\end{aligned}$$

imply according to (1.9) that

$$\limsup_{n\to\infty}\Big|\int_X f\,d(\lambda\mu_{m_n})-\int_X f\,d\nu_\lambda\Big|\le K\|\lambda-\lambda_{k_{p_q}}\|\cdot\|f\|_\infty+\Big|\int_X f\,d\nu_{k_{p_q}}-\int_X f\,d\nu_\lambda\Big|$$

for $f\in C(X)$ and $q\in\mathbb{N}$. From this and conditions (1.10), (1.11), we obtain (1.6) when q tends to infinity. This finishes the proof of Step I.

STEP II. Fix a strictly increasing sequence (m_n) of positive integers with property (P). Clearly, for every $\lambda\in E^*$ condition (1.6) determines the real-valued measure ν_λ on $\mathcal{B}_X$ uniquely. We now prove that there exists $\mu\in\mathcal{M}_K$ such that

$$\lambda\mu=\nu_\lambda\quad\text{ for }\lambda\in E^*. \tag{1.12}$$

Given $f\in C(X)$ consider a functional Λ_f on E^* defined by

$$\Lambda_f\lambda=\int_X f\,d\nu_\lambda. \tag{1.13}$$

Clearly, it is linear. Moreover, according to (1.13) and (1.5) we have

$$|\Lambda_f\lambda|\le\|f\|_\infty|\nu_\lambda|(X)\le K\|f\|_\infty\|\lambda\|\quad\text{ for }\lambda\in E^*. \tag{1.14}$$

Hence $\Lambda_f\in E^{**}$ and, since E is reflexive, there exists $T(f)\in E$ such that

$$\Lambda_f\lambda=\lambda T(f)\quad\text{ for }\lambda\in E^*. \tag{1.15}$$

Obviously, the operator $T:C(X)\to E$ defined by (1.15) is linear. It is also continuous: If $f\in C(X)$ then choosing $\lambda\in E^*$ such that $\|T(f)\|=\lambda T(f)$ and $\|\lambda\|\le 1$ and using (1.15) and (1.14) we obtain

$$\|T(f)\|=\lambda T(f)=\Lambda_f\lambda\le K\|f\|_\infty. \tag{1.16}$$

Finally, by reflexivity of E, the operator T is weakly compact. According to the Riesz Representation Theorem [3; VI.2] there exists a σ-additive measure $\mu : \mathcal{B}_X \to E$ such that $\|\mu\|(X) = \|T\|$ and

$$T(f) = \int_X f\, d\mu \quad \text{for } f \in C(X). \tag{1.17}$$

From this and (1.16) it follows that $\mu \in \mathcal{M}_K$. To prove (1.12) fix $\lambda \in E^*$. Now using (1.17), (1.15) and (1.13) we obtain

$$\int_X f\, d(\lambda\mu) = \lambda T(f) = \int_X f\, d\nu_\lambda \quad \text{for } f \in C(X).$$

Thus $\lambda\mu = \nu_\lambda$, which completes the proof of Theorem 1.1.

Now let us see what Theorem 1.1 says about the space $\mathcal{M}_K$ defined by (1.3) and endowed with the Fortet–Mourier metric

$$\|\mu_1 - \mu_2\|_{\mathcal{F}}. \tag{1.18}$$

Corollary 1.1. *The space $\mathcal{M}_K$ with metric* (1.18) *is complete.*

Proof. Let (μ_n) be a Cauchy sequence in the space under consideration. From Theorem 1.1 it follows that there exists $\mu \in \mathcal{M}_K$ and a strictly increasing sequence (m_n) of positive integers such that (1.4) holds. Thus the sequence of functions

$$(f, \lambda) \mapsto \lambda \int_X f\, d\mu_n \tag{1.19}$$

defined on $\{f \in \mathrm{Lip}_1(X) : \|f\|_\infty \le 1\} \times \{\lambda \in E^* : \|\lambda\| \le 1\}$ satisfies the uniform Cauchy condition and contains a subsequence converging pointwise to the function

$$(f, \lambda) \mapsto \lambda \int_X f\, d\mu. \tag{1.20}$$

Consequently, this convergence is uniform, which means $\lim_{n\to\infty} \|\mu_n - \mu\|_{\mathcal{F}} = 0$. This completes the proof.

The functionals (1.19) and (1.20) are evidently bilinear and continuous on $C(X) \times E^*$. Thus the convergence (1.4) is uniform on every compact subset of $C(X) \times E^*$. In particular, it is uniform on $\{f \in \mathrm{Lip}_1(X) : \|f\|_\infty \le 1\} \times \mathcal{L}$ where $\mathcal{L}$ is a compact subset of E^*. Thus, as an immediate consequence of Theorem 1.1 we have the following corollary.

Corollary 1.2. *For every sequence (μ_n) in $\mathcal{M}_K$ there exists $\mu \in \mathcal{M}_K$ and a strictly increasing sequence (m_n) of positive integers such that*

$$\lim_{n\to\infty} \|\lambda(\mu_{m_n} - \mu)\|_{\mathcal{F}} = 0 \quad \textit{for } \lambda \in E^*.$$

In addition to the semivariation $\|\mu\|$ of a vector-valued measure μ which was defined by formula (1.2) we will consider another set function $|\mu|$ called *variation*. It is given by

$$|\mu|(B) = \sup_{\pi} \sum_{P\in\pi} \|\mu(P)\| \quad \text{for } B \in \mathcal{B}_X \tag{1.21}$$

where the supremum is taken over all finite partitions of B into Borel subsets. It is evident that $\|\mu\| \le |\mu|$ for every measure $\mu \in \mathcal{M}$. Consequently, the set

$$\{\mu \in \mathcal{M} : |\mu|(X) \le K\} \tag{1.22}$$

is a subset of $\mathcal{M}_K$.

THEOREM 1.2. *The set* (1.22) *is a closed subset of the space* $\mathcal{M}_K$ *with metric* (1.18).

The proof of Theorem 1.2 is based on two lemmas.

LEMMA 1.2. *If a sequence* (μ_n) *in* $\mathcal{M}$ *converges in the norm* $\|\cdot\|_{\mathcal{F}}$ *to a measure* $\mu \in \mathcal{M}$, *then for every closed set* $F \subset X$ *and for every* $\varepsilon > 0$ *there exists an open set* $G \subset X$ *such that*

$$F \subset G, \quad \varrho(x, F) < \varepsilon \quad \textit{for } x \in G, \tag{1.23}$$

and

$$\|\mu(F)\| \le \liminf_{n\to\infty} \|\mu_n\|(G) + \varepsilon. \tag{1.24}$$

P r o o f. Let $F \subset X$ be a closed set and $\varepsilon > 0$. Fix $\lambda \in E^*$ such that

$$\|\mu(F)\| = \lambda\mu(F), \quad \|\lambda\| \le 1,$$

and $N > 1/\varepsilon$ such that the set

$$G = \{x \in X : \varrho(x, F) < 1/N\}$$

satisfies

$$|\lambda\mu|(G \setminus F) < \varepsilon.$$

Clearly G is open and (1.23) holds. Now choose a Lipschitzian $f : X \to [0,1]$ such that $f(x) = 1$ for $x \in F$ and $f(x) = 0$ for $x \in X \setminus G$. Evidently

$$\lim_{n\to\infty} \lambda \int_X f\, d\mu_n = \lambda \int_X f\, d\mu$$

and

$$\lambda \int_X f\, d\mu_n = \lambda \int_G f\, d\mu_n \le \Big\| \int_G f\, d\mu_n \Big\| \le \|\mu_n\|(G)$$

for $n \in \mathbb{N}$. Hence

$$\lambda \int_X f\, d\mu \le \liminf_{n\to\infty} \|\mu_n\|(G).$$

Moreover,

$$\lambda \int_X (1_F - f)\, d\mu = - \int_{G\setminus F} f\, d(\lambda\mu) \le |\lambda\mu|(G \setminus F) < \varepsilon$$

and, consequently,

$$\|\mu(F)\| = \lambda\mu(F) = \lambda \int_X (1_F - f)\, d\mu + \lambda \int_X f\, d\mu < \varepsilon + \liminf_{n\to\infty} \|\mu_n\|(G).$$

The proof of the next lemma is straightforward and will be omitted.

LEMMA 1.3. *If $\mu \in \mathcal{M}$, then for every Borel set $B \subset X$ and for every $\varepsilon > 0$ there exists a closed set $F \subset B$ such that $\|\mu(B)\| \le \|\mu(F)\| + \varepsilon$.*

Proof of Theorem 1.2. Let (μ_n) be a sequence in the set (1.22) which converges in the Fortet–Mourier norm to a measure $\mu \in \mathcal{M}$. We have to show that $|\mu|(X) \le K$.

Fix a finite Borel partition $B_1, \ldots, B_N$ of X and $\varepsilon > 0$. It follows from Lemma 1.3 that there exist closed subsets $F_1, \ldots, F_N$ of X such that

$$F_j \subset B_j, \quad \|\mu(B_j)\| \le \|\mu(F_j)\| + \frac{\varepsilon}{2N} \quad \text{for } j = 1, \ldots, N.$$

Since $F_1, \ldots, F_N$ are compact and disjoint, there exists a positive nuber $\varepsilon_0 \le \varepsilon/2$ such that

$$\{x \in X : \varrho(x, F_j) < \varepsilon_0\} \cap \{x \in X : \varrho(x, F_k) < \varepsilon_0\} = \emptyset \quad \text{for } j \ne k.$$

Finally, according to Lemma 1.2 there exist open subsets $G_1, \ldots, G_N$ of X such that $F_j \subset G_j$, $\varrho(x, F_j) < \varepsilon_0$ for $x \in G_j$, and

$$\|\mu(F_j)\| \le \liminf_{n\to\infty} \|\mu_n\|(G_j) + \varepsilon_0/N \quad \text{for } j = 1, \ldots, N.$$

Evidently the sets $G_1, \ldots, G_N$ are also disjoint. Consequently,

$$\sum_{j=1}^N \|\mu_n\|(G_j) \le \sum_{j=1}^N |\mu_n|(G_j) = |\mu_n|\Big(\bigcup_{j=1}^N G_j\Big) \le |\mu_n|(X) \le K$$

for $n \in \mathbb{N}$ and

$$\sum_{j=1}^N \|\mu(B_j)\| \le \sum_{j=1}^N \|\mu(F_j)\| + \frac{\varepsilon}{2} \le \sum_{j=1}^N \liminf_{n\to\infty} \|\mu_n\|(G_j) + \varepsilon_0 + \frac{\varepsilon}{2}$$

$$\le \liminf_{n\to\infty} \sum_{j=1}^N \|\mu_n\|(G_j) + \varepsilon \le K + \varepsilon.$$

This completes the proof.

2. The adjoint space. In this section we study the properties of continuous functionals on $(\mathcal{M}, \|\cdot\|_{\mathcal{F}})$. We start with the definition of an integral of the form

$$\int_X \psi(x,\mu(dx)) \tag{2.1}$$

where $\psi : X \times E \to \mathbb{R}$ is a function such that

$$\psi(x,\cdot) \in E^* \quad \text{for } x \in X. \tag{2.2}$$

The integral (2.1) is defined as the only real number c satisfying the following condition (C).

(C) For every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\Big|\sum_{i=1}^{m} \psi(x_i,\mu(B_i)) - c\Big| < \varepsilon$$

holds for every finite partition $B_1,\ldots,B_m$ of X into nonempty Borel sets of diameter less than δ and for all $x_1 \in B_1,\ldots,x_m \in B_m$.

In what follows we will exploit the following condition:

(A) There exists a constant $L \ge 0$ such that

$$\Big|\sum_{i=1}^{m} \psi(x_i,a_i) - \psi(z_i,a_i)\Big| \le L\varrho(x,z)\|a\|$$

for all finite sequences $x_1,\ldots,x_m \in X$, $z_1,\ldots,z_m \in X$ and $a_1,\ldots,a_m \in E$, where

$$\varrho(x,z) := \max\{\varrho(x_i,z_i) : i = 1,\ldots,m\}$$

and

$$\|a\| = \sup\Big\{\Big\|\sum_{i=1}^{m} \varepsilon_i a_i\Big\| : |\varepsilon_1| \le 1,\ldots,|\varepsilon_m| \le 1\Big\}.$$

Conditions (2.2) and (A) guarantee the existence of the integral (2.1) for any $\mu \in \mathcal{M}$. We omit the routine proof of this fact.

Define

$$\delta_x(B) = 1_B(x) \quad \text{for } x \in X \text{ and } B \in \mathcal{B}_X.$$

Clearly $a\delta_x \in \mathcal{M}$ and, according to (1.1), (1.2) and (1.21),

$$\|a\delta_x\|_{\mathcal{F}} = \|a\delta_x\|(X) = |a\delta_x|(X) = \|a\| \quad \text{for } x \in X \text{ and } a \in E. \tag{2.3}$$

THEOREM 2.1. *If φ is a continuous linear functional on $(\mathcal{M},\|\cdot\|_{\mathcal{F}})$, then the function $\psi : X \times E \to \mathbb{R}$ defined by*

$$\psi(x,a) = \varphi(a\delta_x) \tag{2.4}$$

satisfies conditions (2.2) *and* (A). *Moreover*,

$$\varphi(\mu) = \int_X \psi(x,\mu(dx)) \quad \text{for } \mu \in \mathcal{M}. \tag{2.5}$$

Proof. From (2.4) it follows that $\psi(x,\cdot)$ is linear for every $x \in X$. Applying (2.3) we also have

$$|\psi(x,a)| \le \|\varphi\| \cdot \|a\delta_x\|_{\mathcal{F}} = \|\varphi\| \cdot \|a\| \quad \text{for } a \in E,$$

which proves (2.2).

To show (A) (with $L = \|\varphi\|$) it is enough to prove the inequality

$$\Big\| \sum_{i=1}^{m} (a_i\delta_{x_i} - a_i\delta_{z_i}) \Big\| \le \varrho(x,z)\|a\|. \tag{2.6}$$

Of course we may (and do) assume that $\varrho(x,z) > 0$. Then, for $f \in \mathrm{Lip}_1(X)$ with $\|f\|_\infty \le 1$, and $\varepsilon_i = (f(x_i) - f(z_i))/\varrho(x,z)$ for $i = 1,\ldots,m$, we have

$$\begin{aligned}\Big\| \int_X f\, d\sum_{i=1}^{m}(a_i\delta_{x_i} - a_i\delta_{z_i})\Big\| = \Big\| \sum_{i=1}^{m} f(x_i)a_i - f(z_i)a_i \Big\| &= \varrho(x,z)\Big\| \sum_{i=1}^{m} \varepsilon_i a_i \Big\| \\ &\le \varrho(x,z)\|a\|,\end{aligned}$$

and (2.6) follows.

It remains to verify (2.5). Fix $\mu \in \mathcal{M}$ and $\varepsilon > 0$. Let $B_1,\ldots,B_m$ be a finite partition of X into nonempty Borel sets with diameters less than ε and such that

$$\Big| \sum_{i=1}^{m} \psi(x_i,\mu(B_i)) - \int_X \psi(x,\mu(dx)) \Big| < \varepsilon \tag{2.7}$$

for any $x_1 \in B_1,\ldots,x_m \in B_m$. Let $a_i = \mu(B_i)$ for $i = 1,\ldots,m$. We claim that

$$\Big\| \mu - \sum_{i=1}^{m} a_i\delta_{x_i} \Big\|_{\mathcal{F}} \le 2\varepsilon\|\mu\|(X). \tag{2.8}$$

To prove this fix $f \in \mathrm{Lip}_1(X)$ with $\|f\|_\infty \le 1$. Defining

$$g = \sum_{i=1}^{m} f(x_i)1_{B_i}$$

we have $|f(x) - g(x)| < \varepsilon$ for $x \in X$ and $\int_X g\, d(\mu - \sum_{i=1}^m a_i\delta_{x_i}) = 0$. Hence

$$\Big\| \int_X f d\Big(\mu - \sum_{i=1}^{m} a_i\delta_{x_i}\Big)\Big\| = \Big\| \int_X (f-g) d\Big(\mu - \sum_{i=1}^{m} a_i\delta_{x_i}\Big)\Big\| \le \varepsilon\Big\|\mu - \sum_{i=1}^{m} a_i\delta_{x_i}\Big\|(X)$$

and, consequently,

$$\Big\| \mu - \sum_{i=1}^{m} a_i\delta_{x_i} \Big\|_{\mathcal{F}} \le \varepsilon\Big\|\mu - \sum_{i=1}^{m} a_i\delta_{x_i}\Big\|(X). \tag{2.9}$$

Now let $C_1, \ldots, C_n$ be a finite Borel partition of X and let $\varepsilon_1, \ldots, \varepsilon_n \in [-1, 1]$. Then

$$\Big|\sum_{j=1}^{n} \varepsilon_j \delta_{x_i}(C_j)\Big| \le \sum_{j=1}^{n} \delta_{x_i}(C_j) = \delta_{x_i}(X) = 1 \quad \text{for } i = 1, \ldots, m,$$

and

$$\Big\|\sum_{j=1}^{n} \varepsilon_j \Big(\sum_{i=1}^{m} a_i \delta_{x_i}\Big)(C_j)\Big\| = \Big\|\sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \varepsilon_j \delta_{x_i}(C_j) \mu(B_i)\Big)\Big\|.$$

Taking the supremum over all partitions $C_1, \ldots, C_n$ and all $\varepsilon_1, \ldots, \varepsilon_n$ we obtain (cf. [3; p.4, Proposition 11])

$$\Big\|\sum_{i=1}^{m} a_i \delta_{x_i}\Big\|(X) \le \|\mu\|(X).$$

The last inequality jointly with (2.9) implies (2.8).

Finally, using (2.4), (2.8) and (2.7) we obtain

$$\begin{aligned}\Big|\varphi(\mu) - \int_X \psi(x, \mu(dx))\Big| &\le \Big|\varphi\Big(\mu - \sum_{i=1}^{m} a_i \delta_{x_i}\Big)\Big| \\ &\quad + \Big|\sum_{i=1}^{m} \psi(x_i, \mu(B_i)) - \int_X \psi(x, \mu(dx))\Big| \\ &\le \varepsilon(2\|\varphi\| \cdot \|\mu\|(X) + 1),\end{aligned}$$

which implies (2.5) and completes the proof of Theorem 2.1.

Unfortunately, Theorem 2.1 does not give a precise description of the space adjoint to $(\mathcal{M}, \|\cdot\|_{\mathcal{F}})$. Namely, we know that every continuous linear functional φ on $\mathcal{M}$ is of the form (2.5) with ψ satisfying (2.2) and (A) but we do not know if the converse is true. Of course every functional φ given by (2.5) where ψ satisfies conditions (2.2) and (A) is linear. It remains, however, an open question if φ is continuous.

We close this section with a corollary concerning functionals on

$$\mathcal{M}_{\text{fin}} = \{\mu \in \mathcal{M} : |\mu|(X) < \infty\}. \tag{2.10}$$

COROLLARY 2.1. *If φ is a continuous linear functional on the space $(\mathcal{M}_{\text{fin}}, \|\cdot\|_{\mathcal{F}})$, then the function $\psi : X \times E \to \mathbb{R}$ defined by* (2.4) *satisfies conditions* (2.2) *and* (A) *and*

$$\varphi(\mu) = \int_X \psi(x, \mu(dx)) \quad \textit{for } \mu \in \mathcal{M}_{\text{fin}}.$$

Proof. Extend φ onto the whole $\mathcal{M}$ and apply Theorem 2.1.

3. Markov operators. We consider two types of Markov operators. The first are defined on $\mathcal{M}$ and satisfy a normalization condition stated in terms of the semivariation. The Markov operators of the second type act on $\mathcal{M}_{\text{fin}}$ and are related to the variation of a measure. Their theory is quite analogous and will be sketched rather briefly.

Fix $e \in E$. A linear operator $P : \mathcal{M} \to \mathcal{M}$ will be called a *Markov operator* if it satisfies the following conditions:

$$\|P\mu\|(X) \le \|\mu\|(X) \quad \text{for } \mu \in \mathcal{M}, \tag{3.1}$$

$$P\mu(X) = e \quad \text{for } \mu \in \mathcal{M} \text{ with } \mu(X) = e. \tag{3.2}$$

In the class of Markov operators we distinguish contractive Feller operators. To define them denote by Ψ_1 the family of functions $\psi : X \times E \to \mathbb{R}$ satisfying (2.2) and the following condition:

(A_1) If $m \in \mathbb{N}$, $x_1, \ldots, x_m, z_1, \ldots, z_m \in X$ and $a_1, \ldots, a_m \in E$, then

$$\Big|\sum_{i=1}^{m} \psi(x_i, a_i) - \psi(z_i, a_i)\Big| \le \varrho(x, z)\|a\|,$$

where

$$\varrho(x, z) := \max\{\varrho(x_i, z_i) : i = 1, \ldots, m\}$$

and

$$\|a\| := \sup\Big\{\Big\|\sum_{i=1}^{m} \varepsilon_i a_i\Big\| : |\varepsilon_1| \le 1, \ldots, |\varepsilon_m| \le 1\Big\}.$$

REMARK 3.1. *If $f \in \mathrm{Lip}_1(X)$, $\lambda \in E^*$ and $\|\lambda\| \le 1$, then the function $\psi : X \times E \to \mathbb{R}$ defined by*

$$\psi(x, a) = f(x)\lambda(a) \tag{3.3}$$

belongs to Ψ_1 and

$$\int_X \psi(x, \mu(dx)) = \lambda \int_X f\, d\mu \quad \textit{for } \mu \in \mathcal{M}. \tag{3.4}$$

Proof. Clearly (2.2) holds. If $x_1, \ldots, x_m, z_1, \ldots, z_m \in X$, $a_1, \ldots, a_m \in E$, and $\varrho(x, z) > 0$, then

$$\Big|\sum_{i=1}^{m} \psi(x_i, a_i) - \psi(z_i, a_i)\Big| = \varrho(x, z)\Big|\lambda\Big(\sum_{i=1}^{m} \frac{f(x_i) - f(z_i)}{\varrho(x, z)} a_i\Big)\Big| \le \varrho(x, z)\|a\|.$$

Thus $\psi \in \Psi_1$. The proof of (3.4) is straightforward.

REMARK 3.2. *If $\psi \in \Psi_1$, $\mu \in \mathcal{M}$ and $\mu(X) = 0$, then*

$$\Big|\int_X \psi(x, \mu(dx))\Big| \le \operatorname{diam}(X)\|\mu\|(X).$$

P r o o f. Fix $\varepsilon > 0$ and a finite partition $B_1, \ldots, B_m$ of X into nonempty Borel sets such that (2.7) holds for all $x_1 \in B_1, \ldots, x_m \in B_m$. Then, fixing $x_1 \in B_1, \ldots, x_m \in B_m$ and $x_0 \in X$, we have

$$\begin{aligned}\Big|\int_X \psi(x,\mu(dx))\Big| &< \varepsilon + \Big|\sum_{i=1}^m \psi(x_i,\mu(B_i)) - \psi(x_0,\mu(B_i))\Big| \\ &\le \varepsilon + \max_{i=1,\ldots,m} \varrho(x_i,x_0)\cdot \sup_{|\varepsilon_1|\le 1,\ldots,|\varepsilon_m|\le 1} \Big\|\sum_{i=1}^m \varepsilon_i\mu(B_i)\Big\| \\ &\le \varepsilon + \operatorname{diam}(X)\|\mu\|(X),\end{aligned}$$

as required.

A Markov operator $P : \mathcal{M} \to \mathcal{M}$ is called a *contractive Feller operator* if there exist: a family $\Psi \subset \Psi_1$ containing the functions of the form (3.3) where $f \in \operatorname{Lip}_1(X)$, $\lambda \in E^*$ and $\|\lambda\| \le 1$, a number $\vartheta \in (0,1)$, and a mapping $U : \Psi \to \vartheta\Psi$ such that

$$\Big|\int_X \psi(x,P\mu(dx))\Big| \le \Big|\int_X U\psi(x,\mu(dx))\Big| \tag{3.5}$$

for $\psi \in \Psi$ and $\mu \in \mathcal{M}$ with $\mu(X) = 0$.

PROPOSITION 3.1. *Let N be a positive integer and let $S_j : X \to X$, $j = 1, \ldots, N$, be Lipschitzian mappings with Lipschitz constants L_j. Further, let $T_j : E \to E$ be a linear and continuous operator such that*

$$\sum_{j=1}^N \|T_j\| \le 1, \qquad \sum_{j=1}^N T_j e = e, \qquad \sum_{j=1}^N L_j\|T_j\| < 1. \tag{3.6}$$

Then the operator $P : \mathcal{M} \to \mathcal{M}$ defined by

$$P\mu(B) = \sum_{j=1}^N T_j\mu(S_j^{-1}(B)) \tag{3.7}$$

is a contractive Feller operator.

P r o o f. If $B_1, \ldots, B_m$ is a finite Borel partition of X and $\varepsilon_1, \ldots, \varepsilon_m \in [-1,1]$, then

$$\begin{aligned}\Big\|\sum_{i=1}^m \varepsilon_i P\mu(B_i)\Big\| &= \Big\|\sum_{j=1}^N T_j\Big(\sum_{i=1}^m \varepsilon_i\mu(S_j^{-1}(B_i))\Big)\Big\| \\ &\le \sum_{j=1}^N \|T_j\|\Big\|\sum_{i=1}^m \varepsilon_i\mu(S_j^{-1}(B_i))\Big\| \le \sum_{j=1}^N \|T_j\|\cdot\|\mu\|(X) \le \|\mu\|(X),\end{aligned}$$

whence (3.1) follows. Property (3.2) is evident. Clearly, P defined by (3.7) is linear. Hence P is a Markov operator.

Let Ψ denote the family of functions $\psi \in \Psi_1$ which can be written in the form

$$\psi(x,a) = \sum_{k=1}^{n} f_k(x)\lambda_k(a) \tag{3.8}$$

with a positive integer n, Lipschitzian $f_k : X \to \mathbb{R}$ and $\lambda_k \in E^*$ for $k = 1,\ldots,n$. According to Remark 3.1 the family Ψ contains the functions of the form (3.3) where $f \in \mathrm{Lip}_1(X)$, $\lambda \in E^*$ and $\|\lambda\| \le 1$. Evidently

$$\int_X \psi(x,\mu(dx)) = \sum_{k=1}^{n} \lambda_k \int_X f_k\,d\mu \quad \text{for } \psi \in \Psi \text{ and } \mu \in \mathcal{M}.$$

Fix now $\psi \in \Psi$ and define $U\psi : X \times E \to \mathbb{R}$ by

$$U\psi(x,a) = \sum_{j=1}^{N} \psi(S_j(x), T_j a).$$

Clearly, $U\psi(x,\cdot) \in E^*$ for $x \in X$. Let $x_1,\ldots,x_m,z_1,\ldots,z_m \in X$ and $a_1,\ldots,a_m \in E$. Applying condition (A_1) we obtain

$$\begin{aligned}\Big|\sum_{i=1}^{m} U\psi(x_i,a_i) - U\psi(z_i,a_i)\Big| &= \Big|\sum_{j=1}^{N}\sum_{i=1}^{m} \psi(S_j(x_i),T_j a_i) - \psi(S_j(z_i),T_j a_i)\Big| \\ &\le \sum_{j=1}^{N} \max_{i=1,\ldots,m} \varrho(S_j(x_i),S_j(z_i)) \cdot \sup_{|\varepsilon_1|\le 1,\ldots,|\varepsilon_m|\le 1} \Big\|\sum_{i=1}^{m} \varepsilon_i T_j a_i\Big\| \\ &\le \sum_{j=1}^{N} L_j \varrho(x,z)\|T_j\| \cdot \|a\| = \vartheta\varrho(x,z)\|a\|,\end{aligned}$$

where

$$\vartheta = \sum_{j=1}^{N} L_j\|T_j\| < 1.$$

Thus $U\psi \in \vartheta\Psi_1$. Finally, if $\psi \in \Psi_1$ is given by (3.8), then

$$U\psi(x,a) = \sum_{j=1}^{N}\sum_{k=1}^{n} f_k(S_j(x))\lambda_k(T_j a).$$

Hence $U\psi \in \vartheta\Psi$ and

$$\begin{aligned}\int_X \psi(x,P\mu(dx)) &= \sum_{k=1}^{n} \lambda_k \int_X f_k\,dP\mu = \sum_{k=1}^{n} \lambda_k \sum_{j=1}^{N} \int_X f_k \circ S_j\,d(T_j\mu) \\ &= \int_X U\psi(x,\mu(dx)).\end{aligned}$$

Now we are in a position to state the main result of this section.

THEOREM 3.1. *If $P : \mathcal{M} \to \mathcal{M}$ is a contractive Feller operator then there exists $\mu_* \in \mathcal{M}$ such that $\mu_*(X) = e$ and*

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\|_{\mathcal{F}} = 0 \quad \textit{for } \mu \in \mathcal{M} \textit{ with } \mu(X) = e. \tag{3.9}$$

Proof. Consider a function $\|\cdot\|_0 : \mathcal{M} \to [0,\infty]$ defined by

$$\|\mu\|_0 = \sup\Big\{\Big|\int_X \psi(x,\mu(dx))\Big| : \psi \in \Psi\Big\},$$

where Ψ is a fixed subfamily of Ψ_1 appearing in the definition of the contractive Feller operator. We claim that

$$\|\mu\|_{\mathcal{F}} \le \|\mu\|_0 \quad \text{for } \mu \in \mathcal{M}. \tag{3.10}$$

Let $\mu \in \mathcal{M}$, $f \in \mathrm{Lip}_1(X)$ with $\|f\|_\infty \le 1$, and $\lambda \in E^*$ be such that

$$\Big\|\int_X f\,d\mu\Big\| = \lambda\int_X f\,d\mu \quad \text{and} \quad \|\lambda\| \le 1.$$

Defining $\psi : X \times E \to \mathbb{R}$ by (3.3) and applying Remark 3.1 we obtain

$$\Big\|\int_X f\,d\mu\Big\| = \int_X \psi(x,\mu(dx)) \le \|\mu\|_0,$$

which implies (3.10).

Since $\frac{1}{\vartheta}U\psi \in \Psi$ for $\psi \in \Psi$, according to (3.5) we have

$$\Big|\int_X \psi(x,P\mu(dx))\Big| \le \vartheta\Big|\int_X \frac{1}{\vartheta}U\psi(x,\mu(dx))\Big| \le \vartheta\|\mu\|_0$$

for $\psi \in \Psi$ and $\mu \in \mathcal{M}$ with $\mu(X) = 0$. Hence

$$\|P\mu\|_0 \le \vartheta\|\mu\|_0 \quad \text{for } \mu \in \mathcal{M} \text{ with } \mu(X) = 0. \tag{3.11}$$

Consider now the set

$$\mathcal{M}_e = \{\mu \in \mathcal{M} : \mu(X) = e\}.$$

It follows from Remark 3.2 and inequality (3.10) that the function

$$\|\mu_1 - \mu_2\|_0 \tag{3.12}$$

is a metric in $\mathcal{M}_e$. According to (3.2) the operator P maps $\mathcal{M}_e$ into itself. Moreover, by (3.11),

$$\|P\mu_1 - P\mu_2\|_0 \le \vartheta\|\mu_1 - \mu_2\|_0 \quad \text{for } \mu_1,\mu_2 \in \mathcal{M}_e. \tag{3.13}$$

Let $\mu \in \mathcal{M}_e$. Inequalities (3.13) and (3.10) imply that $(P^n\mu)$ is a Cauchy sequence with respect to the Fortet–Mourier norm. Let $K = \|\mu\|(X)$. From (3.1) it follows that $P^n\mu \in \mathcal{M}_K$ for all n. Thus according to Corollary 1.1 the sequence $(P^n\mu)$ converges in the Fortet–Mourier norm to a measure $\mu_* \in \mathcal{M}$. This convergence and the conditions $P^n\mu(X) = e$ imply that

$\mu_*(X) = e$. From (3.10) and (3.13) it follows that the limiting measure μ_* does not depend on the initial measure $\mu \in \mathcal{M}_e$.

A Markov operator $P : \mathcal{M} \to \mathcal{M}$ is called *asymptotically stable* if there exists a measure $\mu_* \in \mathcal{M}$ such that $\mu_*(X) = e$, $P\mu_* = \mu_*$ and (3.9) holds.

COROLLARY 3.1. *If a contractive Feller operator is continuous with respect to the Fortet–Mourier norm then it is asymptotically stable.*

The next corollary concerns operators of the form (3.7).

COROLLARY 3.2. *Let N be a positive integer and let $S_j : X \to X$, $j = 1, \ldots, N$, be Lipschitzian mappings with Lipschitz constants L_j. Further, let $T_j : E \to E$ be a linear continuous operator such that* (3.6) *holds. Then the operator $P : \mathcal{M} \to \mathcal{M}$ defined by* (3.7) *is asymptotically stable.*

Proof. According to Proposition 3.1 and Corollary 3.1 it is sufficient to prove that P is continuous with respect to the Fortet–Mourier norm $\|\cdot\|_{\mathcal{F}}$. We show that

$$\|P\mu\|_{\mathcal{F}} \le 2\|\mu\|_{\mathcal{F}} \quad \text{for } \mu \in \mathcal{M}.$$

For, if $\mu \in \mathcal{M}$, $f \in \mathrm{Lip}_1(X)$ and $\|f\|_\infty \le 1$, then setting $J_1 = \{j : L_j \le 1\}$ and $J_2 = \{j : L_j > 1\}$, we have

$$\begin{aligned}
\Big\|\int_X f\,dP\mu\Big\| &= \Big\|\sum_{j=1}^N T_j \int_X f \circ S_j\,d\mu\Big\| \le \sum_{j=1}^N \|T_j\| \Big\|\int_X f \circ S_j\,d\mu\Big\| \\
&= \sum_{j\in J_1} \|T_j\| \Big\|\int_X f \circ S_j\,d\mu\Big\| + \sum_{j\in J_2} \|T_j\| L_j \Big\|\int_X \frac{f\circ S_j}{L_j}\,d\mu\Big\| \\
&\le \sum_{j\in J_1} \|T_j\| \cdot \|\mu\|_{\mathcal{F}} + \sum_{j\in J_2} \|T_j\| L_j \|\mu\|_{\mathcal{F}} \le 2\|\mu\|_{\mathcal{F}}
\end{aligned}$$

and the corollary is proved.

Finally, we consider operators acting on the space $\mathcal{M}_{\mathrm{fin}}$ defined by (2.10).

Fix $e \in E$. A linear operator $P : \mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$ is called a *Markov operator of the second type* if

$$|P\mu|(X) \le |\mu|(X) \quad \text{for } \mu \in \mathcal{M}_{\mathrm{fin}}$$

and

$$P\mu(X) = e \quad \text{for } \mu \in \mathcal{M}_{\mathrm{fin}} \text{ with } \mu(X) = e.$$

Let Ψ_2 denote the family of functions $\psi : X \times E \to \mathbb{R}$ satisfying condition (2.2) and the inequality

$$|\psi(x,a) - \psi(z,a)| \le \varrho(x,z)\|a\| \quad \text{for } x, z \in X \text{ and } a \in E.$$

Clearly, $\Psi_1 \subset \Psi_2$. Note also that for $\psi \in \Psi_2$ and $\mu \in \mathcal{M}_{\mathrm{fin}}$ the integral (2.1) exists, and if $\mu(X) = 0$ then

$$\Big|\int_X \psi(x, \mu(dx))\Big| \le \operatorname{diam}(X)|\mu|(X).$$

A Markov operator $P : \mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$ of the second type is called a *contractive Feller operator of the second type* if there exist: a family $\Psi \subset \Psi_2$ containing the functions of the form (3.3) where $f \in \mathrm{Lip}_1(X)$, $\lambda \in E^*$ and $\|\lambda\| \le 1$, a number $\vartheta \in (0,1)$, and a mapping $U : \Psi \to \vartheta\Psi$ such that (3.5) holds for $\psi \in \Psi$ and $\mu \in \mathcal{M}_{\mathrm{fin}}$ with $\mu(X) = 0$.

Arguing as for Proposition 3.1 we can prove the following.

PROPOSITION 3.2. *Let N be a positive integer and let $S_j : X \to X$, $j = 1, \ldots, N$, be Lipschitzian mappings with Lipschitz constants L_j. Further, let $T_j : E \to E$ be a linear and continuous operator. If* (3.6) *holds then the formula* (3.7) *defines a contractive Feller operator $P : \mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$ of the second type.*

Further, arguing as in the proof of Theorem 3.1 and using Theorem 1.2 we also obtain the following result.

THEOREM 3.2. *If $P : \mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$ is a contractive Feller operator of the second type then there exists $\mu_* \in \mathcal{M}_{\mathrm{fin}}$ such that $\mu_*(X) = e$ and*

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\|_{\mathcal{F}} = 0 \quad \text{for } \mu \in \mathcal{M}_{\mathrm{fin}} \text{ with } \mu(X) = e. \tag{3.14}$$

A Markov operator $P : \mathcal{M}_{\mathrm{fin}} \to \mathcal{M}_{\mathrm{fin}}$ of the second type is called *asymptotically stable* if there exists $\mu_* \in \mathcal{M}_{\mathrm{fin}}$ such that $\mu_*(X) = e$, $P\mu_* = \mu_*$ and (3.14) holds.

Clearly, Corollary 3.1 remains valid for contractive Feller operators of the second type, and Corollary 3.2 can be strengthened as follows.

COROLLARY 3.3. *Let N be a positive integer and let $S_j : X \to X$, $j = 1, \ldots, N$, be Lipschitzian mappings with Lipschitz constants L_j. Further, let $T_j : E \to E$ be a linear and continuous operator. If* (3.6) *holds and $P : \mathcal{M} \to \mathcal{M}$ is defined by* (3.7), *then there exists a measure $\mu_* \in \mathcal{M}$ such that $|\mu_*|(X) < \infty$, $\mu_*(X) = e$, $P\mu_* = \mu_*$ and* (3.9) *is satisfied.*

Corollaries 3.2 and 3.3 extend the well known criteria of asymptotic stability of Iterated Function Systems acting on real-valued measures [2], [4], [5]. For these measures the operator (3.7) has the form

$$P\mu(B) = \sum_{j=1}^{N} p_j \mu(S_j^{-1}(B)) \tag{3.15}$$

where $p_1, \ldots, p_N$ are nonnegative numbers such that $\sum_{j=1}^{N} p_j = 1$. In this case $\mathcal{M}_e$ is replaced by the family of probability measures and condition (3.6) reduces to $\sum_{j=1}^{N} p_j L_j < 1$.

Note that due to the special properties of probabilistic measures (e.g. Prokhorov criterion for compactness) the asymptotic stability of the operator (3.15) can be proved not only for compact X (see for example [7]).

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