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On certain subclasses of multivalently meromorphic close-to-convex maps

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Abstract. Let M_p denote the class of functions f of the form $f(z) = 1/z^p + \sum_{k=0}^{\infty} a_k z^k$, p a positive integer, in the unit disk $E = \{|z| < 1\}$, f being regular in $0 < |z| < 1$. Let $L_{n,p}(\alpha) = \{f : f \in M_p,\ \mathrm{Re}\{-(z^{p+1}/p)(D^n f)'\} > \alpha\}$, $\alpha < 1$, where $D^n f = (z^{n+p} f(z))^{(n)}/(z^p n!)$. Results on $L_{n,p}(\alpha)$ are derived by proving more general results on differential subordination. These results reduce, by putting $p = 1$, to the recent results of Al-Amiri and Mocanu.

1. Introduction. Let M_p denote the class of meromorphic functions f in the unit disk $E = \{z : |z| < 1\}$ having only a pole of order p at $z = 0$, of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=0}^{\infty} a_k z^k, \qquad p \text{ a positive integer.} \tag{1}$$

We define

$$D^n f(z) = \frac{1}{z^p(1-z)^{n+1}} * f(z),$$

where n is a non-negative integer and $*$ denotes Hadamard product. It can be verified that

$$D^n f(z) = \frac{(z^{n+p} f(z))^{(n)}}{z^p n!},$$

where $f^{(n)}$ denotes the nth derivative of f in the usual sense. Let $L_{n,p}(\alpha)$, $\alpha < 1$, denote the class of functions $f \in M_p$ such that

$$\mathrm{Re}\left\{\frac{-z^{p+1}}{p}(D^n f)'\right\} > \alpha. \tag{2}$$

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Let

$$I_c(f)(z)=\frac{c-p+1}{z^{c+1}}\int_0^z t^c f(t)\,dt,\qquad \operatorname{Re} c>0.$$

For $p=1$, H. Al-Amiri and P. T. Mocanu [1] have shown that $L_{n+1,1}(\alpha)\subset L_{n,1}(r)$, $r>\alpha$; and $I_c[L_{n,1}(\alpha)]\subset L_{n,1}(\delta)$, $\delta>\alpha$. Furthermore, they proved that if $f\in M_1$ and $\operatorname{Re} c>0$ then

$$\operatorname{Re}[-z^2(D^n f(z))']>\alpha-(1-\alpha)\operatorname{Re}\frac{1}{c}\ \Rightarrow\ I_c(f)\in L_{n,1}(\alpha).$$

Indeed they have proved certain more general results on differential subordination from which the above results are deduced.

In this paper, we obtain analogous results for $L_{n,p}(\alpha)$ when p is any positive integer and the results obtained by Al-Amiri and Mocanu [1] can be deduced from our results when we put $p=1$. We also establish general results on differential subordination from which the results of Al-Amiri and Mocanu are deducible.

2. Preliminary definitions and lemmas

Definition 1. If f and g are analytic in E and g is univalent in E, then f is said to be *subordinate* to g, written $f\prec g$, if $f(0)=g(0)$ and $f(E)\subset g(E)$.

Lemma A [3, 4]. *Let $p(z)=p(0)+p_n z^n+\ldots$ be analytic in E and q analytic and univalent in E. If p is not subordinate to the analytic function q in E, then there exist points $z_0\in E$ and $\zeta_0\in\partial E$ such that* (i) $p(z_0)=q(\zeta_0)$, (ii) $z_0p'(z_0)=m\zeta_0q'(\zeta_0)$, *where $m\ge n$.*

Lemma B [3, 4]. *Let the function $H:\mathbb{C}^2\to\mathbb{C}$ satisfy $\operatorname{Re} H(is,t)\le 0$ for real s and $t\le -n(1+s^2)/2$, where n is a positive integer. If $p(z)=1+p_nz^n+\ldots$ is analytic in E and $\operatorname{Re} H(p(z),zp'(z))>0$ for $z\in E$, then $\operatorname{Re} p(z)>0$ in E.*

Definition 2. Let $z\in E$, $t\ge 0$. A function $L(z,t)$ is called a *subordination chain* if $L(\cdot,t)$ is analytic and univalent on E for all $t\ge 0$, $L(z,\cdot)$ is continuously differentiable on $[0,\infty)$ for each $z\in E$, and $L(z,s)\prec L(z,l)$ for $0\le s<l$.

Lemma C [6]. *The function $L(z,t)=a_1(t)z+\ldots$, with $a_1(t)\ne 0$ and $\lim_{t\to\infty}|a_1(t)|=\infty$, is a subordination chain if and only if*

$$\operatorname{Re}\left[z\frac{\partial L}{\partial z}\Big/\frac{\partial L}{\partial t}\right]>0\qquad \textit{for } z\in E \textit{ and } t\ge 0.$$

LEMMA D [7]. *Let* $\alpha, \beta \in \mathbb{R}$, $\alpha > 0$, *g starlike univalent in* E, *and* h *analytic in* E *with* $h(0) = 1$ *and* $\operatorname{Re} h > 0$ *in* E. *Define*

$$B(\alpha,\beta) = \Big\{F : F(z) = \Big(\int_0^z h(t) g^{\alpha}(t) t^{i\beta-1}\, dt\Big)^{1/(\alpha+i\beta)},\ z \in E\Big\}.$$

Then $G \in B(\alpha, \beta)$, *where*

$$G(z) = \Big[z^{-c}\int_0^z t^{c-1} F^{\alpha+i\beta}(t)\, dt\Big]^{1/(\alpha+i\beta)}, \qquad c \in \mathbb{C},\ \operatorname{Re} c > 0.$$

REMARK. $F \in B(\alpha, \beta)$ is univalent and analytic in E and is called a *Bazilevič function*. The class $B(1, 0)$ consists of close-to-convex functions in E.

DEFINITION 3. Let $H(p(z), zp'(z)) \prec h(z)$ be a first order differential subordination. Then a univalent function q is called its *dominant* if $p \prec q$ for all analytic functions p that satisfy the differential subordination. A dominant $\overline{q}$ is called *the best dominant* if $\overline{q} \prec q$ for all dominants q. For the general theory of differential subordination and its applications we refer to [5].

LEMMA 1. *Let* q *be a convex univalent function in* E *and* $\operatorname{Re} c > 0$. *Let*

$$h(z) = q(z) + \frac{p+1}{c} zq'(z),$$

where p *is a positive integer. If* $p(z) = 1 + a_{p+1}z^{p+1} + \ldots$ *is analytic in* E *and*

$$p(z) + \frac{1}{c} zp'(z) \prec h(z),$$

then $p(z) \prec q(z)$ *and* q *is the best dominant.*

Proof. We can assume that q is analytic and convex on $\overline{E}$ without any loss of generality, because otherwise we replace $q(z)$ by $q_r(z) = q(rz)$, $0 < r < 1$. These functions satisfy the conditions of the lemma on $\overline{E}$. We can prove that $p_r(z) \prec q_r(z)$, which enables us to obtain $p \prec q$ on letting $r \to 1$. Consider

$$L(z,t) = q(z) + \frac{p+1+t}{c} zq'(z), \qquad z \in E,\ t \ge 0.$$

Then

$$\frac{\partial L}{\partial t} = \frac{zq'(z)}{c}, \qquad \frac{\partial L}{\partial z} = q'(z) + \frac{p+1+t}{c} zq''(z) + \frac{p+1+t}{c} q'(z).$$

We have

$$\operatorname{Re}\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right) = \operatorname{Re}\{c + (p+1+t)(1 + zq''(z)/q'(z))\} > 0,$$

since q is convex and $\operatorname{Re} c > 0$. Hence $L(z,t)$ is a subordination chain by Lemma C. We have $L(z,0) = h(z) \prec L(z,t)$ for $t > 0$ and $L(\zeta,t) \notin h(E)$ for $|\zeta| = 1$ and $t \geq 0$. If p is not subordinate to q, then by Lemma A, there exist points $z_0 \in E$, $\zeta_0 \in \partial E$ and $m \geq p+1$ such that $p(z_0) = q(\zeta_0)$, $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$ and so

$$p(z_0) + \frac{1}{c} z_0 p'(z_0) = q(\zeta_0) + \frac{m}{c}\zeta_0 q'(\zeta_0) = L(\zeta_0, m-p-1) \notin h(E),$$

which contradicts our assumption that $p(z) + \frac{1}{c} z p'(z) \prec h(z)$. So we conclude that $p \prec q$. Consider $p(z) = q(z^{p+1})$ to see that q is the best dominant.

Lemma 2. *Let*

$$w = \frac{(p+1)^2 + |c|^2 - |(p+1)^2 - c^2|}{4(p+1)\operatorname{Re} c}, \qquad \operatorname{Re} c > 0. \tag{3}$$

If h is analytic in E with $h(0) = 1$ and

$$\operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -w, \tag{4}$$

and if $p(z) = 1 + a_{p+1}z^{p+1} + \ldots$ is analytic in E and satisfies

$$p(z) + \frac{1}{c} zp'(z) \prec h(z), \tag{5}$$

then $p(z) \prec q(z)$, where $q(z)$ is the solution of

$$q(z) + \frac{p+1}{c} zq'(z) = h(t), \qquad q(0) = 1, \tag{6}$$

given by

$$q(z) = \frac{c}{(p+1)z^{c/(p+1)}} \int_0^z t^{c/(p+1)-1} h(t)\, dt. \tag{7}$$

Also q is the best dominant of (5).

Proof. Using Lemma 1, we see that it is sufficient to show that q is convex. First we note that $w \leq 1/2$. To see this we observe that $\operatorname{Re} c > 0$ implies $|c-(p+1)| < |c+(p+1)|$. Multiplying by $|c-(p+1)|$ and simplifying, we get

$$(p+1)^2 + |c|^2 - |(p+1)^2 - c^2| < 2\operatorname{Re} c \cdot (p+1)$$

whence $w \leq 1/2$.

If $c = p+1$, then $w = 1/2$, and (4) implies that h is close-to-convex and, by Lemma D, (7) implies that q is also close-to-convex. So $q'(z) \neq 0$ for $z \in E$ and the function

$$P(z) = 1 + \frac{zq''(z)}{q'(z)} = 1 + P_1 z + P_2 z^2 + \ldots$$

is analytic in E, with $P(0) = 1$. From (6), on differentiation, we get

$$(p+1)P(z) + c = ch'(z)/q'(z).$$

Again logarithmic differentiation and substitution for $zq''(z)/q'(z)$ in terms of $P(z)$ yields

$$P(z) + zP'(z)\Big/\left(P(z) + \frac{c}{p+1}\right) = 1 + \frac{zh''(z)}{h'(z)}. \tag{8}$$

Now let

$$H(u,v) = u + \frac{v}{u + \frac{c}{p+1}} + w. \tag{9}$$

Then

$$\begin{aligned} \operatorname{Re} H(is,t) &= \operatorname{Re}\left\{is + \frac{t}{is + \frac{c}{p+1}} + w\right\} \\ &= \operatorname{Re}\left\{\frac{t(p+1)(\overline{c} - (p+1)is)}{|c + (p+1)is|^2} + w\right\} \\ &= \frac{(p+1)t \operatorname{Re} c}{|c + (p+1)is|^2} + w. \end{aligned}$$

From (8), (9) and (4) we obtain

$$\operatorname{Re} H(P(z), zP'(z)) > 0, \qquad z \in E. \tag{10}$$

We proceed to show that $\operatorname{Re} H(is,t) \le 0$ for all real s and $t \le -(1+s^2)/2$:

$$\begin{aligned} \operatorname{Re} H(is,t) &= \frac{(p+1)t \operatorname{Re} c}{|c + (p+1)is|^2} + w \\ &\le -\frac{1}{|c + (p+1)is|^2}\left\{s^2\left(\frac{p+1}{2}\operatorname{Re} c - (p+1)^2 w\right)\right. \\ &\quad \left. - 2s(p+1)w \cdot \operatorname{Im} c + \operatorname{Re} c \cdot \frac{p+1}{2} - w|c|^2\right\}. \end{aligned} \tag{11}$$

For w given by (3), the coefficient of s^2 of the quadratic expression in s in braces is positive. To check this, put $c = c_1 + ic_2$ so that $\operatorname{Re} c = c_1$, $\operatorname{Im} c = c_2$. We have to verify that

$$c_1 > 2(p+1)w = \frac{(p+1)^2 + |c|^2 - |(p+1)^2 - c^2|}{2c_1}.$$

This inequality will hold if

$$2c_1^2 + |(p+1)^2 - c^2| > (p+1)^2 + |c| = (p+1)^2 + c_1^2 + c_2^2,$$

that is, if

$$|(p+1)^2 - c^2| > (p+1)^2 - \operatorname{Re} c^2,$$

which is obviously true. Further, the quadratic expression in s is a perfect square for the assumed value of w. So from (11) we see that $\operatorname{Re} H(is,t) \le 0$.

Lemma B enables us to conclude from (10) that $\operatorname{Re} P(z) > 0$, $z \in E$, that is,

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > 0, \qquad z \in E.$$

So q is convex and the proof is complete.

REMARK. If $c > 0$, then $w = \frac{c}{2(p+1)}$ for $0 < c \le p+1$, and $w = \frac{p+1}{2c}$ for $c > p+1$.

3. Theorems and their proofs

THEOREM 1. *Let q be a convex analytic function in E with $q(0) = 1$ and let*

$$h(z) = q(z) + \frac{(p+1)zq'(z)}{n+1}, \qquad n \text{ a positive integer.}$$

If $f \in M_p$ and

$$D^n f(z) = \frac{1}{z^p(1-z)^{n+1}} * f(z),$$

then

$$-\frac{z^{p+1}}{p}(D^{n+1}f)' \prec h \;\Rightarrow\; -\frac{z^{p+1}}{p}(D^n f)' \prec q$$

and the latter subordination is best possible.

Proof. One can verify without difficulty the relation

$$(n+1)D^{n+1}f = z(D^n f)' + (n+p+1)D^n f. \tag{12}$$

Set $P(z) = -\frac{z^{p+1}}{p}(D^n f)'$. Differentiation gives

$$pzP'(z) = p(p+1)P(z) - z^{p+2}(D^n f)''. \tag{13}$$

Differentiating (12) we obtain

$$(n+1)(D^{n+1}f)' = (n+p+2)(D^n f)' + z(D^n f)''. \tag{14}$$

Multiplying (14) by $-z^{p+1}$ and using (13) gives

$$\frac{zP'}{n+1} + P(z) = -\frac{z^{p+1}}{p}(D^{n+1}f)' \prec h(z). \tag{15}$$

Moreover, $P(0) = 1$ and $P'(0) = 0$. Indeed, $P(z) = 1 + P_{p+1}z^{p+1} + P_{p+2}z^{p+2} + \ldots$ By Lemma 1, we conclude that $P(z) \prec q(z)$ and q is the best dominant.

THEOREM 2. *Let h be analytic in E with*

$$h(0) = 1, \qquad \operatorname{Re}\left(1 + \frac{zh''(z)}{h'(z)}\right) > -w,$$

where

$$w = \begin{cases} (n+1)/(2(p+1)), & n = 0, 1, \ldots, p-1, \\ (p+1)/(2(n+1)), & n \geq p, \end{cases}$$

n being a positive integer. If $f \in M_p$, then

$$-\frac{z^{p+1}}{p}(D^{n+1}f)' \prec h \;\Rightarrow\; -\frac{z^{p+1}}{p}(D^n f)' \prec q,$$

where q is the solution of

$$q(z) + (p+1)\frac{zq'(z)}{n+1} = h(z), \qquad q(0) = 1.$$

In fact, q is given by

$$q(z) = \frac{n+1}{(p+1)z^{(n+1)/(p+1)}} \int_0^z t^{(n+1)/(p+1)-1} h(t)\, dt$$

and it is the best dominant.

Proof. The proof is immediate from Lemma 2, with $c = n+1$, if we note that the value of w for positive c is given by the remark following the proof of Lemma 2.

COROLLARY 1. *$L_{n+1,p}(\alpha) \subset L_{n,p}(r)$ for $\alpha < 1$, where the best possible value of r is given by*

$$r = r(\alpha, n) = 2\alpha - 1 + \frac{2(1-\alpha)}{p+1}(n+1)\int_0^1 \frac{t^{(n-p)/(p+1)}}{1+t}\, dt > \alpha.$$

Proof. Choose

$$h(z) = \frac{1 + z(2\alpha - 1)}{1+z}$$

in Theorem 2. Then h is convex and

$$q(z) = \frac{n+1}{(p+1)z^{(n+1)/(p+1)}} \int_0^z t^{(n-p)/(p+1)} \frac{1 + (2\alpha-1)t}{1+t}\, dt. \tag{16}$$

Since $\operatorname{Re} h(z) > \alpha$, the theorem asserts that

$$\operatorname{Re}\left\{-\frac{z^{p+1}}{p}(D^{n+1}f)'\right\} > \alpha \;\Rightarrow\; -\frac{z^{p+1}}{p}(D^n f)' \prec q(z).$$

But q is convex as observed in Lemma 2, has real coefficients in its Taylor expansion and is real for real z. Hence $q(E)$ is symmetric with respect to the real and thus $\operatorname{Re} q(z) > q(1)$ for $z \in E$. Moreover,

$$q(1) = \frac{n+1}{p+1} \int_0^1 t^{(n-p)/(p+1)} \frac{1+(2\alpha-1)t}{1+t}\, dt$$
$$= 2\alpha - 1 + \frac{2(1-\alpha)}{p+1}(n+1) \int_0^1 \frac{t^{(n-p)/(p+1)}}{1+t}\, dt$$
$$= r, \quad \text{say.}$$

Evidently

$$r > 2\alpha - 1 + \frac{2(1-\alpha)}{p+1}(n+1) \int_0^1 \frac{t^{(n-p)/(p+1)}}{2}\, dt = \alpha.$$

So, if $f \in L_{n+1,p}(\alpha)$ then

$$\operatorname{Re}\left\{-\frac{z^{(p+1)}}{p}(D^{n+1} f)'\right\} > \min_{|z|<1} \operatorname{Re} q(z) = q(1) = r,$$

which means $f \in L_{n,p}(r)$.

Remark. For $n = 0$,

$$r = 2\alpha - 1 + \frac{2(1-\alpha)}{p+1} \int_0^1 \frac{t^{1/(p+1)-1}}{1+t}\, dt.$$

Denoting the integral by $I\left(\frac{1}{p+1}\right)$, we have $r = 0$ if

$$2\alpha - 1 + \frac{2(1-\alpha)}{p+1} I\left(\frac{1}{p+1}\right) = 0,$$

that is, if

$$2\alpha\left(1 - \frac{I(1/(p+1))}{p+1}\right) = 1 - \frac{2}{p+1} I\left(\frac{1}{p+1}\right).$$

Denoting this value of α by α_0, we find $L_{m,p}(\alpha_0) \subset L_{0,p}(0)$, $m > 0$. Now $L_{0,p}(0)$ consists of the functions f for which

$$\operatorname{Re}\left\{-\frac{z^{p+1}}{p} f'\right\} > 0,$$

since $D^0 f = z^{-p}(1-z)^{-1} * f(z) = f(z)$. So $L_{0,p}(0)$ is a subclass of the class of multivalently close-to-convex meromorphic functions in the unit disk introduced by A. E. Livingston [2], the associated meromorphically starlike function being $-1/z^p$.

Theorem 3. *Let h be defined on E by*

$$h(z) = q(z) + \frac{p+1}{c-p+1} z q'(z),$$

where q is convex univalent in E, $h(0) = 1$, and c is a complex number with $\operatorname{Re} c > p - 1$. *If $f \in M_p$ and $F = I_c(f)$, where*

$$I_c(f)(z) = \frac{c-p+1}{z^{c+1}} \int_0^z t^c f(t)\,dt, \tag{17}$$

then

$$-\frac{z^{p+1}}{p}(D^n f(z))' \prec h(z) \ \Rightarrow\ -\frac{z^{p+1}}{p}(D^n F(z))' \prec q(z)$$

and the subordination is sharp.

P r o o f. From (17) we get

$$(c+1)F(z) + zF'(z) = (c-p+1)f(z). \tag{18}$$

If we use the facts $D^n(zF') = z(D^n F)'$ and

$$z(D^n F)' = (n+1)D^{n+1}F - (n+p+1)D^n F, \tag{19}$$

then (18) yields

$$(c+1)D^n F + (n+1)D^{n+1}F - (n+p+1)D^n F = (c-p+1)D^n f$$

or

$$(c-n-p)D^n F + (n+1)D^{n+1}F = (c-p+1)D^n f. \tag{20}$$

Set

$$P(z) = -\frac{z^{p+1}}{p}(D^n F)'$$

so that

$$pP'(z) = -(p+1)z^p(D^n F)' - z^{p+1}(D^n F)''. \tag{21}$$

Differentiating (19) and using (21) we obtain

$$pzP'(z) + p(n+1)P(z) = -(n+1)z^{p+1}(D^{n+1}F)'. \tag{22}$$

Differentiating (20) and using (21), we can rewrite (22) in the form

$$\frac{zp'(z)}{c-p+1} + P(z) = -\frac{z^{p+1}}{p}(D^n f)' \prec h(z). \tag{23}$$

Since $P(z) = 1 + P_1 z^{p+1} + \ldots$, application of Lemma 1 shows that (23) implies $P(z) \prec q(z)$ and q is the best dominant.

THEOREM 4. *Let*

$$w = \frac{(p+1)^2 + |c'|^2 - |(p+1)^2 - c'^2|}{4(p+1)\operatorname{Re} c'}, \qquad \operatorname{Re} c' > 0,\ c' = c-p+1.$$

Let h be analytic in E and satisfy

$$h(0) = 1, \qquad \operatorname{Re}\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -w.$$

If $f \in M_p$ *and* $F = I_c(f)$ *is defined by* (17), *then*

$$-\frac{z^{p+1}}{p}(D^n f(z))' \prec h(z) \ \Rightarrow\ -\frac{z^{p+1}}{p}(D^n F(z))' \prec q(z),$$

where q *is the solution of the differential equation*

$$q(z) + \frac{p+1}{c-p+1} z q'(z) = h(z), \qquad q(0) = 1,$$

given by

$$q(z) = \frac{c-p+1}{(p+1) z^{(c-p+1)/(p+1)}} \int_0^z t^{(c-p+1)/(p+1)-1} h(t)\, dt. \tag{24}$$

Moreover, q *is the best dominant.*

Proof. Setting $P(z) = -\frac{z^{p+1}}{p}(D^n F)'$ as in the proof of Theorem 3, we find

$$\frac{zP'(z)}{c-p+1} + P(z) = -\frac{z^{p+1}}{p}(D^n F)' \prec h(z).$$

An application of Lemma 2 with c replaced by $c' = c - p + 1$ gives $P(z) \prec q(z)$, where q is given by (24). The proof is complete.

Corollary. *If* $\alpha < 1$, $\operatorname{Re} c > p - 1$, *and* I_c *is defined by* (17), *then*

$$I_c(L_{n,p}(\alpha)) \subset L_{n,p}(\delta),$$

where

$$\delta = \min_{|z|=1} \operatorname{Re} q(z) = \delta(\alpha, c)$$

and

$$q(z) = \frac{c-p+1}{(p+1) z^{(c-p+1)/(p+1)}} \int_0^z t^{(c-p+1)/(p+1)} \left\{ \frac{1+(2\alpha-1)t}{1+t} \right\} dt, \tag{25}$$

and the result is sharp. Also if c *is real and* $c > p-1$, *then*

$$\delta(\alpha, c) = q(1) = 2\alpha - 1 + \frac{2(1-\alpha)}{p+1}(c-p+1) \int_0^1 \frac{t^{(c-p+1)/(p+1)-1}}{1+t}\, dt. \tag{26}$$

Proof. If we choose

$$h(z) = \frac{1 + z(2\alpha - 1)}{1+z}$$

in the theorem, then h is convex and we deduce from the theorem that

$$\operatorname{Re}\left\{ -\frac{z^{p+1}}{p}(D^n f(z))' \right\} > \alpha \ \Rightarrow\ -\frac{z^{p+1}}{p}(D^n F(z))' \prec q(z),$$

where q is given by (25), and so $I_c(L_{n,p}(\alpha)) \subset L_{n,p}(\delta)$. If c is real and $c > p-1$, then observing that $q(E)$ is convex and symmetric with respect to the real axis, we get $\operatorname{Re} q(z) > \delta = q(1)$ given by (26).

REMARK. If we take $c = (3p-1)/2$, then the integral in (26) reduces to

$$\int_0^1 \frac{t^{-1/2}}{1+t}\,dt = \frac{\pi}{2}.$$

We have $\delta = 2\alpha - 1 + (1-\alpha)\pi/2$, and $\delta = 0$ if $\alpha = -(\pi-2)/(4-\pi)$. If

$$\operatorname{Re}\left\{-\frac{z^{p+1}}{p}(D^n f)'\right\} > -\frac{\pi-2}{4-\pi},$$

then

$$\operatorname{Re}\left\{-\frac{z^{p+1}}{p}(D^n F)'\right\} > 0,$$

where

$$F(z) = \frac{p+1}{2}\cdot\frac{1}{z^{(3p+1)/2}}\int_0^z t^{(3p-1)/2} f(t)\,dt.$$

THEOREM 5. *Let $f \in M_p$ and let $I_c(f)$ be defined by* (17). *Let $\alpha < 1$. If*

$$\operatorname{Re}\left\{-\frac{z^{p+1}}{p}(D^n f)'\right\} > \alpha - (1-\alpha)\operatorname{Re}\frac{1}{c-p+1}$$

then $I_c(f) \in L_{n,p}(\alpha)$.

Proof. Denote $I_c(f)$ by F and put

$$\frac{-z^{p+1}(D^n F(z))'}{p} = (1-\alpha)P(z) + \alpha. \tag{27}$$

Using (20) and (12) we obtain after differentiation and simplification

$$(c+2)(D^n F)' + z(D^n F)'' = (c-p+1)(D^n f)'. \tag{28}$$

Multiplying both sides of (28) by z^{p+1} and using (27) we obtain

$$\begin{aligned}-\{(1-\alpha)P(z)+\alpha\}p(c+2) + p(p+1)\{(1-\alpha)P(z)+\alpha\}&\\ - (1-\alpha)pzP'(z)&\\ = (c-p+1)z^{p+1}(D^n f)',&\end{aligned}$$

or

$$-\frac{z^{p+1}}{p}(D^n f)' = (1-\alpha)P(z) + \alpha + (1-\alpha)\frac{zP'(z)}{c-p+1}. \tag{29}$$

So the inequality in the assumptions of the theorem becomes

$$\operatorname{Re}\left\{(1-\alpha)P(z) + \frac{1-\alpha}{c-p+1}(zP'(z)+1)\right\} > 0, \qquad z \in E. \tag{30}$$

Since $P(z) = 1 + P_{p+1}z^{p+1} + \ldots$, in order to show that (30) implies that $\operatorname{Re} P(z) > 0$ in E, it suffices to prove the inequality

$$\operatorname{Re}\left\{(1-\alpha)is + \frac{1-\alpha}{c-p+1}(t+1)\right\} \le 0$$

for all real s and

$$t \le -(1+s^2)\frac{p+1}{2} \le -(1+s^2),$$

by Lemma B. Since $\operatorname{Re}(c-p+1) > 0$, the inequality holds and so $\operatorname{Re} P(z) > 0$. In other words,

$$\operatorname{Re}\left\{-\frac{z^{p+1}}{p}(D^n F)'\right\} = \operatorname{Re}(1-\alpha)P(z) + \alpha > \alpha,$$

or $F \in L_{n,p}(\alpha)$. The proof is complete.

Remark. If $\alpha = 0$, we conclude that

$$\operatorname{Re}\left\{-\frac{z^{p+1}}{p}(D^n f)'\right\} > -\operatorname{Re}\frac{1}{c-p+1} \ \Rightarrow\ \operatorname{Re}\left\{-\frac{z^{p+1}}{p}(D^n F)'\right\} > 0.$$

If moreover $n = 0$ and $c = p$, we obtain the result: For $f \in M_p$,

$$\operatorname{Re}\left\{-\frac{z^{p+1}}{p}f'(z)\right\} > -1 \ \Rightarrow\ \operatorname{Re}\left\{-\frac{z^{p+1}}{p}F'(z)\right\} > 0,$$

where $F(z) = z^{-p-1}\int_0^z t^p f(t)\,dt$.

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