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Dini continuity of the first derivatives of generalized solutions to the Dirichlet problem for linear elliptic second order equations in nonsmooth domains

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Abstract. We consider generalized solutions to the Dirichlet problem for linear elliptic second order equations in a domain bounded by a Dini–Lyapunov surface and containing a conical point. For such solutions we derive Dini estimates for the first order generalized derivatives.

1. Introduction. We consider generalized solutions to the Dirichlet problem for a linear uniformly elliptic second order equation in divergence form

$$
\text{(DL)}\qquad \begin{cases} \dfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j}+a^i(x)u)+b^i(x)u_{x_i}+c(x)u \\ \qquad = g(x)+\dfrac{\partial f^j(x)}{\partial x_j}, \quad x\in G, \\ u(x)=\varphi(x), \quad x\in\partial G \end{cases}
$$

(summation over repeated indices from 1 to n is understood), where $G\subset\mathbb{R}^n$ is a bounded domain with boundary ∂G and ∂G is a Dini–Lyapunov surface containing the origin $\mathcal{O}$ as a conical point. This last means that $\partial G\setminus\mathcal{O}$ is a smooth manifold but near $\mathcal{O}$ the domain G is diffeomorphic to a cone.

Hölder estimates for the first derivatives of generalized solutions to the problem (DL) are well known in the case where the leading coefficients $a^{ij}(x)$ are Hölder continuous (see e.g. [5, 8.11] for smooth domains and [1] for domains with angular points). Here we derive Dini estimates for the first derivatives of generalized solutions of the problem (DL) in a domain with a conical boundary point under *minimal* smoothness conditions on the leading coefficients (Dini continuity). It should be noted that interior Dini continuity

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of the first and second derivatives of generalized solutions to the problem (DL) was investigated in [3, 7] under the condition of Dini continuity of the first derivatives of the leading coefficients.

We introduce the following notations and definitions:

- $[l]$: the integral part of l (if l is not an integer);
- $r = |x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$;
- $G' \Subset G$: G' has compact closure contained in G;
- $\operatorname{mes} G$: volume of G;
- S^{n-1}: the unit sphere in $\mathbb{R}^n$;
- $B_r(x_0)$: the open ball in $\mathbb{R}^n$ of radius r centered at x_0;
- $\omega_n = 2\pi^{n/2}/(n\Gamma(n/2))$: the volume of the unit ball in $\mathbb{R}^n$;
- $\sigma_n = n\omega_n$: the area of the n-dimensional unit sphere;
- $\mathbb{R}^n_+$: the half-space $x_n > 0$;
- Σ: the hyperplane $\{x_n = 0\}$;
- $B_r^+ = B_r \cap \mathbb{R}^n_+$, where $x_0 \in \overline{\mathbb{R}^n_+}$;
- (r, ω): the spherical coordinates of $x \in \mathbb{R}^n$ with pole $\mathcal{O}$;
- Ω: a domain in S^{n-1} with smooth $(n-2)$-dimensional boundary $\partial\Omega$;
- $G_a^b = G \cap \{(r,\omega) \mid 0 \le a < r < b,\ \omega \in \Omega\}$: a layer in $\mathbb{R}^n$;
- $\Gamma_a^b = \partial G \cap \{(r,\omega) \mid 0 \le a < r < b,\ \omega \in \partial\Omega\}$: the lateral surface of the layer G_a^b;
- $D_i u = u_{x_i} = \partial u/\partial x_i,\ D_{ij} u = u_{x_i x_j} = \partial^2 u/\partial x_i \partial x_j$;
- $\nabla u \equiv u_x = (u_{x_1}, \ldots, u_{x_n})$: the gradient of $u(x)$;
- $\mathbf{n} = \mathbf{n}(x) = \{\nu_1, \ldots, \nu_n\}$: the unit outward normal to ∂G at the point x;
- $d\Omega$: the $(n-1)$-dimensional area element of the unit sphere;
- $d\sigma$: the $(n-1)$-dimensional area element of ∂G;
- Δ: the Laplacian in $\mathbb{R}^n$;
- Δ_ω: the Laplace–Beltrami operator on the unit sphere S^{n-1};
- $d(x) = \operatorname{dist}(x, \partial G \setminus \mathcal{O})$;
- $\Phi(x)$: any possible extension into G of a boundary function $\varphi(x)$, i.e., $\Phi(x) = \varphi(x)$ for $x \in \partial G$;
- $\mathcal{A}(t)$: a function defined for $t \ge 0$, nonnegative, increasing, continuous at zero, with $\lim_{t\to+0} \mathcal{A}(t) = 0$.

Definition 1.1. The function $\mathcal{A}$ is called *Dini continuous at zero* if $\int_0^d t^{-1}\mathcal{A}(t)\,dt < \infty$ for some $d > 0$.

Definition 1.2. The function $\mathcal{A}$ is called an *α-function*, $0 < \alpha < 1$, on $(0, d]$ if $t^{-\alpha}\mathcal{A}(t)$ is decreasing on $(0, d]$, i.e.

$$\mathcal{A}(t) \le t^\alpha \tau^{-\alpha} \mathcal{A}(\tau), \quad 0 < \tau \le t \le d. \tag{1.1}$$

In particular, setting $t = c\tau,\ c > 1$, we have

$$\mathcal{A}(c\tau) \le c^\alpha \mathcal{A}(\tau), \quad 0 < \tau \le c^{-1} d. \tag{1.2}$$

If an α-function $\mathcal{A}$ is Dini continuous at zero, then we say that $\mathcal{A}$ is an *α-Dini function*. In that case we also define the function $\mathcal{B}(t) = \int_0^t (\mathcal{A}(\tau)/\tau)\,d\tau$. It is obvious that $\mathcal{B}$ is increasing and continuous on $[0,d]$, and $\mathcal{B}(0)=0$. We integrate the inequality (1.1) over τ from 0 to t:

$$\mathcal{A}(t) \le \alpha\mathcal{B}(t). \tag{1.3}$$

Similarly from (1.1) we derive

$$\int_\delta^d (\mathcal{A}(t)/t^2)\,dt = \int_\delta^d t^{\alpha-2}(\mathcal{A}(t)/t^\alpha)\,dt \le \delta^{-\alpha}\mathcal{A}(\delta)\int_\delta^d t^{\alpha-2}\,dt \le (1-\alpha)^{-1}\mathcal{A}(\delta)/\delta,$$

whence by (1.3),

$$\begin{aligned}\delta\int_\delta^d (\mathcal{A}(t)/t^2)\,dt &\le (1-\alpha)^{-1}\mathcal{A}(\delta)\\ &\le \alpha(1-\alpha)^{-1}\mathcal{B}(\delta), \qquad \forall\alpha\in(0,1),\ 0<\delta<d.\end{aligned} \tag{1.4}$$

Definition 1.3. The function $\mathcal{B}$ is called *equivalent* to $\mathcal{A}$, written $\mathcal{A}\sim\mathcal{B}$, if there exist positive constants C_1 and C_2 such that

$$C_1\mathcal{A}(t) \le \mathcal{B}(t) \le C_2\mathcal{A}(t) \qquad \text{for all } t\ge 0.$$

An *equivalence test* is known [4, theorem of Sec. 1]: $\mathcal{A}\sim\mathcal{B}$ if and only if

$$\varliminf_{t\to 0} \mathcal{A}(2t)/\mathcal{A}(t) > 1. \tag{1.5}$$

In some cases we shall consider functions $\mathcal{A}$ such that also

$$\sup_{0<\tau\le 1} \mathcal{A}(\tau t)/\mathcal{A}(\tau) \le c\mathcal{A}(t), \qquad \forall t\in(0,d], \tag{1.6}$$

with some constant c independent of t. Examples of α-Dini functions $\mathcal{A}$ which satisfy (1.5), (1.6) with $c=1$ are:

$$t^\alpha, \quad 0\le t<\infty;$$
$$t^\alpha\ln(1/t), \quad t\in(0,d],\ d=\min(e^{-e}, e^{-1/\alpha}),\ e^{-1}<\alpha<1.$$

We will consider the following function spaces:

- $C^l(\overline{G})$: the Banach space of functions having all the derivatives of order at most l (if $l=$ integer ≥ 0) and of order $[l]$ (if l is noninteger) continuous in $\overline{G}$ and whose $[l]$th order partial derivatives are uniformly Hölder continuous with exponent $l-[l]$ in $\overline{G}$; $|u|_{l;G}$ is the norm of the element $u\in C^l(\overline{G})$; if $l\ne[l]$ then

$$|u|_{l;G} = \sum_{j=0}^{[l]} \sup_G |D^j u| + \sup_{|\alpha|=[l]} \sup_{\substack{x,y\in G\\ x\ne y}} \frac{|D^\alpha u(x)-D^\alpha u(y)|}{|x-y|^{l-[l]}}.$$

- $C_0^k(G)$: the set of functions in $C^k(G)$ with compact support in G.

- $C^{0,\mathcal{A}}(G)$: the set of bounded and continuous functions f on G with
$$[f]_{\mathcal{A};G} = \sup_{\substack{x,y\in G\\ x\neq y}} \frac{|f(x)-f(y)|}{\mathcal{A}(|x-y|)} < \infty;$$

equipped with the norm
$$\|f\|_{0,\mathcal{A};G} = |f|_{0;G} + [f]_{\mathcal{A};G},$$
this set is a Banach space. We also define the quantity
$$[f]_{\mathcal{A},x} = \sup_{y\in G\setminus\{x\}} \frac{|f(x)-f(y)|}{\mathcal{A}(|x-y|)}.$$
It is not difficult to see that if $\mathcal{A}\sim\mathcal{B}$ then $[f]_{\mathcal{A}}\sim[f]_{\mathcal{B}}$.

If $k\geq 1$ is an integer then we denote by $C^{k,\mathcal{A}}(G)$ the subspace of $C^k(G)$ consisting of functions whose $(k-1)$th order partial derivatives are uniformly Lipschitz continuous and each kth order derivative belongs to $C^{0,\mathcal{A}}(G)$; it is a Banach space with the norm
$$\|f\|_{k,\mathcal{A};G} = |f|_{k;G} + \sum_{|\beta|=k} [D^\beta f]_{\mathcal{A};G}.$$

The *interpolation inequality* (see [8, (10.1)]) will be needed: *if the domain has a Lipschitz boundary, then for any* $\varepsilon>0$ *there exists a constant* $c(\varepsilon,G)$ *such that for every* $f\in C^{1,\mathcal{A}}(G)$,

$$\sum_{i=1}^n |D_i f|_{0;G} \leq \varepsilon\sum_{i=1}^n [D_i f]_{\mathcal{A};G} + c(\varepsilon,G)|f|_{0;G}. \tag{1.7}$$

- $L_p(G)$: the Banach space of p-integrable functions u on G ($p\geq 1$) with norm $\| u \|_{p;G}$.

Moreover, $\lambda=\lambda(\Omega)$ will stand for the smallest positive eigenvalue of the problem

$$\text{(EVP)}\qquad \begin{cases} \Delta_\omega\psi + \lambda(\lambda+n-2)\psi = 0, & \omega\in\Omega\subset S^{n-1},\\ \psi(\omega)=0, & \omega\in\partial\Omega,\end{cases}$$

and $c(\dots)$ will be different constants depending only on the quantities appearing in parentheses.

Let $T\subset\partial G$ be a nonempty set. Following [5, Sec. 6.2] and [8, Sec. 3] we shall say that the boundary portion T is *of class* $C^{1,\mathcal{A}}$ if for each point $x_0\in T$ there are a ball $B=B(x_0)$, a one-to-one mapping ψ of B onto a ball B' and a constant $K>0$ such that:

(i) $B\cap\partial G\subset T$, $\psi(B\cap G)\subset\mathbb{R}^n_+$;
(ii) $\psi(B\cap\partial G)\subset\Sigma$;
(iii) $\psi\in C^{1,\mathcal{A}}(B)$, $\psi^{-1}\in C^{1,\mathcal{A}}(B')$;

(iv) $\|\psi\|_{1,\mathcal{A};B} \le K,\ \|\psi^{-1}\|_{1,\mathcal{A};B'} \le K.$

It is not difficult to see that for $y = \psi(x)$ one has

$$K^{-1}|y-y'| \le |x-x'| \le K|y-y'|, \qquad \forall x, x' \in B. \tag{1.8}$$

Lemma [8, Sec. 7, (iv)]. *Let $\mathcal{A}$ be an α-function and $f \in C^{0,\mathcal{A}}(B)$, $\psi^{-1} \in C^{1,\mathcal{A}}(B')$. Then $f \circ \psi^{-1} \in C^{1,\mathcal{A}}(B)$ and*

$$[f \circ \psi^{-1}]_{\mathcal{A};B} \le K^{\alpha}[f]_{\mathcal{A};B}. \tag{1.9}$$

2. Dini estimates of the first derivatives for the generalized Newtonian potential (cf. [5, Ch. 4]). We shall consider the Dirichlet problem for the Poisson equation

$$\begin{cases} \Delta v = \mathcal{G} + \sum_{j=1}^{n} D_j \mathcal{F}^j, & x \in G, \\ v(x) = 0, & x \in \partial G. \end{cases} \tag{PE}$$

Let $\Gamma(x-y)$ be the normalized fundamental solution of Laplace's equation. The following estimates are known (see e.g. [5, (2.12), (2.14)]):

$$\begin{aligned} |\Gamma(x-y)| &= |x-y|^{2-n}/(n(n-2)\omega_n), \qquad n \ge 3, \\ |D_i\Gamma(x-y)| &\le |x-y|^{1-n}/(n\omega_n), \\ |D_{ij}\Gamma(x-y)| &\le |x-y|^{-n}/\omega_n, \\ |D^{\beta}\Gamma(x-y)| &\le C(n,\beta)|x-y|^{2-n-|\beta|}. \end{aligned} \tag{2.1}$$

We define the functions

$$z(x) = \int_G \Gamma(x-y)\mathcal{G}(y)\,dy, \qquad w(x) = D_j \int_G \Gamma(x-y)\mathcal{F}^j(y)\,dy, \tag{2.2}$$

assuming that the functions $\mathcal{G}(x)$ and $\mathcal{F}^j(x)$, $j = 1, \ldots, n$, are integrable on G. The function z is called the *Newtonian potential* with density function $\mathcal{G}$, and w is called the *generalized Newtonian potential* with density function $\operatorname{div}\mathcal{F}$. We now give estimates for these potentials.

Let $B_1 = B_R(x_0)$, $B_2 = B_{2R}(x_0)$ be concentric balls in $\mathbb{R}^n$ and z, w be Newtonian potentials in B_2.

Lemma 1. *Suppose $\mathcal{G} \in L_p(B_2)$, $p > n/2$, and $\mathcal{F}^j \in L_\infty(B_2)$, $j = 1, \ldots, n$. Then*

$$\begin{aligned} |z|_{0;B_1} &\le c(p)R^{2/p'} \ln^{1/p'}(1/(2R)) \|\mathcal{G}\|_{p;B_2}, \qquad && n = 2, \\ |z|_{0;B_1} &\le c(p,n)R^{2-n+n/p'} \|\mathcal{G}\|_{p;B_2}, && n \ge 3, \end{aligned} \tag{2.3}$$

$$|w|_{0;B_1} \le 2R\sum_{j=1}^{n} |\mathcal{F}^j|_{0;B_2}, \tag{2.4}$$

where $1/p + 1/p' = 1$.

Proof. The estimates follow from inequalities (2.1), Hölder's inequality and [5, Lemma 4.1].

In the following the D operator is always taken with respect to the x variable.

LEMMA 2 [5, Lemmas 4.1, 4.2]. *Let* $\partial G \in C^{1,\mathcal{A}}$, $\mathcal{G} \in L_p(G)$, $p > n$, $\mathcal{F}^j \in C^{0,\mathcal{A}}(G)$, *where* $\mathcal{A}$ *is an* α*-function Dini continuous at zero. Then for any* $x \in G$,

$$D_i z(x) = \int_G D_i\Gamma(x-y)\mathcal{G}(y)\,dy, \tag{2.5}$$

$$\begin{aligned} D_i w(x) = &\int_{G_0} D_{ij}\Gamma(x-y)(\mathcal{F}^j(y) - \mathcal{F}^j(x))\,dy \\ &- \mathcal{F}^j(x)\int_{\partial G_0} D_i\Gamma(x-y)\nu_j\,d_y\sigma \end{aligned} \tag{2.6}$$

$(i = 1,\ldots,n)$; *here* G_0 *is any domain containing* G *for which the Gauss divergence theorem holds and* $\mathcal{F}^j$ *are extended to vanish outside* G.

LEMMA 3 (cf. [5, Lemma 4.4]). *Let* $\mathcal{G} \in L_p(B_2)$, $p > n$, $\mathcal{F}^j \in C^{0,\mathcal{A}}(\overline{B}_2)$, *where* $\mathcal{A}$ *is an* α*-function Dini continuous at zero. Then* $z, w \in C^{1,\mathcal{B}}(\overline{B}_1)$ *and*

$$\|z\|_{1,\mathcal{B};B_1} \le c(n,p,R,\mathcal{A}^{-1}(2R))\,\|\mathcal{G}\|_{p;B_2}, \tag{2.7}$$

$$\|w\|_{1,\mathcal{B};B_1} \le c(n,p,\alpha,R,\mathcal{A}^{-1}(2R),\mathcal{B}(2R))\sum_{j=1}^{n}\|\mathcal{F}^j\|_{0,\mathcal{A};B_2}. \tag{2.8}$$

Proof. Let $x, \overline{x} \in B_1$ and $G = B_2$. By formulas (2.5), (2.6), taking into account (2.1) and Hölder's inequality and setting $|x-y| = t$, $y - x = t\omega$, $dy = t^{n-1}dt\,d\Omega$, we have

$$\begin{aligned} |D_i z| &\le (n\omega_n)^{-1}\int_{B_2} |x-y|^{1-n}|\mathcal{G}(y)|\,dy \\ &\le (n\omega_n)^{-1}\|\mathcal{G}\|_{p;B_2}\Big\{\int_{B_2} |x-y|^{(1-n)p'}\,dy\Big\}^{1/p'} \\ &= \frac{p-1}{p-n}(2R)^{(p-n)/(p-1)}\|\mathcal{G}\|_{p;B_2}, \end{aligned} \tag{2.9}$$

$$|D_i w(x)| \le (n\omega_n)^{-1}R^{1-n}\sum_{j=1}^{n}|\mathcal{F}^j(x)|\int_{\partial B_2} d_y\sigma \tag{2.10}$$

$$+\,\omega_n^{-1}\sum_{j=1}^{n}[\mathcal{F}^j]_{\mathcal{A},x}\int\limits_{B_2}\frac{\mathcal{A}(x-y)}{|x-y|^n}\,dy$$

$$\le 2^{n-1}\sum_{j=1}^{n}|\mathcal{F}^j(x)| + n\sum_{j=1}^{n}[\mathcal{F}^j]_{\mathcal{A},x}\int\limits_0^{2R}\frac{\mathcal{A}(t)}{t}\,dt$$

$$\le c(n)\mathcal{B}(2R)\sum_{j=1}^{n}(|\mathcal{F}^j(x)|+[\mathcal{F}^j]_{\mathcal{A},x}).$$

Taking into account (2.5) we obtain by subtraction

$$|D_i z(x)-D_i z(\overline{x})| \le \int\limits_{B_2}|D_i\Gamma(x-y)-D_i\Gamma(\overline{x}-y)|\cdot|\mathcal{G}(y)|\,dy.$$

We set $\delta=|x-\overline{x}|$, $\xi=\frac{1}{2}(x-\overline{x})$ and write $B_2=B_\delta(\xi)\cup\{B_2\setminus B_\delta(\xi)\}$. Then

$$\begin{aligned}
(2.11)\qquad &\int\limits_{B_\delta(\xi)}|D_i\Gamma(x-y)-D_i\Gamma(\overline{x}-y)|\cdot|\mathcal{G}(y)|\,dy\\
&\le \int\limits_{B_\delta(\xi)}|D_i\Gamma(x-y)|\cdot|\mathcal{G}(y)|\,dy+\int\limits_{B_\delta(\xi)}|D_i\Gamma(\overline{x}-y)|\cdot|\mathcal{G}(y)|\,dy\\
&\le (n\omega_n)^{-1}\Big\{\int\limits_{B_\delta(\xi)}|x-y|^{1-n}|\mathcal{G}(y)|\,dy+\int\limits_{B_\delta(\xi)}|\overline{x}-y|^{1-n}|\mathcal{G}(y)|\,dy\Big\}\\
&\le 2(n\omega_n)^{-1}\int\limits_{B_{3\delta/2}(x)}|x-y|^{1-n}|\mathcal{G}(y)|\,dy\\
&\le 2(n\omega_n)^{-1}\|\mathcal{G}\|_{p;B_2}\Big(\int\limits_{B_{3\delta/2}(x)}|x-y|^{(1-n)p'}\,dy\Big)^{1/p'}\\
&\le 2(n\omega_n)^{-1/p}\|\mathcal{G}\|_{p;B_2}\left(\frac{3\delta}{2}\right)^{1-n/p}\{n+(1-n)p'\}^{-1/p'}\\
&\le \frac{2(n\omega_n)^{-1/p}(2R)^{1-n/p}}{\{n+(1-n)p'\}^{-1/p'}}\cdot\frac{\mathcal{A}(|\overline{x}-x|)}{\mathcal{A}(2R)}\|\mathcal{G}\|_{p;B_2},\qquad 1/p+1/p'=1
\end{aligned}$$

(here we take into account that $\delta^\alpha\le(2R)^\alpha\mathcal{A}(\delta)/\mathcal{A}(2R)$ for all $\alpha>0$ by (1.1), since $\delta\le 2R$). Similarly,

$$\begin{aligned}
(2.12)\qquad &\int\limits_{B_2\setminus B_\delta(\xi)}|D_i\Gamma(x-y)-D_i\Gamma(\overline{x}-y)|\cdot|\mathcal{G}(y)|\,dy\\
&\le |x-\overline{x}|\int\limits_{B_2\setminus B_\delta(\xi)}|DD_i\Gamma(\widetilde{x}-y)|\cdot|\mathcal{G}(y)|\,dy\\
&\qquad\qquad\text{(for some } \widetilde{x} \text{ between } x \text{ and } \overline{x})
\end{aligned}$$

$$\le \delta\omega_n^{-1} \int\limits_{|y-\xi|\ge\delta} |y-\widetilde{x}|^{-n}|\mathcal{G}(y)|\,dy$$

$$\le 2^n\delta\omega_n^{-1} \int\limits_{|y-\xi|\ge\delta} |y-\xi|^{-n}|\mathcal{G}(y)|\,dy \qquad (\text{since } |y-\xi|\le 2|y-\widetilde{x}|)$$

$$\le 2^n\delta\omega_n^{-1}\|\mathcal{G}\|_{p;B_2}\Big(\int\limits_{|y-\xi|\ge\delta} |y-\xi|^{-np'}\,dy\Big)^{1/p'}$$

$$\le 2^n\omega_n^{-1/p}(p-1)^{1/p'}\delta^{1-n/p}\|\mathcal{G}\|_{p;B_2}$$

$$\le 2^n\omega_n^{-1/p}(p-1)^{1/p'}(2R)^{1-n/p}\,\frac{\mathcal{A}(|x-\overline{x}|)}{\mathcal{A}(2R)}\|\mathcal{G}\|_{p;B_2}.$$

From (2.11) and (2.12), taking into account (1.3), we obtain

$$\begin{aligned} (2.13)\qquad |D_iz(x)-D_iz(\overline{x})| &\le c(n,p,R)\mathcal{A}^{-1}(2R)\|\mathcal{G}\|_{p;B_2}\mathcal{A}(|x-\overline{x}|)\\ &\le c(n,p,R)\mathcal{A}^{-1}(2R)\|\mathcal{G}\|_{p;B_2}\mathcal{B}(|x-\overline{x}|), \qquad \forall x,\overline{x}\in B_1. \end{aligned}$$

The first of the required estimates, (2.7), follows from (2.3) and (2.13). Now we derive the estimate (2.8).

By (2.6) for all $x,\overline{x}\in B_1$ we have

$$\begin{aligned} (2.14)\qquad D_iw(\overline{x})-D_iw(x) &= \sum_{j=1}^n(\mathcal{F}^j(x)\mathcal{J}_{1j}+(\mathcal{F}^j(x)-\mathcal{F}^j(\overline{x}))\mathcal{J}_{2j})+\mathcal{J}_3\\ &\quad+\mathcal{J}_4+\sum_{j=1}^n(\mathcal{F}^j(x)-\mathcal{F}^j(\overline{x}))\mathcal{J}_{5j}+\mathcal{J}_6, \end{aligned}$$

where

$$\mathcal{J}_{1j} = \int\limits_{\partial B_2}(D_i\varGamma(x-y)-D_i\varGamma(\overline{x}-y))\nu_j(y)\,d_y\sigma,$$

$$\mathcal{J}_{2j} = \int\limits_{\partial B_2} D_i\varGamma(\overline{x}-y)\nu_j(y)\,d_y\sigma,$$

$$\mathcal{J}_3 = \int\limits_{B_\delta(\xi)} D_{ij}\varGamma(x-y)(\mathcal{F}^j(x)-\mathcal{F}^j(y))\,dy,$$

$$\mathcal{J}_4 = \int\limits_{B_\delta(\xi)} D_{ij}\varGamma(\overline{x}-y)(\mathcal{F}^j(y)-\mathcal{F}^j(\overline{x}))\,dy,$$

$$\mathcal{J}_{5j} = \int\limits_{B_2\setminus B_\delta(\xi)} D_{ij}\varGamma(x-y)\,dy,$$

$$\mathcal{J}_6 = \int\limits_{B_2\setminus B_\delta(\xi)} (D_{ij}\Gamma(x-y) - D_{ij}\Gamma(\overline{x}-y))(\mathcal{F}^j(\overline{x}) - \mathcal{F}^j(y))\,dy.$$

(Here we set again $\delta = |x-\overline{x}|$, $\xi = \frac{1}{2}(x-\overline{x})$ and write $B_2 = B_\delta(\xi) \cup \{B_2 \setminus B_\delta(\xi)\}$.)

We estimate these integrals by analogy with [5, pp. 58–59]:

$$\begin{aligned}
|\mathcal{J}_{1j}| &\le |x-\overline{x}| \int\limits_{\partial B_2} |DD_i\Gamma(\widetilde{x}-y)|\,d_y\sigma \\
&\qquad\qquad\qquad\qquad \text{(for some point } \widetilde{x} \text{ between } x \text{ and } \overline{x}) \\
&\le |x-\overline{x}| n\omega_n^{-1} \int\limits_{\partial B_2} |\widetilde{x}-y|^{-n} d_y\sigma \\
&\le n^2 2^{n-1}|x-\overline{x}|R^{-1} \quad (\text{since } |\widetilde{x}-y| \ge R \text{ for } y \in \partial B_2) \\
&\le n^2 2^{n-1}\mathcal{A}(|x-\overline{x}|)R^{-1}\delta/\mathcal{A}(\delta) \\
&\le n^2 2^n \mathcal{A}(|x-\overline{x}|)/\mathcal{A}(2R) \\
&\qquad (\text{since } \delta = |x-\overline{x}| \le 2R \text{ and } \delta/\mathcal{A}(\delta) \le 2R/\mathcal{A}(2R) \text{ by (1.1)}) \\
&\le n^2 2^n \alpha\mathcal{B}(\delta)/\mathcal{A}(2R) \quad (\text{by (1.3)}).
\end{aligned}$$

Next,

$$\begin{aligned}
|\mathcal{J}_{2j}| &\le 2^{n-1}, \\
|\mathcal{J}_3| &\le \omega_n^{-1}[\mathcal{F}^j]_{\mathcal{A},x} \int\limits_{B_\delta(\xi)} |x-y|^{-n}\mathcal{A}(|x-y|)\,dy \\
&\le \omega_n^{-1}[\mathcal{F}^j]_{\mathcal{A},x} \int\limits_{B_{3\delta/2}(x)} |x-y|^{-n}\mathcal{A}(|x-y|)\,dy \\
&= n[\mathcal{F}^j]_{\mathcal{A},x} \int\limits_0^{3\delta/2} t^{-1}\mathcal{A}(t)\,dt \\
&\le (3/2)^\alpha n[\mathcal{F}^j]_{\mathcal{A},x}\mathcal{B}(\delta) \quad (\text{by (1.2)}).
\end{aligned}$$

By analogy with the estimate for $\mathcal{J}_3$ we obtain

$$|\mathcal{J}_4| \le (3/2)^\alpha n[\mathcal{F}^j]_{\mathcal{A},\overline{x}}\mathcal{B}(\delta), \qquad |\mathcal{J}_{5j}| \le 2^n \qquad (\text{see [5, p. 59]}),$$

and

$$\begin{aligned}
|\mathcal{J}_6| &\le |x-\overline{x}| \int\limits_{B_2\setminus B_\delta(\xi)} |DD_{ij}\Gamma(\widetilde{x}-y)| \cdot |\mathcal{F}^j(\overline{x}) - \mathcal{F}^j(y)|\,dy \\
&\qquad\qquad\qquad\qquad \text{(for some } \widetilde{x} \text{ between } x \text{ and } \overline{x}) \\
&\le |x-\overline{x}|c(n) \int\limits_{|y-\xi|\ge\delta} |\mathcal{F}^j(\overline{x}) - \mathcal{F}^j(y)| \cdot |\widetilde{x}-y|^{-n-1}\,dy
\end{aligned}$$

$$\le c(n)\delta[\mathcal{F}^j]_{\mathcal{A},\overline{x}} \int\limits_{|y-\xi|\ge\delta} \mathcal{A}(|\overline{x}-y|)|\widetilde{x}-y|^{-n-1}\,dy$$

$$\le 2^{n+1}c(n)\delta[\mathcal{F}^j]_{\mathcal{A},\overline{x}} \int\limits_{|y-\xi|\ge\delta} \mathcal{A}((3/2)|\xi-y|)|\xi-y|^{-n-1}\,dy$$

$$\text{(since } |\overline{x}-y|\le(3/2)|\xi-y|\le 3|x-\widetilde{y}|)$$

$$\le 2^{n+1}n\omega_n c(n)(3/2)^\alpha\delta[\mathcal{F}^j]_{\mathcal{A},\overline{x}} \int\limits_\delta^R t^{-2}\mathcal{A}(t)\,dt$$

$$\text{(since } \mathcal{A}((3/2)t)\le(3/2)^\alpha\mathcal{A}(t) \text{ by (1.2))}$$

$$\le \frac{\alpha}{1-\alpha}(3/2)^\alpha n\omega_n 2^{n+1}c(n)[\mathcal{F}^j]_{\mathcal{A},\overline{x}}\mathcal{B}(\delta) \qquad \text{(by (1.4))}.$$

Now from (2.14) and the above estimates we obtain

$$|D_iw(\overline{x})-D_iw(x)| \le c(n,\alpha)\sum_{j=1}^n(|\mathcal{F}^j(x)|\mathcal{A}^{-1}(2R)+[\mathcal{F}^j]_{\mathcal{A},x}+[\mathcal{F}^j]_{\mathcal{A},\overline{x}})\mathcal{B}(|x-\overline{x}|), \quad \forall x,\overline{x}\in B_1. \tag{2.15}$$

Finally, from (2.10) and (2.15) it follows that $w\in C^{1,\mathcal{B}}(B_1)$ and the estimate (2.8) holds. Lemma 3 is proved.

Theorem 1. *Let v be a generalized solution of equation* (PE) *in B_2^+ with $\mathcal{G}\in L_{n/(1-\alpha)}(B_2^+)$, $\mathcal{F}^j\in C^{0,\mathcal{A}}(\overline{B_2^+})$, where $\mathcal{A}$ is an α-function satisfying the Dini condition at zero, and let $v=0$ on $B_2\cap\Sigma$. Then $v\in C^{1,\mathcal{B}}(\overline{B_1^+})$ and*

$$\|v\|_{1,\mathcal{B};B_1^+}\le c\Big(|v|_{0;B_2^+}+|\!|\!|\mathcal{G}|\!|\!|_{n/(1-\alpha);B_2^+}+\sum_{j=1}^n\|\mathcal{F}^j\|_{0,\mathcal{A};B_2^+}\Big),$$

where $c=c(n,\alpha,R,\mathcal{A}^{-1}(2R),\mathcal{B}(2R))$.

Theorem 1 follows from (2.7), (2.8), representation of solutions of (PE) by means of the fundamental solution and by the same argument as in [5, 4.4–4.5] (see also [5, 8.11]).

3. Dini continuity near a smooth portion of the boundary

Theorem 2 (cf. [5, Corollary 8.36]). *Let $\mathcal{A}$ be an α-Dini function* ($0<\alpha<1$) *satisfying the condition* (1.5). *Let $T\subset\partial G$ be of class $C^{1,\mathcal{A}}$. Let $u\in W^1(G)$ be a weak solution of the problem* (DL) *with $\varphi\in C^{1,\mathcal{A}}(\partial G)$. Suppose the coefficients of the equation in* (DL) *satisfy the conditions*

$$a^{ij}(x)\xi_i\xi_j \ge \nu|\xi|^2, \quad \forall x \in \overline{G},\ \xi \in \mathbb{R}^n,$$
$$a^{ij}, a^i, f^i \in C^{0,\mathcal{A}}(\overline{G}) \quad (i,j = 1,\dots,n),$$
$$b^i, c \in L_\infty(G), \quad g \in L_{n/(1-\alpha)}(G).$$

Then $u \in C^{1,\mathcal{B}}(G \cup T)$ *and for every* $G' \Subset G \cup T$,

$$\|u\|_{1,\mathcal{B};G'} \le c(n,T,\nu,k,d')\Big(|u|_{0;G} + [\![g]\!]_{n/(1-\alpha);G} + \sum_{i=1}^n \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G}\Big), \tag{3.1}$$

where $d' = \operatorname{dist}(G', \partial G \setminus T)$ *and*

$$k = \max_{i,j=1,\dots,n}\{\|a^{ij}, a^i\|_{0,\mathcal{A};G}, |b^i, c|_{0;G}\}.$$

P r o o f. We use the perturbation method. We freeze the leading coefficients $a^{ij}(x)$ at $x_0 \in G \cup T$ by setting, without loss of generality, $a^{ij}(x_0) = \delta_i^j$ (see [5, Lemma 6.1]), and rewrite the equation of (DL) in the form (PE) for the function $v(x) = u(x) - \varphi(x)$ with

$$\mathcal{G}(x) = g(x) - b^i(x)(D_i v + D_i\varphi) - c(x)(v(x) + \varphi(x)), \tag{3.2}$$
$$\mathcal{F}^i(x) = (a^{ij}(x_0) - a^{ij}(x))D_j v - a^{ij}(x)D_j\varphi - a^i(x)(v(x) + \varphi(x)) + f^i(x) \quad (i = 1,\dots,n). \tag{3.3}$$

It is not difficult to observe that the conditions on the coefficients of the equation and on T are invariant under maps of class $C^{1,\mathcal{A}}$. Therefore after a preliminary rectification of T by means of a diffeomorphism $\psi \in C^{1,\mathcal{A}}$ it is sufficient to prove the theorem in the case $T \subset \Sigma$. This is carried out using Theorem 1 in a standard way (see [5, Chs. 6, 8]). In this connection we use the following estimates for the functions (3.2), (3.3):

$$\begin{aligned} [\![\mathcal{G}]\!]_{n/(1-\alpha);B_2^+} &\le [\![g]\!]_{n/(1-\alpha);B_2^+} + k\Big(\sum_{i=1}^n |D_i v|_{0;B_2^+} + |v|_{0;B_2^+} + \sum_{i=1}^n |D_i\varphi|_{0;B_2^+} + |\varphi|_{0,B_2^+}\Big) \\ &\le [\![g]\!]_{n/(1-\alpha);B_2^+} + k\Big(\varepsilon\sum_{i=1}^n [D_i v]_{\mathcal{A};B_2^+} + c_\varepsilon|v|_{0;B_2^+} + |\varphi|_{1,B_2^+}\Big) \quad \text{(by (1.7))}, \end{aligned} \tag{3.4}$$

$$(3.5)\qquad \sum_{j=1}^{n}\|\mathcal{F}^j\|_{0,\mathcal{A};B_2^+} \le nk\mathcal{A}(2R)\|\nabla v\|_{0,\mathcal{A};B_2^+} + k\sum_{i=1}^{n}|D_iv|_{0,B_2^+} + c(k)(|v|_{0;B_2^+} + \|\varphi\|_{1,\mathcal{A};B_2^+}) + \sum_{j=1}^{n}\|f^j\|_{0,\mathcal{A};B_2^+}.$$

Taking into account once more the inequality (1.7) and the condition (1.5) that ensures the equivalence $[\ ]_{\mathcal{A}} \sim [\ ]_{\mathcal{B}}$, from (3.4)–(3.5) we finally obtain

$$(3.6)\qquad [\![\mathcal{G}]\!]_{n/(1-\alpha);B_2^+} + \sum_{j=1}^{n}\|\mathcal{F}^j\|_{0,\mathcal{A};B_2^+} \le k(\varepsilon + n\mathcal{A}(2R))\|v\|_{1,\mathcal{B};B_2^+} + c_\varepsilon(k)(|v|_{0;B_2^+} + \|\varphi\|_{1,\mathcal{A};B_2^+}) + \sum_{j=1}^{n}\|f^j\|_{0,\mathcal{A};B_2^+} + [\![g]\!]_{n/(1-\alpha);B_2^+} \quad \text{for all } \varepsilon > 0.$$

Since $\mathcal{A}$ is continuous, choosing $\varepsilon, R > 0$ sufficiently small we obtain the desired assertion and the estimate (3.1) in a standard way from (2.16) and (3.6).

4. Dini continuity near the conical point. We consider the problem (DL) under the following assumptions:

(i) ∂G is a Dini–Lyapunov surface with conical point $\mathcal{O}$;

(ii) the uniform ellipticity holds:

$$\nu\xi^2 \le a^{ij}(x)\xi_i\xi_j \le \mu\xi^2, \qquad \forall x \in G,\ \xi \in \mathbb{R}^n,$$

where $\nu, \mu = \text{const} > 0$ and $a^{ij}(0) = \delta_i^j$ $(i, j = 1, \ldots, n)$;

(iii) $a^{ij}, a^i \in C^{0,\mathcal{A}}(G)$ $(i, j = 1, \ldots, n)$ where $\mathcal{A}$ is an α-Dini function on $(0, d]$, $\alpha \in (0, 1)$, satisfying the conditions (1.5)–(1.6) and also

$$(4.1)\qquad \sup_{0<\varrho\le 1} \varrho^{\lambda-1}/\mathcal{A}(\varrho) \le \text{const},$$

$$|x|\Big(\sum (b^i(x))^2\Big)^{1/2} + |x|^2|c(x)| \le \mathcal{A}(|x|);$$

(iv) $g \in L_{n/(1-\alpha)}(G)$, $\varphi \in C^{1,\mathcal{A}}(\partial G)$, $f^j \in C^{0,\mathcal{A}}(\overline{G})$, $j = 1, \ldots, n$;

(v) $\int_G r^{4-n-2\lambda}\mathcal{H}^{-1}(r)g^2(x)\,dx < \infty$,

$$\int_G r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\Big(\sum_{j=1}^{n}|f^j|^2 + |\nabla\Phi|^2 + r^{-2}\Phi^2\Big)\,dx < \infty,$$

where $\mathcal{H}$ is a continuous increasing function satisfying the Dini condition at $t = 0$.

THEOREM 3. *Let u be a generalized solution of* (DL) *and suppose that assumptions* (i)–(v) *are satisfied. Then there exist $d > 0$ and a constant $c > 0$ independent of u and depending only on parameters and norms of the given functions appearing in assumptions* (i)–(v), *such that*

$$
\begin{aligned}
(4.2)\qquad |u(x)| \le c|x|\mathcal{A}(|x|)\Big(& |\!|\!| g |\!|\!|_{n/(1-\alpha);G} + \sum_{i=1}^{n} \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} \\
& + \Big\{ \int_G \Big(r^{4-n-2\lambda}\mathcal{H}^{-1}(r) g^2(x) + r^{2-n-2\lambda}\mathcal{H}^{-1}(r) \\
& \times \sum_{i=1}^{n} |f^i(x)|^2 + r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla \Phi|^2 \\
& + |u|^2 + |\nabla u|^2 \Big)\, dx \Big\}^{1/2} \Big), \qquad \forall x \in G_0^d,
\end{aligned}
$$

$$
\begin{aligned}
(4.3)\quad |\nabla u(x)| \le c\mathcal{A}(|x|)\Big(& |\!|\!| g |\!|\!|_{n/(1-\alpha);G} + \sum_{i=1}^{n} \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} \\
& + \Big\{ \int_G \Big(r^{4-n-2\lambda}\mathcal{H}^{-1}(r) g^2(x) + r^{2-n-2\lambda}\mathcal{H}^{-1}(r) \\
& \times \sum_{i=1}^{n} |f^i(x)|^2 + r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla \Phi|^2 \\
& + |u|^2 + |\nabla u|^2 \Big)\, dx \Big\}^{1/2} \Big), \qquad \forall x \in G_0^d.
\end{aligned}
$$

P r o o f. We use Kondrat'ev's method of layers: we move away from the conical point of $\varrho > 0$ and work in $G_{\varrho/4}^{2\varrho}$; after the change of variables $x = \varrho x'$ the layer $G_{\varrho/4}^{2\varrho}$ takes the position of a fixed domain $G_{1/4}^{2}$ with smooth boundary.

1°. We consider a solution u in the domain G_0^{2d} with some positive $d \ll 1$; then u is a weak solution in G_0^{2d} of the problem

$$
(\mathrm{DL})_{0,2d}\qquad
\begin{cases}
\dfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j} + a^i(x)u) + b^i(x)u_{x_i} + c(x)u \\[2mm]
\qquad = g(x) + \dfrac{\partial f^j}{\partial x_j}, \qquad x \in G_0^{2d}, \\[2mm]
u(x) = \varphi(x), \qquad x \in \Gamma_0^{2d} \subset \partial G_0^{2d}.
\end{cases}
$$

We make the change of variables $x = \varrho x'$ and set $v(x') = \varrho^{-1}\mathcal{A}^{-1}(\varrho)u(\varrho x')$,

$\varrho \in (0,d)$, $0 < d \ll 1$. Then v satisfies in $G^2_{1/4}$ the problem

$$\begin{cases} \frac{\partial}{\partial x'_i}(a^{ij}(\varrho x')v_{x'_j} + \varrho a^i(\varrho x')v) + \varrho b^i(\varrho x')v_{x'_i} + \varrho^2 c(\varrho x')v \\ \qquad = \mathcal{A}^{-1}(\varrho)\sum_{j=1}^{n}\frac{\partial f^j(\varrho x')}{\partial x'_j} + \varrho\mathcal{A}^{-1}(\varrho)g(\varrho x'), \quad x' \in G^2_{1/4}, \\ v(x') = \varrho^{-1}\mathcal{A}^{-1}(\varrho)\varphi(\varrho x'), \quad x' \in \Gamma^2_{1/4}. \end{cases}$$

To solve this problem we use Theorem 2. We check its assumptions. Since under assumption (ii), $\mathcal{A}$ is increasing, $\varrho \in (0,d)$ and $0 < d \ll 1$, from the inequality $\varrho^{-1}|x-y| \geq |x-y|$ for $\varrho \in (0,d)$ it follows that

$$\mathcal{A}(|x'-y'|) = \mathcal{A}(\varrho^{-1}|x-y|) \geq \mathcal{A}(|x-y|)$$

and by (iii) we have

$$\sum_{i,j}\|a^{ij}(\varrho\,\cdot)\|_{0,\mathcal{A};G^2_{1/4}} + \varrho\sum_i\|a^i(\varrho\,\cdot)\|_{0,\mathcal{A};G^2_{1/4}} \leq \sum_{i,j}\|a^{ij}\|_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} + d\sum_i\|a^i\|_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} < \infty.$$

Further, let Φ be a regularity preserving extension of the boundary function φ into a domain G^d_ε for $\varepsilon > 0$ (such an extension exists; see e.g. [5, Lemma 6.38]).

Since $\varphi \in C^{1,\mathcal{A}}(\partial G)$ we have

$$\|\Phi\|_{1,\mathcal{A};G^{2\varrho}_{\varrho/4}} \leq c(G)\|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}} \leq \text{const}.$$

By definition of the norm in $C^{1,\mathcal{A}}$ we obtain

$$\sup_{\substack{x,y\in G^{2\varrho}_{\varrho/4}\\ x\neq y}}\frac{|\nabla\Phi(x)-\nabla\Phi(y)|}{\mathcal{A}(|x-y|)} \leq \|\Phi\|_{1,\mathcal{A};G^{2\varrho}_{\varrho/4}} \leq c(G)\|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}}. \tag{4.4}$$

Now we show that by (v) and the smoothness of φ,

$$|\varphi(x)| \leq c|x|\mathcal{A}(|x|), \quad |\nabla\Phi(x)| \leq c\mathcal{A}(|x|), \quad \forall x \in G^{2\varrho}_{\varrho/4}. \tag{4.5}$$

Indeed, from

$$\varphi(x) - \varphi(0) = \int_0^1 \frac{d}{d\tau}\Phi(\tau x)\,d\tau = x_i\int_0^1\frac{\partial\Phi(\tau x)}{\partial(\tau x_i)}\,d\tau$$

by Hölder's inequality we have

$$|\varphi(x) - \varphi(0)| \leq r|\nabla\Phi|. \tag{4.6}$$

From (iv) it follows that

$$\begin{aligned}(4.7)\qquad &\int_{G_0^\varrho} (r^{2-n}|\nabla\Phi|^2 + r^{-n}|\varphi|^2)\,dx\\ &= \int_{G_0^\varrho} (r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 + r^{-n-2\lambda}\mathcal{H}^{-1}(r)|\varphi|^2)(r^{2\lambda}\mathcal{H}(r))\,dx\\ &\le \operatorname{const}\varrho^{2\lambda}\mathcal{H}(\varrho).\end{aligned}$$

Since $|\varphi(0)| \le |\varphi(x)| + |\varphi(x) - \varphi(0)|$, by (4.6) we obtain

$$|\varphi(0)| \le |\varphi(x)| + r|\nabla\Phi|.$$

Squaring both sides, multiplying by r^{-n} and integrating over G_0^ϱ we obtain

$$(4.8)\qquad \varphi^2(0)\int_{G_0^\varrho} r^{-n}\,dx \le 2\int_{G_0^\varrho} (r^{-n}\varphi^2(x) + r^{2-n}|\nabla\Phi|^2)\,dx < \infty$$

by (4.7). Since

$$\int_{G_0^\varrho} r^{-n}\,dx = \operatorname{mes}\Omega\int_0^\varrho \frac{dr}{r} = \infty,$$

the assumption $\varphi(0) \ne 0$ contradicts (4.8). Thus $\varphi(0) = 0$. Then from (4.4) we have

$$\begin{aligned}|\nabla\Phi(x) - \nabla\Phi(y)| &\le \operatorname{const}\mathcal{A}(|x-y|)\|\varphi\|_{1,\mathcal{A};\Gamma_{\varrho/4}^{2\varrho}}, \qquad \forall x,y \in G_{\varrho/4}^{2\varrho},\\ |\nabla\Phi(y)| &\le |\nabla\Phi(x) - \nabla\Phi(y)| + |\nabla\Phi(x)|\\ &\le c\mathcal{A}(|x-y|)\|\varphi\|_{1,\mathcal{A};\Gamma_{\varrho/4}^{2\varrho}} + |\nabla\Phi(x)|.\end{aligned}$$

Hence considering y to be fixed in $G_{\varrho/4}^{2\varrho}$ and x variable, we get

$$\begin{aligned}|\nabla\Phi(y)|^2 \int_{G_{\varrho/4}^{2\varrho}} r^{2-n}\,dx &\le 2c^2\|\varphi\|_{1,\mathcal{A};\Gamma_{\varrho/4}^{2\varrho}} \int_{G_{\varrho/4}^{2\varrho}} r^{2-n}\mathcal{A}^2(|x-y|)\,dx\\ &\quad + 2\int_{G_{\varrho/4}^{2\varrho}} r^{2-n}|\nabla\Phi(x)|^2\,dx\end{aligned}$$

or by (4.7),

$$\varrho^2|\nabla\Phi(y)|^2 \le c(\operatorname{mes}\Omega, k_1)(\varrho^2\mathcal{A}^2(\varrho) + \varrho^{2\lambda}\mathcal{H}(\varrho)), \qquad \forall y \in G_{\varrho/4}^{2\varrho}.$$

Hence the assumption (4.1) yields the second inequality of (4.5). Now the first inequality of (4.5) follows from (4.6) and $\varphi(0) = 0$. Thus (4.5) is proved.

Now we obtain

$$(4.9)\quad \begin{aligned}&\varrho^{-1}\mathcal{A}^{-1}(\varrho)\|\varphi(\varrho\,\cdot)\|_{1,\mathcal{A};\Gamma^2_{1/4}}\\
&\le c\varrho^{-1}\mathcal{A}^{-1}(\varrho)\|\Phi(\varrho\,\cdot)\|_{1,\mathcal{A};G^2_{1/4}}\\
&= c\varrho^{-1}\mathcal{A}^{-1}(\varrho)\Big\{\sup_{x'\in G^2_{1/4}}|\Phi(\varrho x')| + \sup_{x'\in G^2_{1/4}}|\nabla'\Phi(\varrho x')|\\
&\quad + \sup_{\substack{x',y'\in G^2_{1/4}\\ x'\neq y'}}\frac{|\nabla'\Phi(\varrho x')-\nabla'\Phi(\varrho y')|}{\mathcal{A}(|x'-y'|)}\Big\}\\
&\le c_1 + c\mathcal{A}^{-1}(\varrho)\sup_{x,y\in G^{2\varrho}_{\varrho/4}}\frac{|\nabla\Phi(x)-\nabla\Phi(y)|}{\mathcal{A}(\varrho^{-1}|x-y|)}\\
&= c_1 + c[\nabla\Phi]_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}}\mathcal{A}^{-1}(\varrho)\sup_{0<t<4\varrho}\frac{\mathcal{A}(t)}{\mathcal{A}(\varrho^{-1}t)}\\
&\le \text{ const},\qquad \forall\varrho\in(0,d),\end{aligned}$$

by (4.5), since by (1.6),

$$\sup_{0<t<4\varrho}\frac{\mathcal{A}(t)}{\mathcal{A}(\varrho^{-1}t)} = \sup_{0<\tau<4}\frac{\mathcal{A}(\tau\varrho)}{\mathcal{A}(\tau)} \le c\mathcal{A}(\varrho).$$

In the same way we have

$$(4.10)\quad \begin{aligned}&\mathcal{A}^{-1}(\varrho)\|f^j\|_{0,\mathcal{A};G^2_{1/4}}\\
&= \mathcal{A}^{-1}(\varrho)\Big(|f^j|_{0;G^{2\varrho}_{\varrho/4}} + \sup_{\substack{x,y\in G^{2\varrho}_{\varrho/4}\\ x\neq y}}\frac{|f^j(x)-f^j(y)|}{\mathcal{A}(\varrho^{-1}|x-y|)}\Big).\end{aligned}$$

Since $f^j\in C^{0,\mathcal{A}}(\overline{G})$, we get

$$(4.11)\quad |f^j(x)-f^j(y)| \le \widetilde{c}_j\mathcal{A}(|x-y|),\qquad \forall x,y\in G^{2\varrho}_{\varrho/4},$$

$$(4.12)\quad \begin{aligned}\int_{G^\varrho_0} r^{2-n}|f^j(x)|^2\,dx &= \int_{G^\varrho_0}(r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|f^j(x)|^2)(\mathcal{H}(r)r^{2\lambda})\,dx\\
&\le \text{ const}\,\varrho^{2\lambda}\mathcal{H}(\varrho)\end{aligned}$$

by (v). Now fix y in $G^{2\varrho}_{\varrho/4}$. Then

$$|f^j(y)| \le |f^j(x)| + |f^j(x)-f^j(y)| \le |f^j(x)| + \widetilde{c}_j\mathcal{A}(|x-y|).$$

Hence

$$|f^j(y)|^2\int_{G^{2\varrho}_{\varrho/4}} r^{2-n}\,dx \le 2\int_{G^{2\varrho}_{\varrho/4}} r^{2-n}|f^j(x)|^2\,dx + 2\widetilde{c}_j^{\,2}\int_{G^{2\varrho}_{\varrho/4}} r^{2-n}\mathcal{A}^2(|x-y|)\,dx.$$

Calculations and (4.12) give

$$\varrho^2|f^j(y)|^2 \le c(\widetilde{c}_j, k_1, \operatorname{mes}\Omega)(\varrho^2\mathcal{A}^2(\varrho) + \varrho^{2\lambda}\mathcal{H}(\varrho)), \qquad \forall y \in G^{2\varrho}_{\varrho/4}.$$

Hence by the assumption (4.1) it follows that

$$|f^j(x)| \le c_j\mathcal{A}(\varrho), \qquad \forall x \in G^{2\varrho}_{\varrho/4},\ j = 1, \ldots, n. \tag{4.13}$$

Further, in the same way as in the proof of (4.9),

$$\begin{aligned} \sup_{\substack{x,y\in G^{2\varrho}_{\varrho/4}\\ x\neq y}} \frac{|f^j(x) - f^j(y)|}{\mathcal{A}(\varrho^{-1}|x-y|)} &\le [f^j]_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} \sup_{0<t<4\varrho} \frac{\mathcal{A}(t)}{\mathcal{A}(\varrho^{-1}t)} \\ &\le c\mathcal{A}(\varrho)[f^j]_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}}. \end{aligned} \tag{4.14}$$

Now from (4.10), (4.13) and (4.14) we obtain

$$\mathcal{A}^{-1}(\varrho)\sum_{j=1}^n |f^j|_{0,\mathcal{A};G^2_{1/4}} \le \text{const.} \tag{4.15}$$

It remains to verify the finiteness of $\boldsymbol{|}\varrho\mathcal{A}(\varrho)^{-1}g(\varrho x')\boldsymbol{|}_{n/(1-\alpha);G^2_{1/4}}$. We have

$$\begin{aligned} &\varrho\mathcal{A}^{-1}(\varrho)\Big(\int\limits_{G^2_{1/4}} |g(\varrho x')|^{n/(1-\alpha)}\,dx'\Big)^{(1-\alpha)/n} \\ &\qquad = \varrho^\alpha\mathcal{A}^{-1}(\varrho)\Big(\int\limits_{G^{2\varrho}_{\varrho/4}} |g(x)|^{n/(1-\alpha)}\,dx\Big)^{(1-\alpha)/n} \\ &\qquad \le d^\alpha\mathcal{A}^{-1}(d)\Big(\int\limits_{G^{2\varrho}_{\varrho/4}} |g(x)|^{n/(1-\alpha)}\,dx\Big)^{(1-\alpha)/n} \\ &\qquad \le \text{const}, \qquad \forall\varrho \in (0,d), \end{aligned}$$

by the condition (1.1). Thus the conditions of Theorem 2 are satisfied.

By this theorem we have

$$\begin{aligned} &\|v\|_{1,\mathcal{B};G^1_{1/2}} \\ &\le c\{n,\nu,G,\max_{i,j=1,\ldots,n}(\|a^{ij}(\varrho\,\cdot)\|_{0,\mathcal{A};G^2_{1/4}}, \varrho\|a^i(\varrho\,\cdot)\|_{0,\mathcal{A};G^2_{1/4}}), \mathcal{A}(2\varrho)\} \\ &\quad\times\Big(|v|_{0;G^2_{1/4}} + \varrho^{-1}\mathcal{A}^{-1}(\varrho)\|\varphi(\varrho\,\cdot)\|_{1,\mathcal{A};\Gamma^2_{1/4}} + \varrho\mathcal{A}^{-1}(\varrho)\boldsymbol{|}g(\varrho\,\cdot)\boldsymbol{|}_{n/(1-\alpha);G^2_{1/4}} \\ &\quad+ \mathcal{A}^{-1}(\varrho)\sum_{j=1}^n \|f^j(\varrho\,\cdot)\|_{0,\mathcal{A};G^2_{1/4}}\Big), \qquad \forall\varrho\in(0,d). \end{aligned} \tag{4.16}$$

2°. To estimate $|v|_{0;G^2_{1/4}}$ we use the local estimate at the boundary of the maximum of the modulus of a solution [5, Theorem 8.25]. We check the assumptions of that theorem. To this end, set

$$z(x') = v(x') - \varrho^{-1}\mathcal{A}^{-1}(\varrho)\Phi(\varrho x')$$

and write the problem for the function z:

$$\begin{cases} \dfrac{\partial}{\partial x'_i}(a^{ij}(\varrho x')z_{x'_j} + \varrho a^i(\varrho x')z) + \varrho b^i(\varrho x')z_{x'_i} + \varrho^2 c(\varrho x')z \\ \qquad = G(x') + \dfrac{\partial F^j(x')}{\partial x'_j}, \quad x' \in G^2_{1/4}, \\ z(x') = 0, \quad x' \in \Gamma^2_{1/4}, \end{cases}$$

where

$$\begin{aligned} G(x') &\equiv \varrho\mathcal{A}^{-1}(\varrho)g(\varrho x') - \mathcal{A}^{-1}(\varrho)b^i(\varrho x')\Phi_{x'_i}(\varrho x') \\ &\quad - \varrho\mathcal{A}^{-1}(\varrho)c(\varrho x')\Phi(\varrho x'), \end{aligned} \tag{4.17}$$

$$\begin{aligned} F^i(x') &\equiv \mathcal{A}^{-1}(\varrho)f^i(\varrho x') - \varrho^{-1}\mathcal{A}^{-1}(\varrho)a^{ij}(\varrho x')\Phi_{x'_j}(\varrho x') \\ &\quad - \mathcal{A}^{-1}(\varrho)a^i(\varrho x')\Phi(\varrho x') \quad (i = 1, \ldots, n). \end{aligned} \tag{4.18}$$

First we verify the smoothness of the coefficients (see the remark at the end of [5, 8.10]). Let $q > n$. We have

$$\begin{aligned} \int\limits_{G^2_{1/4}} |\varrho a^i(\varrho x')|^q\,dx' &= \varrho^{q-n} \int\limits_{G^{2\varrho}_{\varrho/4}} |a^i(x)|^q\,dx \\ &\le c_2(G)d^q\|a^i\|^q_{0,\mathcal{A};G}, \quad \forall \varrho \in (0,d). \end{aligned} \tag{4.19}$$

By (iii) we also obtain

$$\begin{aligned} \int\limits_{G^2_{1/4}} |\varrho b^i(\varrho x')|^q\,dx' &= \varrho^{q-n} \int\limits_{G^{2\varrho}_{\varrho/4}} |b^i(x)|^q\,dx \le 4^q\varrho^{-n} \int\limits_{G^{2\varrho}_{\varrho/4}} |rb^i(x)|^q\,dx \\ &\le 4^q\varrho^{-n} \int\limits_{G^{2\varrho}_{\varrho/4}} \mathcal{A}^q(r)\,dx \le 2^{n+2q} \int\limits_{G^{2\varrho}_{\varrho/4}} r^{-n}\mathcal{A}^q(r)\,dx \\ &= 2^{n+2q}\operatorname{mes}\Omega \int\limits_{\varrho/4}^{2\varrho} \frac{\mathcal{A}^q(r)}{r}\,dr \\ &\le 2^{n+2q}\operatorname{mes}\Omega\cdot\mathcal{A}^{q-1}(2d) \int\limits_0^{2d} \frac{\mathcal{A}(r)}{r}\,dr, \end{aligned} \tag{4.20}$$

$$\int\limits_{G^2_{1/4}} |\varrho^2 c(\varrho x')|^{q/2}\,dx' = \varrho^{q-n} \int\limits_{G^{2\varrho}_{\varrho/4}} |c(x)|^{q/2}\,dx \tag{4.21}$$

$$\le 4^q \varrho^{-n} \int\limits_{G^{2\varrho}_{\varrho/4}} |r^2 c(x)|^{q/2}\,dx$$

$$\le 2^{2q+n} \int\limits_{G^{2\varrho}_{\varrho/4}} r^{-n}\mathcal{A}^{q/2}(r)\,dx$$

$$\le 2^{2q+n} \operatorname{mes}\Omega \cdot \mathcal{A}^{(q-2)/2}(2d) \int\limits_0^{2d} \frac{\mathcal{A}(r)}{r}\,dr,$$

for $q > n$ and all $\varrho \in (0, d)$.

In the same way from (4.17) we get

$$\begin{aligned}(4.22)\quad & \varrho\mathcal{A}^{-1}(\varrho)\|G(x')\|_{q/2;G^2_{1/4}} \\ &= \varrho\mathcal{A}^{-1}(\varrho)\Big(\int\limits_{G^{2\varrho}_{\varrho/4}} \varrho^{-n}\Big\{|g(x)|^{q/2} \\ &\quad + \Big(\sum_{i=1}^n |b^i(x)|\Big)^{q/2}|\nabla\Phi|^{q/2} + |c(x)|^{q/2}|\Phi(x)|^{q/2}\Big\}\,dx\Big)^{2/q}.\end{aligned}$$

By (iv) setting $q = n/(1-\alpha) > n$ and applying Hölder's inequality we obtain

$$\begin{aligned}(4.23)\quad & \varrho\mathcal{A}^{-1}(\varrho)\Big(\int\limits_{G^{2\varrho}_{\varrho/4}} \varrho^{-n}|g(x)|^{q/2}\,dx\Big)^{2/q} \\ &\le c\varrho^{\alpha}\mathcal{A}^{-1}(\varrho)\Big(\int\limits_{G^{2\varrho}_{\varrho/4}} \varrho^{-n/2}|g(x)|^{q/2}\,dx\Big)^{2/q} \\ &\le c\varrho^{\alpha}\mathcal{A}^{-1}(\varrho)\|g\|_{q;G^{2\varrho}_{\varrho/4}}(\operatorname{mes}\Omega \ln 8)^{1/q} \\ &\le c(d,\alpha,q,\operatorname{mes}\Omega,\mathcal{A}(d))\|g\|_{q;G^{2\varrho}_{\varrho/4}},\end{aligned}$$

since by (1.1), $\varrho^{\alpha}\mathcal{A}^{-1}(\varrho) \le d^{\alpha}\mathcal{A}^{-1}(d)$ for all $\varrho \in (0, d)$. Similarly,

$$\begin{aligned}(4.24)\quad & \varrho\mathcal{A}^{-1}(\varrho)\Big(\int\limits_{G^{2\varrho}_{\varrho/4}} r^{-n} \\ &\quad \times \Big\{\Big(\sum_{i=1}^n |b^i(x)|\Big)^{q/2}|\nabla\Phi|^{q/2} + |c(x)|^{q/2}|\Phi(x)|^{q/2}\Big\}\,dx\Big)^{2/q} \\ &\le c(\operatorname{mes}\Omega)^{2/q}\|\varphi\|_{1,\mathcal{A};G^{2\varrho}_{\varrho/4}}\mathcal{A}^{(q-2)/q}(\varrho)\int\limits_{\varrho/4}^{2\varrho}\frac{\mathcal{A}(r)}{r}\,dr.\end{aligned}$$

From (4.22)–(4.24) we obtain

$$(4.25)\quad |G(\varrho\,\cdot)|_{q/2;G^2_{1/4}} \le \mathrm{const}\Big(q,\alpha,d,\operatorname{mes}\Omega,\mathcal{A}(d),\int\limits_{\varrho/4}^{2\varrho}\frac{\mathcal{A}(r)}{r}\,dr\Big) \times (|g|_{q;G^{2\varrho}_{\varrho/4}} + \|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}}),\qquad q=n/(1-\alpha)>n.$$

Finally, in the same way from (4.18) we have

$$(4.26)\quad \sum_{i=1}^{n}\int\limits_{G^2_{1/4}}|F^i(x')|^q\,dx' \le c\Big(q,G,\max_{j=1,\dots,n}\Big\{\sum_{i=1}^{n}\|a^{ij}\|^q_{0,\mathcal{A};G},\sum_{i=1}^{n}\|a^i\|^q_{0,\mathcal{A};G}\Big\}\Big) \times \int\limits_{G^{2\varrho}_{\varrho/4}} r^{-n}\mathcal{A}^{-q}(r)\Big(\sum_{i=1}^{n}|f^i(x)|^q+|\nabla\Phi|^q+|\Phi|^q\Big)\,dx.$$

It follows from (4.5) as $\varrho\to+0$ that $|\nabla\Phi(0)|=0$. Therefore

$$|\nabla\Phi(x)| = |\nabla\Phi(x)-\nabla\Phi(0)| \le \mathcal{A}(|x|)\|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}},\qquad \forall x\in G^{2\varrho}_{\varrho/4},$$

and hence

$$|\Phi(x)| \le r|\nabla\Phi| \le |x|\mathcal{A}(|x|)\|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}},\qquad \forall x\in G^{2\varrho}_{\varrho/4}.$$

Similarly it follows from (4.13) as $\varrho\to+0$ that $f^j(0)=0$ for $j=1,\dots,n$. Therefore we have for all $x\in G^{2\varrho}_{\varrho/4}$,

$$|f^j(x)| = |f^j(x)-f^j(0)| \le \mathcal{A}(r)[f^j]_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}}.$$

Consequently, estimating the right side of (4.26) and taking into account the inequalities obtained, we have

$$(4.27)\quad \sum_{i=1}^{n}|F^i|_{q;G^2_{1/4}} \le c\Big(q,G,\max_{j=1,\dots,n}\Big\{\sum_{i=1}^{n}\|a^{ij}\|_{0,\mathcal{A};G},\sum_{i=1}^{n}\|a^i\|_{0,\mathcal{A};G}\Big\}\Big) \times \operatorname{mes}\Omega\cdot\Big(\sum_{i=1}^{n}\|f^i\|_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}}+\|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}}\Big).$$

So all conditions of [5, Theorem 8.25] are satisfied. By this theorem we get

$$
(4.28)\qquad \sup_{x'\in G^1_{1/2}} |z(x')|
\le c\Big(\|z\|_{2;G^2_{1/4}} + \|G\|_{n/(2(1-\alpha));G^2_{1/4}} + \sum_{i=1}^n \|F^i\|_{n/(1-\alpha);G^2_{1/4}}\Big)
\le c\Big(\|z\|_{2;G^2_{1/4}} + \|g\|_{n/(1-\alpha);G^{2\varrho}_{\varrho/4}} + \sum_{i=1}^n \|f^i\|_{0,\mathcal{A};G^{2\varrho}_{\varrho/4}} + \|\varphi\|_{1,\mathcal{A};\Gamma^{2\varrho}_{\varrho/4}}\Big).
$$

Setting $w(x) = u(x) - \varphi(x)$ we have for $w(x)$ the problem

$$
(\mathrm{DL})_{0,2d}\qquad \begin{cases} \dfrac{\partial}{\partial x_i}(a^{ij}(x)w_{x_j} + a^i(x)w) + b^i(x)w_{x_i} + c(x)w \\ \qquad = G(x) + \dfrac{\partial F^j}{\partial x_j}, \quad x\in G^{2d}_0, \\ w(x) = 0, \quad x\in \varGamma^{2d}_0 \subset \partial G^{2d}_0, \end{cases}
$$

where

$$
\begin{aligned} G(x) &= g(x) - b^i(x)\varPhi_{x_i} - c(x)\varPhi(x),\\ F^i(x) &= f^i(x) - a^{ij}(x)\varPhi_{x_j} - a^i(x)\varPhi(x). \end{aligned}
$$

Moreover, by assumptions (i), (ii),

$$
|a^{ij}(x) - \delta^j_i| \le \|a^{ij}\|_{0,\mathcal{A};G}\mathcal{A}(|x|), \qquad x\in G.
$$

By [6, Theorem 1] there is a constant $c>0$ independent of w, G, F^i such that

$$
(4.29)\qquad \int_{G^\varrho_0} r^{2-n}|\nabla w|^2\,dx \le c\varrho^{2\lambda}\int_{G^{2d}_0}\Big\{|w(x)|^2 + |\nabla w|^2 + G^2(x) + \sum_{i=1}^n |F^i(x)|^2 + r^{4-n-2\lambda}\mathcal{H}^{-1}(r)G^2(x) + r^{2-n-2\lambda} \times \mathcal{H}^{-1}(r)\sum_{i=1}^n |F^i(x)|^2\Big\}\,dx, \qquad \forall\varrho\in(0,d).
$$

Our assumptions guarantee that the integral on the right side is finite. Since $z(x') = \varrho^{-1}\mathcal{A}^{-1}(\varrho)w(\varrho x')$ we obtain from (4.29),

$$
(4.30)\qquad \int_{G^2_{1/4}} |\nabla' z|^2\,dx' \le 2^{n-2}\varrho^{-2}\mathcal{A}^{-2}(\varrho)\int_{G^{2\varrho}_{\varrho/4}} r^{2-n}|\nabla w|^2\,dx
\le c\varrho^{2\lambda-2}\mathcal{A}^{-2}(\varrho)\int_G \Big\{|w|^2 + |\nabla w|^2 + G^2(x)
$$

$$+\sum_{i=1}^{n}|F^i(x)|^2+r^{4-n-2\lambda}\mathcal{H}^{-1}(r)G^2(x)$$
$$+r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{n}|F^i(x)|^2\Big\}\,dx.$$

By assumptions (i)–(iv) we have

$$|G(x)|^2\le c\{|g|^2+\mathcal{A}^2(r)(r^{-2}|\nabla\Phi|^2+r^{-4}\Phi^2)\},$$
$$\sum_{i=1}^{n}|F^i(x)|^2\le c\Big\{\sum_{i=1}^{n}|f^i(x)|^2+\max_{i,j=1,\ldots,n}(\|a^{ij}\|_{0,\mathcal{A};G},\|a^i\|_{0,\mathcal{A};G})(|\nabla\Phi|^2+\Phi^2)\Big\}. \tag{4.31}$$

Applying now the Friedrichs inequality and taking into account (4.1), we obtain from (4.30), (4.31),

$$\begin{aligned}
\|z\|^2_{2;G^2_{1/4}} &\le c_0\|\nabla' z\|^2_{2;G^2_{1/4}}\\
&\le c\varrho^{2\lambda-2}\mathcal{A}^{-2}(\varrho)\int_G\Big\{|w|^2+|\nabla w|^2+g^2(x)\\
&\quad+\sum_{i=1}^{n}|f^i(x)|^2+|\nabla\Phi|^2+\Phi^2+r^{4-n-2\lambda}\mathcal{H}^{-1}(r)g^2(x)\\
&\quad+r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{n}|f^i(x)|^2\\
&\quad+r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2+r^{-2}\mathcal{A}^2(r)|\nabla\Phi|^2\Big\}\,dx\\
&\le \text{const}\Big\{\|g\|^2_{n/(1-\alpha);G}+\sum_{i=1}^{n}\|f^i\|^2_{0,\mathcal{A};G}+\|\varphi\|^2_{1,\mathcal{A};G}\\
&\quad+\int_G\Big(|w|^2+|\nabla w|^2+r^{4-n-2\lambda}\mathcal{H}^{-1}(r)g^2(x)\\
&\quad+r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{n}|f^i(x)|^2\\
&\quad+r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2\Big)\,dx\Big\}
\end{aligned} \tag{4.32}$$

by assumptions (iii)–(v). By the definition of $z(x')$, inequalities (4.28), (4.32) and assumptions (i)–(v) we finally obtain

$$|v|_{0;G^2_{1/4}}\le|z|_{0;G^2_{1/4}}+\varrho^{-1}\mathcal{A}^{-1}(\varrho)|\varphi(\varrho\,\cdot)|_{0;G^2_{1/4}} \tag{4.33}$$

$$\le c\Big(|\!|\!| g |\!|\!|_{n/(1-\alpha);G} + \sum_{i=1}^{n} \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} + \Big\{ \int_G \Big(|w|^2 + |\nabla w|^2 + r^{4-n-2\lambda}\mathcal{H}^{-1}(r) g^2(x) + r^{2-n-2\lambda}\mathcal{H}^{-1}(r) \sum_{i=1}^{n} |f^i(x)|^2 + r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 \Big)\,dx \Big\}^{1/2} \Big).$$

3°. Returning to the variables $x, u(x)$, we now obtain from inequalities (4.16), (4.33),

$$\varrho^{-1}\mathcal{A}^{-1}(\varrho) \sup_{x\in G^{\varrho}_{\varrho/2}} |u(x)| + \mathcal{A}^{-1}(\varrho) \sup_{x\in G^{\varrho}_{\varrho/2}} |\nabla u(x)| + \sup_{\substack{x,y\in G^{\varrho}_{\varrho/2}\\ x\neq y}} \frac{|\nabla u(x) - \nabla u(y)|}{\mathcal{A}(\varrho)\mathcal{B}(|x-y|)}$$
$$\le c\Big(|\!|\!| g |\!|\!|_{n/(1-\alpha);G} + \sum_{i=1}^{n} \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G} + \Big\{ \int_G \Big(|u|^2 + |\nabla u|^2 + r^{4-n-2\lambda}\mathcal{H}^{-1}(r) g^2(x) + r^{2-n-2\lambda}\mathcal{H}^{-1}(r) \sum_{i=1}^{n} |f^i(x)|^2 + r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2 \Big)\,dx \Big\}^{1/2} \Big). \tag{4.34}$$

Setting $|x| = 2\varrho/3$ we deduce from (4.34) the validity of (4.2), (4.3). This completes the proof of Theorem 3.

Remark. As an example of $\mathcal{A}$ that satisfies all the conditions of Theorem 3, besides the function r^{α}, one may take $\mathcal{A}(r) = r^{\alpha}\ln(1/r)$, provided $\lambda \ge 1+\alpha$. In the case of $\mathcal{A}(r) = r^{\alpha}$ the result of [1] follows from Theorem 3 for a single equation and the estimate (4.2) coincides with [6, (10)].

5. Global regularity and solvability

Theorem 4. *Let $\mathcal{A}$ be an α-Dini function* ($0 < \alpha < 1$) *that satisfies the conditions* (1.5), (1.6), (4.1). *Let $\overline{G} \setminus \{\mathcal{O}\}$ be a domain of class $C^{1,\mathcal{A}}$, and*

$\mathcal{O} \in \partial G$ be a conical point of G. Suppose that assumptions (i)–(iv) *are valid and*

$$\text{(vi)} \qquad \int\limits_G (c(x)\eta - a^i(x)D_i\eta)\,dx \le 0, \qquad \forall \eta \ge 0,\ \eta \in C_0^1(G).$$

Then the generalized problem (DL) *has a unique solution $u \in C^{1,\mathcal{A}}(\overline{G})$ and we have the estimate*

$$\begin{aligned}(5.1)\qquad \|u\|_{1,\mathcal{A};G} \le c\Big(&\| g \|_{n/(1-\alpha);G} + \sum_{i=1}^n \|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G}\\ &+ \Big\{ \int\limits_G \Big(r^{4-n-2\lambda}\mathcal{H}^{-1}(r) g^2(x)\\ &+ r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^n |f^i(x)|^2\\ &+ r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2\Big)\,dx\Big\}^{1/2}\Big).\end{aligned}$$

P r o o f. The inequality (4.34) implies that

$$\begin{aligned}(5.2)\quad &|\nabla u(x) - \nabla u(y)|\\ &\le c\mathcal{B}(|x-y|)\Big(\| g \|_{n/(1-\alpha);G} + \sum_{i=1}^n \|f^i\|_{0,\mathcal{A};G}\\ &\quad + \|\varphi\|_{1,\mathcal{A};\partial G} + \Big\{\int\limits_G \Big(|u|^2 + |\nabla u|^2 + r^{4-n-2\lambda}\mathcal{H}^{-1}(r) g^2(x)\\ &\quad + r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^n |f^i(x)|^2 + r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2\Big)\,dx\Big\}^{1/2}\Big)\end{aligned}$$

for all $x, y \in G^{\varrho}_{\varrho/2}$ and all $\varrho \in (0, d)$.

From (4.34), (5.2) we now infer that $u \in C^{1,\mathcal{B}}(\overline{G_0^d})$. Indeed, let $x, y \in \overline{G_0^d}$ and $\varrho \in (0, d)$. If $x, y \in G^{\varrho}_{\varrho/2}$ then (5.2) holds. If $|x-y| > |\varrho| = |x|$ then by (4.34) we obtain

$$\begin{aligned}&\frac{|\nabla u(x) - \nabla u(y)|}{\mathcal{B}(|x-y|)}\\ &\quad \le 2c\mathcal{A}(|x|)\mathcal{B}^{-1}(|x|)\Big(\| g \|_{n/(1-\alpha);G} + \|\varphi\|_{1,\mathcal{A};\partial G}\\ &\qquad + \sum_{i=1}^n \|f^i\|_{0,\mathcal{A};G} + \Big\{\int\limits_G \Big(|u|^2 + |\nabla u|^2 + r^{4-n-2\lambda}\mathcal{H}^{-1}(r) g^2(x)\end{aligned}$$

$$+ r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{n}|f^i(x)|^2 + r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2\Big)\,dx\Big\}^{1/2}\Big)$$

$$\le 2c\alpha\Big(|\!|\!|g|\!|\!|_{n/(1-\alpha);G} + \|\varphi\|_{1,\mathcal{A};\partial G} + \sum_{i=1}^{n}\|f^i\|_{0,\mathcal{A};G}$$

$$+ \Big\{\int_G \Big(|u|^2 + |\nabla u|^2 + r^{4-n-2\lambda}\mathcal{H}^{-1}(r)g^2(x)$$

$$+ r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{n}|f^i(x)|^2 + r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2\Big)\,dx\Big\}^{1/2}\Big)$$

in view of (1.3). Because of the condition (1.5) for the equivalence of $\mathcal{A}$ and $\mathcal{B}$, we derive $u \in C^{1,\mathcal{A}}(\overline{G_0^d})$ and the estimate

$$\|u\|_{1,\mathcal{A};G_0^d} \le c\Big(|\!|\!|g|\!|\!|_{n/(1-\alpha);G} + \sum_{i=1}^{n}\|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G}$$
$$+ \Big\{\int_G \Big(|u|^2 + |\nabla u|^2 + r^{4-n-2\lambda}\mathcal{H}^{-1}(r)g^2(x)$$
$$+ r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{n}|f^i(x)|^2$$
$$+ r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2\Big)\,dx\Big\}^{1/2}\Big), \tag{5.3}$$

following from the above arguments.

By means of a partition of unity, from the bounds (3.1) of Theorem 2 and (5.3) we derive

$$\|u\|_{1,\mathcal{A};G} \le c\Big(|\!|\!|g|\!|\!|_{n/(1-\alpha);G} + \sum_{i=1}^{n}\|f^i\|_{0,\mathcal{A};G} + \|\varphi\|_{1,\mathcal{A};\partial G}$$
$$+ |u|_{0;G} + \Big\{\int_G \Big(|u|^2 + |\nabla u|^2 + r^{4-n-2\lambda}\mathcal{H}^{-1}(r)g^2(x)$$
$$+ r^{2-n-2\lambda}\mathcal{H}^{-1}(r)\sum_{i=1}^{n}|f^i(x)|^2$$
$$+ r^{2-n-2\lambda}\mathcal{H}^{-1}(r)|\nabla\Phi|^2\Big)\,dx\Big\}^{1/2}\Big). \tag{5.4}$$

By the assumption (vi) that guarantees the uniqueness of the solution for the problem (DL), we have the bound [5, Corollary 8.7]

$$\int_G (|u|^2 + |\nabla u|^2)\,dx \le C\int_G \Big(g^2 + \sum_{i=1}^{n}|f^i|^2 + |\nabla\Phi|^2 + \Phi^2\Big)\,dx,$$

which together with the global boundedness of weak solutions [5, Theorem 8.16], and the bound (5.4), leads to the desired estimate (5.1).

Finally, the global estimate (5.1) leads to the assertion on the unique solvability in $C^{1,\mathcal{A}}(\overline{G})$. This is proved by an approximation argument using the relevant propositions from [8] in the same way as in [5, Theorem 8.34].

Remark. The conclusion of Theorem 4 is best possible. This is shown for $\mathcal{A}(r) = r^{\alpha},\ \lambda \geq 1+\alpha,\ \alpha \in (0,1)$, in [6] (see also examples in [2]).

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