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On extensions of the Mittag-Leffler theorem

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Abstract. The classical Mittag-Leffler theorem on meromorphic functions is extended to the case of functions and hyperfunctions belonging to the kernels of linear partial differential operators with constant coefficients.

1. Introduction and preliminaries. The aim of the present paper is to extend the classical Mittag-Leffler theorem on meromorphic functions to the case of functions belonging to the kernels of linear partial differential operators with constant coefficients.

We use the following convention:

(∗) The expression "differential operator" stands for "linear partial differential operator with constant coefficients in $\mathbb{R}^n$".

We now formulate the following condition on sequences. Let $\varrho = (\varrho_1, \ldots, \varrho_n) \in \mathbb{R}^n$, $\varrho_i \geq 1$, $i = 1, \ldots, n$. Let $I = \mathbb{N}^n$ be the set of multiindices. A sequence $\{c_\alpha\}_{\alpha \in I}$ satisfies *condition* A_ϱ iff for every $\varepsilon > 0$ there exists a constant C_ε such that

$$|c_\alpha| \leq C_\varepsilon \frac{\varepsilon^{|\alpha|}}{\alpha_1^{\varrho_1 \alpha_1} \ldots \alpha_n^{\varrho_n \alpha_n}}, \quad \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in I.$$

If $\varrho = (1, \ldots, 1)$ then we say that the sequence $\{c_\alpha\}_{\alpha \in I}$ satisfies *condition* A. By Stirling's formula, condition A is equivalent to the following: for every ε there exists a constant C'_ε such that

$$|c_\alpha| \leq C'_\varepsilon \frac{\varepsilon^{|\alpha|}}{\alpha_1! \ldots \alpha_n!}, \quad \forall \alpha = (\alpha_1, \ldots, \alpha_n) \in I.$$

Let us make one more convention.

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(∗∗) Ω will always denote an open set in $\mathbb{R}^n$, and $\{x_k\}_{k\in\mathbb{N}}$ a discrete sequence of distinct points of Ω.

We always denote by $B(\Omega)$ the space of hyperfunctions on Ω (see [9], [7]) and by $D'(\Omega)$ the space of distributions on Ω. Our work was inspired by the following theorem on hyperfunctions proved by Chung, Kim and Lee ([4], Theorem 2.6).

Theorem 0. *Let $P(D)$ be a differential operator* (∗), *U be the unit ball in $\mathbb{R}^n$ and u be a hyperfunction on U. Assume that $P(D)u = 0$ on $U\backslash\{0\}$. Then there exists a solution $v \in B(U)$ of the equation $P(D)v = 0$ and a sequence $\{c_\alpha\}_{\alpha\in I}$ satisfying condition* A *such that*

$$u = v + \sum_{\alpha\in I} c_\alpha \partial^\alpha E,$$

where E is a fundamental solution of $P(D)$.

For elliptic differential operators similar theorems were proved earlier by Wachman [13] and Harvey and Polking [6]. These results were extended by Tarkhanov (see [10]–[12]) to the case of elliptic systems.

Nowadays, one usually thinks of the Mittag-Leffler theorem as of a theorem from Čech cohomology theory. This approach permits finding an analogue of it for elliptic complexes (see, for example, [12], 4.2.7).

However, in its original form the Mittag-Leffler theorem concerned the construction of a holomorphic function with a priori given singularities at a priori given isolated points. In the present note, we want to generalize this "constructive side" of the Mittag-Leffler theorem.

We prove a general Mittag-Leffler theorem for hyperfunctions and use it in the case of elliptic operators. We also consider the case of hypoelliptic operators using the results of [7].

2. Mittag-Leffler theorem for hyperfunctions

2.1. Theorem. *Let Ω be an open set in $\mathbb{R}^n$ and let $P(D)$ be a differential operator* (∗). *Assume that $\{\Omega_j\}_{j\in S}$ is an open covering of Ω and hyperfunctions $u_j \in B(\Omega_j)$ are such that $P(D)(u_i - u_j)|_{\Omega_i\cap\Omega_j} = 0$. Then there exists a hyperfunction $u \in B(\Omega)$ such that for every $j \in S$,*

$$P(D)(u - u_j)|_{\Omega_j} = 0.$$

Proof. By the very definition of hyperfunction (see [9]) there exists a hyperfunction v on Ω such that $v|_{\Omega_j} = P(D)u_j$. By Corollary 1 in §1 of Chapter III of [9] there exists $u \in B(U)$ such that $Pu = v$. Hence

$$P(u - u_j) = v - Pu_j = 0 \quad \text{on } \Omega_j.$$

2.2. Corollary. *Let $P(D)$ be as above and let Ω and $\{x_k\}_{k\in\mathbb{N}}$ be as in* (∗∗). *For each $k \in \mathbb{N}$ let V_k be an open neighbourhood of x_k. Assume that*

$V_i \cap V_j = \emptyset$ if $i \neq j$. For every sequence $u_j \in B(V_j)$ with $Pu_j|_{V_j \setminus \{x_j\}} = 0$ there exists $u \in B(\Omega)$ such that $P(D)u|_{\Omega \setminus \{x_1, x_2, \ldots\}} = 0$ and for each $j \in \mathbb{N}$, $P(D)(u - u_j)|_{V_j} = 0$.

Proof. Take $\Omega_0 = \Omega \setminus \{x_1, x_2, \ldots\}$, $u_0 = 0$, $\Omega_j = V_j$ and use Theorem 2.1.

2.3. Remark. Note that by Theorem 0 each u_j must be equal to $v_j + \sum_\alpha c_{\alpha,j} \partial^\alpha E(x - x_j)$ where $\{c_{\alpha,j}\}_{\alpha \in I}$ satisfies condition A and E is a fundamental solution for $P(D)$.

Corollary 2.2 yields immediately the following.

2.4. Theorem (Mittag-Leffler theorem for hyperfunctions). *Let Ω and $\{x_k\}_{k \in \mathbb{N}}$ be as in $(**)$. For each $k \in \mathbb{N}$ let V_k be an open neighbourhood of x_k such that $V_i \cap V_j = \emptyset$ if $i \neq j$. Let $P(D)$ be a differential operator and let E be its fundamental solution. For every $k \in \mathbb{N}$ take a sequence $\{c_{\alpha,k}\}_{\alpha \in I}$ satisfying condition* A *and define*

$$R_k(x) = \sum_{\alpha \in I} c_{\alpha,k} \partial^\alpha E(x).$$

There exists $u \in B(\Omega)$ such that $P(D)u = 0$ on $\Omega \setminus \{x_1, x_2, \ldots\}$ and

$$P(D)(u - R_k(x - x_k)) = 0 \quad \text{on } V_k \text{ for each } k \in \mathbb{N}.$$

2.5. Remark. Let us return to Theorem 0. For a general differential operator $P(D)$ there is a problem of uniqueness of the Laurent expansion $\sum_{\alpha \in I} c_\alpha \partial^\alpha E$ for a given hyperfunction u. The situation is as follows. For a fixed $u \in B(U)$ this expansion is unique. If u is defined on $U \setminus \{0\}$ and we extend it to a hyperfunction $\widetilde{u}$ on U then the Laurent expansion of $\widetilde{u}$ on U depends on the choice of $\widetilde{u}$. (This is why Theorem 2.5 of [4] is incorrectly stated and proved. Compare [6], [13].) However, the restriction of the Laurent expansion to $U \setminus \{0\}$ does not depend on the choice of $\widetilde{u}$. Let $\widetilde{u}_1$ and $\widetilde{u}_2$ be two such extensions. We have $\widetilde{u}_1 - \widetilde{u}_2 = \sum_{\alpha \in I} d_\alpha \partial^\alpha \delta_0$ and hence if $\widetilde{u}_2 = v_2 + \sum_{\alpha \in I} c_\alpha \partial^\alpha E$ then $u_1 = v_1 + \sum_{\alpha \in I} c_\alpha \partial^\alpha E + \sum_{\alpha \in I} d_\alpha P(D) \partial^\alpha E$. Hence the difference between the Laurent expansions for $\widetilde{u}_1$ and $\widetilde{u}_2$ has support contained in $\{0\}$.

The abstract Theorem 2.4 can be useful only if we have some knowledge about the fundamental solution of $P(D)$ and the regularity of solutions of the equation $P(D)u = 0$.

3. The case of elliptic operators. Recall two characterizations of elliptic differential operators $(*)$. The ellipticity of $P(D)$ is equivalent to each of the following two conditions:

1) If $u \in B(\Omega)$ and $P(D)u = 0$ then u is a real-analytic function on Ω.

2) $P(D)$ has a fundamental solution E which is real-analytic outside $\{0\}$ (see [7]).

Suppose now that $\{c_\alpha\}_{\alpha\in I}$ is a sequence satisfying condition A.

It follows from the Cauchy inequalities that if h is a function real-analytic on Ω then so is $g = \sum_{\alpha\in I} c_\alpha \partial^\alpha h$.

Hence Theorem 2.4 yields the following

3.1. Theorem. *Let $P(D)$ be an elliptic differential operator and let E be its fundamental solution in $\mathbb{R}^n$ real-analytic outside zero. Let Ω and $\{x_k\}_{k\in\mathbb{N}}$ be as in $(**)$. For each $n \in \mathbb{N}$ take $\{c_\alpha\}_{\alpha\in I}$ satisfying condition* A *and define*

$$R_k(x) = \sum_{\alpha\in I} c_\alpha \partial^\alpha E(x),$$

which is analytic on $\mathbb{R}^n \setminus \{0\}$. There exists an analytic function u on $\Omega\setminus\{x_1, x_2, \ldots\}$ such that $P(D)u = 0$ on $\Omega\setminus\{x_1, x_2, \ldots\}$, for each $k \in \mathbb{N}$, $u(x) - R_k(x - x_k)$ is analytic on some neighbourhood V_k of x_k, and $P(D)(u(x) - R_k(x - x_k)) = 0$ on V_k.

Theorem 3.1 is particularly interesting when we have an explicit formula for the fundamental solution E.

The most important cases are:

1) $\quad P(D) = \dfrac{\partial}{\partial \bar z}$ on $\mathbb{C}$ $(=\mathbb{R}^2)$, $\qquad E(z) = \dfrac{1}{\pi z}$,

2) $\quad P(D) = \Delta = \displaystyle\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}, \qquad E(x) = \begin{cases} -\ln|x|, & n = 2,\\ c_n/|x|^{n-2}, & n > 2,\end{cases}$

3) $\quad P(D) = \Delta^m$,

$$E(x) = \begin{cases} c_{mn}|x|^{2m-n} ln|x| & \text{if } 2m \ge n \text{ and } n \text{ is even},\\ c_{m,n}/|x|^{n-2m} & \text{if } n \text{ is odd or } n > 2m \text{ and } n \text{ is even}\end{cases}$$

(see [1]).

3.2. Theorem (Mittag-Leffler theorem for holomorphic functions). *Let Ω be an open set in $\mathbb{C}$ and let $\{z_k\}_{k\in\mathbb{N}}$ be a discrete sequence of distinct points of Ω. For each $k \in \mathbb{N}$ put $R_k(z - z_k) = \sum_{i=1}^\infty a_i/(z - z_k)^i$. There exists a function $g(z)$ holomorphic on $\Omega\setminus\{z_1, z_2, \ldots\}$ such that the principal part of the Laurent expansion of g at z_k is equal to $R_k(z - z_k)$ for every $k \in \mathbb{N}$.*

3.3. Theorem (Mittag-Leffler theorem for harmonic functions). *Let Ω and $\{x_k\}_{k\in\mathbb{N}}$ be as in $(**)$. For each $k \in \mathbb{N}$ let $\sum_{j=0}^\infty P_{j,k}(x)$ be a series of j-homogeneous harmonic polynomials, convergent on $\mathbb{R}^n$, and let*

$$R_k(x) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} \frac{P_{j,k}(x)}{|x|^{2j+n-2}} & \text{if } n > 2, \\ \displaystyle\sum_{j=1}^{\infty} \frac{P_{j,k}(x)}{|x|^{2j}} + c_0 \ln|x| & \text{if } n = 2. \end{cases}$$

There exists a function u harmonic on $\Omega \setminus \{x_1, x_2, \ldots\}$ such that for each $k \in \mathbb{N}$, $u(x) - R_k(x - x_k)$ is harmonic on some neighbourhood of x_k.

(Information on convergent Laurent series of harmonic functions can be found in [3], [2] and [5].)

3.4. Theorem (Mittag-Leffler theorem for polyharmonic functions). *Let Ω and $\{x_k\}$ be as above. For each k let $\sum_{j=0}^{\infty} P_{j,k}(x)$ be a series of m-polyharmonic polynomials, convergent on $\mathbb{R}^n$. Let*

$$R_k(x) = \sum_{j=0}^{\infty} \frac{P_{j,k}(x)}{|x|^{2j+n-2m}} \qquad \text{if } n > 2m \text{ or } n \text{ is odd}$$

and

$$R_k(x) = \sum_{j \ge 2m-n}^{\infty} \frac{P_{j,k}(x)}{|x|^{2j+n-2m}} \sum_{|\alpha|=0}^{2m-n} d_{\alpha,k} \partial^\alpha (|x|^{2m-n} \ln|x|)$$
$$\text{if } 2m \ge n \text{ and } n \text{ is even.}$$

There exists a function u, m-polyharmonic on $\Omega \setminus \{x_1, x_2, \ldots\}$, such that for each $k \in \mathbb{N}$ $u(x) - R_k(x - x_k)$ is m-polyharmonic on some neighbourhood of x_k.

3.5. Remark. In the above formulas we have

$$P_{j,k} = |x|^{2j+n-2m} \sum_{|\alpha|=j} c_{\alpha,k} \partial^\alpha E,$$

where $\{c_{\alpha,k}\}$ satisfies condition A.

4. The case of hypoelliptic operators. The hypoelliptic differential operators $(*)$ can be characterized as those operators which have a fundamental solution E, C^∞-smooth on $\mathbb{R}^n \setminus \{0\}$ (see [7], Vol. 2, Theorem 11.1.1).

There are two major obstacles to extending Theorem 3.1 to hypoelliptic operators. First, if $P(D)$ is hypoelliptic but not elliptic, then there exist a domain $\Omega \subset \mathbb{R}^n$ and $u \in B(\Omega)$ such that $Pu = 0$ on Ω and $u \notin D'(\Omega)$ (see [9]).

Moreover, condition A on the sequence $\{c_\alpha\}$ is too weak to guarantee that $\sum_\alpha c_\alpha \partial^\alpha E$ is C^∞-smooth on $\mathbb{R}^n \setminus \{0\}$. (If it is not C^∞-smooth on $\mathbb{R}^n \setminus \{0\}$ then it does not belong to $D'(\Omega)$.)

Hence in order to get an analogue of Theorem 3.1 we must put additional assumptions on the sequence $\{c_\alpha\}$ and the open set Ω. Those assumptions will depend on the operator $P(D)$.

4.1. Definition. Let $\varrho = (\varrho_1, \ldots, \varrho_n)$, with $\varrho_i \geq 1$ for all i. We say that $u \in C^\infty(\Omega)$ belongs to the class $\Gamma_\varrho(\Omega)$ iff for each compact $K \subset \Omega$ there exists C_K such that for every $x \in K$ and multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$,

$$|\partial^\alpha u(x)| \leq \frac{C_K^{|\alpha|+1}}{\alpha_1^{\varrho_1\alpha_1} \ldots \alpha_n^{\varrho_n\alpha_n}}.$$

The class $\Gamma_\varrho(\Omega)$ is called the *anisotropic Gevrey class*.

The Stirling and Newton formulas imply the following.

4.2. Proposition. *If $\{c_\alpha\}_{\alpha\in I}$ satisfies condition A_ϱ and $u \in \Gamma_\varrho(\Omega)$ then $\sum c_\alpha \partial^\alpha u \in \Gamma_\varrho(\Omega)$.*

In Hörmander's book [7], Vol. 2, the following fact was proved:

(∗∗∗) For every hypoelliptic operator $P(D)$ there exists $\varrho = (\varrho_1, \ldots, \varrho_n)$ (and an orthonormal system of coordinates in $\mathbb{R}^n$) such that if $u \in D'(\Omega)$ and $P(D)u = 0$ then $u \in \Gamma_\varrho(\Omega)$ for every open set Ω in $\mathbb{R}^n$ (Theorem 11.4.12 of [7]).

Let us recall some more facts from Hörmander's book. Let $P(D)$ be a linear partial differential operator with constant coefficients and let Ω be an open set in $\mathbb{R}^n$. We say that Ω is *P-convex for supports* if for every compact $K \subset \Omega$ there exists a compact $K' \subset \Omega$ such that $\varphi \in C_0^\infty(\Omega)$ and $\operatorname{supp} P(-D)\varphi \subset K$ implies that $\operatorname{supp} \varphi \subset K'$.

A convex open domain is P-convex for supports for every P and $K' = \operatorname{conv} K$.

If $P(D)$ is as above and Ω is P-convex for supports then for every $v \in C^\infty(\Omega)$ there exists $u \in C^\infty(\Omega)$ such that $P(D)u = v$ ([7], Theorem 10.6.7 and Corollary 10.6.8).

We are now in a position to prove

4.3. Theorem. *Let $P(D)$ be a differential hypoelliptic operator. Let $\varrho = (\varrho_1, \ldots, \varrho_n)$ be as in (∗∗∗). Assume that Ω is P-convex for supports and that $\{x_k\}$ is a discrete sequence of distinct points in Ω. For each $k \in \mathbb{N}$ let $R_k = \sum_\alpha c_{\alpha,k}\partial^\alpha E$, where E is a fundamental solution for $P(D)$, C^∞-smooth on $\mathbb{R}^n \setminus \{0\}$, and $\{c_{\alpha,k}\}$ satisfies condition A_ϱ for each k. Then there exists $u \in \Gamma_\varrho(\Omega \setminus \{x_1, x_2, \ldots\})$ such that $P(D)u = 0$ and for each $k \in \mathbb{N}$ there exist $\Omega \supset V_k \ni x_k$ such that $u(x) - R_k(x - x_k) \in \Gamma_\varrho(V_k)$ and $P(D)[u(x) - R_k(x - x_k)] = 0$ on V_k.*

Proof. For every k take $V'_k = B(x_k, r_k) \subset \Omega$ such that $V'_i \cap V'_j = \emptyset$ for $i \neq j$. Let $V_k = B(x_k, r_k/2)$. Let φ_k be a C^∞ function such that $\varphi_k = 0$ on $\Omega \setminus V'_k$ and $\varphi_k = 1$ on V_k.

Proposition 4.2 implies that $R_k(x - x_k) \in \Gamma_\varrho(\mathbb{R}^n \setminus \{x_k\})$. Note that $P(D)[R_k(x - x_k)] = 0$ in $\mathbb{R}^n \setminus \{x_k\}$. Thus $f(x) = \sum_{i=1}^\infty \varphi_i R_k(x - x_i) \in C^\infty(\Omega \setminus \{x_1, x_2, \ldots\})$. Let $g(x) = P(D)f(x)$ on $\Omega \setminus \bigcup_{i=1}^\infty V_i$ and $g(x) = 0$ on V_i.

Since $g \in C^\infty(\Omega)$ there exists $v \in C^\infty$ such that $P(D)v = g$ on Ω. Then $u = f - v \in C^\infty(\Omega \setminus \{x_1, x_2, \ldots\})$ and $P(D)u = 0$ on $\Omega \setminus \{x_1, x_2, \ldots\}$. On each V_k, $u - R_k(x - x_k) = -v \in C^\infty(V_k)$ and $P(D)(u - R_k) = 0$. It follows from Theorem 11.4.12 of [7] that $u \in \Gamma_\varrho(\Omega \setminus \{x_1, x_2, \ldots\})$ and $u(x) - R_k(x - x_k) \in \Gamma_\varrho(V_k)$.

In order to characterize the isolated singularities of solutions of the equation $P(D)u = 0$ we must know the answer to the following two open questions:

4.4. Problem. Assume that $u \in B(U)$, U is the unit ball in $\mathbb{R}^n$, $P(D)u = 0$ on U and $u|_{U \setminus \{0\}}$ is a C^∞-smooth function on $U \setminus \{0\}$. Is it true that if $P(D)$ is hypoelliptic then u is C^∞-smooth on U?

4.5. Problem. Let $P(D)$ be a hypoelliptic differential operator. Assume that $P(D)f = 0$ on $U \setminus \{0\}$ (U is the unit ball). Can f be extended to a hyperfunction u on U such that $P(D)u = \sum_\alpha c_\alpha \partial^\alpha \delta_0$ where $\{c_\alpha\}_{\alpha \in I}$ satisfies condition A_ϱ for some ϱ from $(***)$?

4.6. Remark. Corollary 10.8.2 of [7] says that if $P(D)$ is elliptic then each open set in $\mathbb{R}^n$ is P-convex for supports. Hence Theorem 4.3 gives us an alternative approach to Theorems 3.1–3.4.

4.7. Remark. Little is known about the isolated singularities of solutions of $P(D)u = 0$ in the case when $P(D)$ is not hypoelliptic. Palamodov [8] proved that if $P(D)u = 0$ on $U \setminus \{0\}$ and $u \in C^\infty(U \setminus \{0\}) \cap D'(U)$ then $u \in C^\infty(U)$. He proved, however, that there can exist isolated essential singularities (that means solutions which cannot be extended to a distribution on U). There are also some papers on isolated singularities of solutions of the Schrödinger equation. However, by Palamodov's result, the fundamental solution E cannot have an isolated singularity at zero and thus we cannot have Laurent expansions of the type described above.

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