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On the class of functions strongly starlike of order α with respect to a point

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Abstract. We consider the class $\mathcal{Z}(k;w)$, $k \in [0,2]$, $w \in \mathbb{C}$, of plane domains Ω called k-starlike with respect to the point w. An analytic characterization of regular and univalent functions f such that $f(U)$ is in $\mathcal{Z}(k;w)$, where $w \in f(U)$, is presented. In particular, for $k=0$ we obtain the well known analytic condition for a function f to be starlike w.r.t. w, i.e. to be regular and univalent in U and have $f(U)$ starlike w.r.t. $w \in f(U)$.

1. Introduction. Let $U_r = \{z \in \mathbb{C} : |z| < r\}$, $0 < r \le 1$, denote the disk of radius r in the complex plane $\mathbb{C}$ and $U = U_1$ denote the unit disk. We denote by $B(\xi,\varrho)$, $\xi \in U$, $\varrho > 0$, the hyperbolic open disk with hyperbolic center at ξ and hyperbolic radius ϱ. We recall that

$$B(\xi,\varrho) = \{z \in U : D(\xi,z) < \varrho\} = \left\{z \in U : \left|\frac{z-\xi}{1-\overline{\xi}z}\right| < R = \tanh\varrho\right\},$$

where

$$D(\xi,z) = \frac{1}{2}\log\frac{|1-\overline{\xi}z| + |z-\xi|}{|1-\overline{\xi}z| - |z-\xi|} = \operatorname{artanh}\left|\frac{z-\xi}{1-\overline{\xi}z}\right|$$

denotes the hyperbolic distance on U between ξ and z.

For each $\alpha \in (0,1]$ we denote by $S^*(\alpha)$ the class of functions f regular in U, normalized by $f(0) = f'(0) - 1 = 0$ and satisfying

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \alpha\frac{\pi}{2} \quad \text{for } z \in U,$$

called *strongly starlike of order* α. For each $\alpha \in (0,1]$ the class $S^*(\alpha)$ is a subset of the class $S^* = S^*(1)$ of starlike functions. Therefore each function in $S^*(\alpha)$ is univalent.

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The classes $S^*(\alpha)$ were introduced by Brannan and Kirwan [1], and independently by Stankiewicz [4, 5] (see also [2, Vol. I, pp. 138–139]).

Brannan and Kirwan found a geometric condition called *δ-visibility* which is sufficient for a function to be in $S^*(\alpha)$. Stankiewicz [5] obtained an external geometric characterization of strongly starlike functions. In [3] Ma and Minda presented an internal geometric characterization of functions in $S^*(\alpha)$ using the concept of *k-starlike* domains.

Using an idea similar to that in the paper of Ma and Minda we introduce the class $\mathcal{Z}(k;w)$, $k \in [0,2]$, $w \in \mathbb{C}$, of domains Ω which will be called *k-starlike with respect to $w \in \Omega$*. For $w = 0$ the class $\mathcal{Z}(k;0)$ consists of the k-starlike domains. For $k = 0$ the class $\mathcal{Z}(0;w)$ consists of the domains Ω starlike w.r.t. w, which means that the line segment joining w and an arbitrary point $\omega \in \Omega$ lies in Ω.

We present an analytic characterization of the class $S^{\mathrm{g}}(k;\xi,w)$ of functions f which are regular and univalent in U and have $f(U) \in \mathcal{Z}(k;w)$, where $w = f(\xi)$ and $\xi \in U$. In other words, the internal geometric property of k-starlikeness w.r.t. an interior point is connected with the class of regular and univalent functions f satisfying an analytic condition (3.1), which are called *strongly starlike of order α w.r.t. w.*

2. Domains and functions k-starlike w.r.t. a point. Let $k \in (0,2]$ be fixed. We denote by $K_1(k)$ and $K_2(k)$ two closed disks of radius $1/k$ each centered at $1/2 - i\sqrt{1/k^2 - 1/4}$ and $1/2 + i\sqrt{1/k^2 - 1/4}$, respectively. For $k = 0$ we set

$$K_1(0) = \{v \in \mathbb{C} : \operatorname{Im} v < 0\} \cup [0,1],$$
$$K_2(0) = \{v \in \mathbb{C} : \operatorname{Im} v > 0\} \cup [0,1].$$

For each $k \in [0,2]$ we define

$$E_k = K_1(k) \cap K_2(k).$$

Of course, $E_0 = [0,1]$. Each set E_k, $k \in (0,2]$, contains the points 0 and 1 on its boundary.

For $A, B \subset \mathbb{C}$ and $\omega \in \mathbb{C}$ we define

$$AB = \{uv \in \mathbb{C} : u \in A \wedge v \in B\}, \qquad A \pm B = \{u \pm v \in \mathbb{C} : u \in A \wedge v \in B\},$$
$$\omega A = \{\omega\}A, \qquad \omega \pm A = \{\omega\} \pm A.$$

For fixed $k \in [0,2]$ define

$$\Gamma_k^+ = \partial E_k \cap \partial K_1(k) \quad \text{and} \quad \Gamma_k^- = \partial E_k \cap \partial K_2(k).$$

Then Γ_k^+ and Γ_k^-, for $k > 0$, are closed circular arcs in the boundary of E_k with endpoints 0 and 1 and with interiors lying in the upper and lower halfplane, respectively. Clearly, $\Gamma_0^+ = \Gamma_0^- = [0,1]$. Throughout, Γ_k^+ and Γ_k^- will be treated as oriented arcs: from 1 to 0 and from 0 to 1, respectively.

For $k > 0$ this means that the boundary of the set E_k is positively oriented, i.e. in counterclockwise direction.

For $w, \omega \in \mathbb{C}$ let

$$\Gamma_k^+(w,\omega) = w + (\omega - w)\Gamma_k^+, \qquad \Gamma_k^-(w,\omega) = w + (\omega - w)\Gamma_k^-,$$
$$E_k(w,\omega) = w + (\omega - w)E_k.$$

Of course, $\Gamma_k^+ = \Gamma_k^+(0,1)$, $\Gamma_k^- = \Gamma_k^-(0,1)$ and $E_k = E_k(0,1)$.

For $w, \omega \in \mathbb{C}$, $w \neq \omega$, $\Gamma_k^+(w,\omega)$ will be oriented from ω to w, and $\Gamma_k^-(w,\omega)$ from w to ω. For $k > 0$ this means that the boundary of $E_k(w,\omega)$ is positively oriented.

For every $z \in \Gamma_k^+ \setminus \{0,1\}$ we denote by $\theta(z) \in [0, \pi/2]$ the directed angle from iz to the tangent vector to Γ_k^+ at z. We also set $\theta(1) = \lim_{\Gamma_k^+ \ni z \to 1} \theta(z) = \arccos(k/2)$ and $\theta(0) = \lim_{\Gamma_k^+ \ni z \to 0} \theta(z) = \pi/2$.

Similarly, for every $z \in \Gamma_k^- \setminus \{0,1\}$ we denote by $\vartheta(z) \in [-\pi/2, 0]$ the directed angle from iz to the tangent vector to Γ_k^- at z and we set $\vartheta(1) = \lim_{\Gamma_k^- \ni z \to 1} \vartheta(z) = -\arccos(k/2)$ and $\vartheta(0) = \lim_{\Gamma_k^- \ni z \to 0} \vartheta(z) = -\pi/2$.

Observation 2.1. 1. *If z moves along Γ_k^+, $k \in (0,2]$, from* 1 *to* 0, *then $\theta(z)$ strictly increases from $\theta(1) = \arccos(k/2)$ to $\theta(0) = \pi/2$. For all $z \in \Gamma_0^+$, $\theta(z) = \pi/2$.*

2. *If z moves along Γ_k^-, $k \in (0,2]$, from* 0 *to* 1, *then $\vartheta(z)$ strictly increases from $\vartheta(0) = -\pi/2$ to $\vartheta(1) = -\arccos(k/2)$. For all $z \in \Gamma_0^-$, $\vartheta(z) = -\pi/2$.*

Definition 2.2. Fix $k \in [0,2]$. A domain Ω in the plane is called *k-starlike with respect to the point $w \in \Omega$* provided that $E_k(w,\omega) \subset \Omega$ for every $\omega \in \Omega$.

The set of all k-starlike domains w.r.t. $w \in \mathbb{C}$ will be denoted by $\mathcal{Z}(k;w)$. For simplicity of notation we denote the set $\mathcal{Z}(0;w)$ by $\mathcal{Z}(w)$ and the set $\mathcal{Z}(0;0)$ of all domains starlike w.r.t. the origin by $\mathcal{Z}$.

Remark 2.3. 1. 0-starlikeness of Ω w.r.t. $w \in \Omega$ is exactly starlikeness w.r.t. w, i.e. the line segment joining w and an arbitrary point $\omega \in \Omega$ lies in Ω.

2. k-starlike domains w.r.t. the origin will be called k-starlike. These domains were considered in [3].

The following lemma is clear.

Lemma 2.4. *If $0 \le k_1 \le k_2 \le 2$, $w \in \mathbb{C}$ and $\Omega \in \mathcal{Z}(k_2;w)$, then $\Omega \in \mathcal{Z}(k_1;w)$.*

Since $\mathcal{Z}(k;w) \subset \mathcal{Z}(w)$ for all $k \in (0,2]$, every domain in $\mathcal{Z}(k;w)$ is simply connected.

LEMMA 2.5. *If* $\Omega \in \mathcal{Z}(k;w)$ *for* $k \in (0,2]$ *and* $w \in \Omega$, *then* $E_k(w,\omega)\setminus\{\omega\} \subset \Omega$ *for every* $\omega \in \partial\Omega$.

Proof. Fix $\omega \in \partial\Omega$. By Lemma 2.4, Ω is starlike w.r.t. w so $[w,\omega) \subset \Omega$. Take the sequence $w_n = w + (1-1/n)(\omega - w)$, $n \ge 2$, in $[w,\omega)$. It is clear that $\lim_{n\to\infty} w_n = \omega$. Since $w_n \in \Omega$ it follows that $E_k(w,w_n) \subset \Omega$ for all $n \ge 2$. Therefore

$$\bigcup_{n=2}^{\infty} E_k(w,w_n) \subset \Omega. \tag{2.1}$$

Notice also that

$$E_k(w,w_n) \subset E_k(w,w_{n+1}) \quad \text{for } n \ge 2. \tag{2.2}$$

Indeed, let $u \in E_k(w,w_n)$. Then there exists $\eta \in E_k$ such that $u = w + (w_n - w)\eta = w + (1-1/n)(\omega - w)\eta$. By starlikeness of E_k we see that $\zeta = (1-1/n^2)\eta \in E_k$. Consequently,

$$\begin{aligned} w + (w_{n+1} - w)\zeta &= w + \left(1 - \frac{1}{n+1}\right)(\omega - w)\left(1 - \frac{1}{n^2}\right)\eta \\ &= w + \left(1 - \frac{1}{n}\right)(\omega - w)\eta = u, \end{aligned}$$

which means that $u \in E_k(w,w_{n+1})$, so (2.2) is proved.

Now we prove that

$$\operatorname{Int} E_k(w,\omega) \subset \bigcup_{n=2}^{\infty} E_k(w,w_n). \tag{2.3}$$

To this end, let $u \in \operatorname{Int} E_k(w,\omega)$. Thus there exists $\eta \in \operatorname{Int} E_k$ such that $u = w + (\omega - w)\eta$. Let $a \in \partial E_k$, $a \ne 0$, be the point of intersection of ∂E_k with the straight line joining the origin and η. It is clear that $\eta \ne a$ and therefore $\zeta = n\eta/(n-1) \in E_k$ for some $n \ge 2$. Hence

$$\begin{aligned} w + (w_n - w)\zeta &= w + \left(1 - \frac{1}{n}\right)(\omega - w)\zeta = w + \frac{n-1}{n}(\omega - w)\frac{n}{n-1}\eta \\ &= w + (\omega - w)\eta = u. \end{aligned}$$

This means that $u \in E_k(w,w_n)$. Therefore (2.3) holds.

From (2.1) and (2.3) we obtain

$$\operatorname{Int} E_k(w,\omega) \subset \Omega. \tag{2.4}$$

It remains to prove that if $v \in \partial E_k(w,\omega)$, $v \ne \omega$, then $v \in \Omega$. Suppose, on the contrary, that there exists $v \in \Gamma_k^+(w,\omega)$, $v \ne \omega$, such that $v \notin \Omega$. By (2.4) we can assume that $v \in \partial\Omega$.

Let w_0 be an arbitrary point lying on the open subarc of $\Gamma_k^+(w,\omega)$ joining ω and v, so $w_0 = w + (\omega - w)\eta$ for some $\eta \in \Gamma_k^+$. The directed angle from the vector $i(w_0 - w)$ to the tangent vector to $\Gamma_k^+(w,\omega)$ at w_0 is equal to $\theta(\eta)$. From Observation 2.1 and since $k > 0$ it follows that $\theta(\eta) > \arccos(k/2)$. But considering the set $E_k(w, w_0)$ we see that the directed angle from the vector $i(w_0 - w)$ to the one-sided tangent vector to $\Gamma_k^+(w, w_0)$ at w_0 is equal to $\arccos(k/2)$. Hence the open subarc of $\Gamma_k^+(w,\omega)$ joining w and w_0 is contained in the interior of $E_k(w, w_0)$. Thus $v \in \operatorname{Int} E_k(w, w_0)$. If now $w_0 \in \Omega$, then $E_k(w, w_0) \subset \Omega$, so $v \in \Omega$. If $w_0 \in \partial\Omega$, then by (2.4) we have $\operatorname{Int} E_k(w, w_0) \subset \Omega$, so $v \in \Omega$ also. Both cases contradict the assumption that $v \in \partial\Omega$.

If we assume that $v \in \Gamma_k^-$, $v \neq \omega$, and $v \notin \Omega$, then a similar analysis leads to a contradiction once again. This ends the proof of the lemma.

Let $\xi \in U$ and $w \in \mathbb{C}$. The set of all functions f regular in U such that $f(\xi) = w$ will be denoted by $\mathcal{A}(\xi, w)$.

Definition 2.6. Fix $k \in [0,2]$. A function $f \in \mathcal{A}(\xi, w)$, where $\xi \in U$ and $w \in \mathbb{C}$, univalent in U will be called *k-starlike w.r.t. w* if the domain $f(U)$ is k-starlike w.r.t. w, i.e. $f(U) \in \mathcal{Z}(k; w)$.

The set of all functions $f \in \mathcal{A}(\xi, w)$, $w = f(\xi)$, which are k-starlike w.r.t. w will be denoted by $S^{\mathrm{g}}(k; \xi, w)$.

We write $S^{\mathrm{g}}(\xi, w)$ for $S^{\mathrm{g}}(0; \xi, w)$. If $\xi = 0$ and $w = f(\xi) = 0$, then k-starlike functions w.r.t. the origin will be called *k-starlike* (see [3]). For $k = 0$, $\xi = 0$ and $w = f(\xi) = 0$ we obtain the well known class $S^{\mathrm{g}}(0, 0; 0)$ of starlike functions. This class will be denoted by S^{g}.

Let us also introduce the following classes:

$$S^{\mathrm{g}}(k; w) = \bigcup_{\xi \in U} S^{\mathrm{g}}(k; \xi, w), \qquad S^{\mathrm{g}}_{\xi}(k) = \bigcup_{w \in \mathbb{C}} S^{\mathrm{g}}(k; \xi, w).$$

The basic property of these classes is preservation of k-starlikeness w.r.t. w on each hyperbolic disk centered at ξ, which can be formulated as follows:

Theorem 2.7. *A regular and univalent function f is in $S^{\mathrm{g}}(k; \xi, w)$, where $k \in [0,2]$, $\xi \in U$ and $w \in \mathbb{C}$, if and only if for every $\varrho > 0$ the domain $f(B(\xi, \varrho))$ is in $\mathcal{Z}(k; w)$, where $w = f(\xi)$.*

Proof. Suppose first that $f \in S^{\mathrm{g}}(k; \xi, w)$, where $k \in [0,2]$, $\xi \in U$ and $w = f(\xi)$. Hence $\Omega = f(U) \in \mathcal{Z}(k; w)$. Fix $\varrho > 0$ and set $\Omega(\xi, \varrho) = f(B(\xi, \varrho))$. We will show that $E_k(w, \omega) \subset \Omega(\xi, \varrho)$ for all $\omega \in \Omega(\xi, \varrho)$.

Since Ω is k-starlike domain w.r.t. w, we see that $w + (\omega - w)v \in \Omega$ for all $\omega \in \Omega$ and $v \in E_k$. Thus the function

$$g(z) = f^{-1}(w + (f(z) - w)v), \qquad z \in U, \tag{2.5}$$

is well defined for each $v \in E_k$, regular in U and $g(U) \subset U$. Since $g(\xi) = \xi$, Pick's Theorem, the invariant formulation of Schwarz's Lemma, shows that

$g(B(\xi,\varrho)) \subset B(\xi,\varrho)$. Moreover, $g(B(\xi,\varrho)) = B(\xi,\varrho)$ only if g is a Möbius transformation which maps the unit circle into itself. From (2.5) we now get

$$w + (\Omega(\xi,\varrho) - w)v = f(g(B(\xi,\varrho))) \subset \Omega(\xi,\varrho)$$

for all $v \in E_k$. This implies that $w + (\Omega(\xi,\varrho) - w)E_k \subset \Omega(\xi,\varrho)$. Consequently, $w + (\omega - w)E_k = E_k(w,\omega) \subset \Omega(\xi,\varrho)$ for all $\omega \in \Omega(\xi,\varrho)$. This means that $\Omega(\xi,\varrho) \in \mathcal{Z}(k;w)$.

Conversely, suppose that $f(B(\xi,\varrho))$ is in $\mathcal{Z}(k;w)$, where $w = f(\xi)$, for every $\varrho > 0$. Since

$$f(U) = \bigcup_{\varrho>0} f(B(\xi,\varrho)),$$

the assertion is immediate. This ends the proof of the theorem.

3. An analytic characterization of the class $S^{\mathrm{g}}(k;\xi,w)$. In this section we present an analytic characterization of functions $f \in S^{\mathrm{g}}(k;\xi,w)$.

The main theorem of this paper is the following.

THEOREM 3.1. *If $f \in S^{\mathrm{g}}(k;\xi,w)$ for $k \in [0,2)$, $\xi \in U$ and $w \in \mathbb{C}$, then*

$$\left|\arg\left\{\frac{(1-\overline{\xi}z)(z-\xi)f'(z)}{f(z)-w}\right\}\right| < \alpha\frac{\pi}{2}, \qquad z \in U, \tag{3.1}$$

where $\alpha = (2/\pi)\arccos(k/2)$.

Conversely, let $\alpha \in (0,1]$, $\xi \in U$ and $w \in \mathbb{C}$. If (3.1) is satisfied for a function f regular in U, then $f \in S^{\mathrm{g}}(k;\xi,w)$ for $k = 2\cos(\alpha\pi/2)$.

Proof. For f regular in U and $\xi \in U$ we set $\Omega = f(U)$, $\Omega(\xi,\varrho) = f(B(\xi,\varrho))$ and $C(\varrho) = \partial B(\xi,\varrho)$ for $\varrho > 0$.

1. We first consider the case $k = 0$.

(i) Assume that $f \in S^{\mathrm{g}}(\xi,w)$, where $\xi \in U$ and $w \in \mathbb{C}$. Thus $w = f(\xi)$ and $\Omega \in \mathcal{Z}(w)$. By Theorem 2.7, also $\Omega(\xi,\varrho) \in \mathcal{Z}(w)$ for every $\varrho > 0$. Therefore $\arg(f(z) - w)$ is well defined locally on the circle $C(\varrho)$. Let us parametrize $C(\varrho)$ as follows:

$$C(\varrho):\quad z = z(t) = \frac{Re^{it}+\xi}{1+\overline{\xi}Re^{it}}, \qquad t \in [0,2\pi), \tag{3.2}$$

where $R = \tanh\varrho \in (0,1)$. Hence we get

$$\begin{aligned} z'(t) &= \frac{i(1-|\xi|^2)Re^{it}}{(1+\overline{\xi}Re^{it})^2} = i\frac{1-|\xi|^2}{1+\overline{\xi}Re^{it}}\cdot\frac{Re^{it}}{1+\overline{\xi}Re^{it}} \\ &= i\left(1-\overline{\xi}\frac{Re^{it}+\xi}{1+\overline{\xi}Re^{it}}\right)\frac{Re^{it}-|\xi|^2Re^{it}}{(1-|\xi|^2)(1+\overline{\xi}Re^{it})} \\ &= \frac{i(1-\overline{\xi}z)(z-\xi)}{1-|\xi|^2}. \end{aligned} \tag{3.3}$$

By starlikeness of $C(\varrho)$ w.r.t. w it follows that the function

$$[0,2\pi) \ni t \to \arg(f(z(t)) - w) \tag{3.4}$$

is nondecreasing. Hence and by (3.3) we have

$$\begin{aligned}\frac{d}{dt}\arg(f(z(t)) - w) &= \frac{d}{dt}\operatorname{Im}\log(f(z(t)) - w) \\ &= \operatorname{Im}\left\{\frac{z'(t)f'(z(t))}{f(z(t)) - w}\right\} \\ &= \frac{1}{1-|\xi|^2}\operatorname{Re}\left\{\frac{(1-\overline{\xi}z)(z-\xi)f'(z)}{f(z)-w}\right\} \geq 0\end{aligned} \tag{3.5}$$

for all $z \in U \setminus \{\xi\}$. As $w = f(\xi)$ the function

$$Q(z,\xi) = \frac{(1-\overline{\xi}z)(z-\xi)f'(z)}{f(z)-w}, \quad z \in U \setminus \{\xi\}, \tag{3.6}$$

has a removable singularity at $z = \xi$ with

$$Q(\xi,\xi) = \lim_{z\to\xi}\frac{(1-\overline{\xi}z)(z-\xi)f'(z)}{f(z)-f(\xi)} = \frac{1}{1-|\xi|^2},$$

where we used the fact that $f'(\xi) \neq 0$ since f is univalent in U. Hence the inequality (3.5) holds for $z = \xi$ also. Since $Q(\xi,\xi) > 0$, the minimum principle for harmonic functions shows that for all $z \in U$ the inequality (3.5) is strict, i.e.

$$\operatorname{Re} Q(z,\xi) > 0 \quad \text{for } z \in U, \tag{3.7}$$

which is equivalent to (3.1) for $\alpha = 1$.

(ii) Conversely, let (3.1) be satisfied for $\alpha = 1$ and fixed f regular in U, i.e. (3.7) holds. From (3.7) we see that Q has no pole and no zero in U. But this holds only when $w = f(\xi)$ and $f'(z) \neq 0$ for all $z \in U$. In consequence, $f \in \mathcal{A}(\xi, w)$ and f is locally univalent in U. Moreover, from (3.7) we have $f(z) \neq w = f(\xi)$ for all $z \in U\setminus\{\xi\}$. We conclude that the equation $f(z) - w = 0$ has a unique simple zero at $z = \xi$ on U. The argument principle now shows that

$$\Delta_{C(\varrho)}\arg(f(z)-w) = \operatorname{Im}\left\{\int_{C(\varrho)}\frac{f'(z)}{f(z)-w}\,dz\right\} = 2\pi$$

for every $\varrho > 0$. Hence applying once more the argument principle we deduce that the equation $f(z) - \omega = 0$ has a unique solution for each $\omega \in \Omega(\xi,\varrho)$, which implies univalence of f in $B(\xi,\varrho)$ for every $\varrho > 0$. In consequence, f is univalent in U.

Further, from (3.7) and (3.5) it follows that the function (3.4) is increasing so the curve $f(C(\varrho))$ and consequently the domain $\Omega(\xi,\varrho)$ are starlike w.r.t. w for every $\varrho > 0$. In this way, by Theorem 2.7 we see that $f(U)$ is starlike w.r.t. w, which means that $f \in S^{\mathrm{g}}(\xi, w)$.

2. (i) Let now $k \in (0,2)$ and $\alpha = (2/\pi)\arccos(k/2)$. Let $f \in S^{\mathrm{g}}(k;\xi,w)$ with $w = f(\xi)$. Hence $\Omega = f(U) \in \mathcal{Z}(k;w)$. We will prove that (3.1) holds, i.e.

$$|\arg Q(z,\xi)| < \alpha\frac{\pi}{2} \quad \text{for } z \in U, \tag{3.8}$$

for $\alpha = (2/\pi)\arccos(k/2)$.

For $z = \xi$ the inequality (3.8) is clear since $Q(\xi,\xi) = 1/(1-|\xi|^2)$ is a positive real number.

Now we prove that (3.8) is true for all points on $C(\varrho)$ for every $\varrho > 0$. Let γ_ϱ denote the curve $\partial\Omega(\xi,\varrho)$ positively oriented. For each $z \in C(\varrho)$ we denote by $\tau(z)$ the tangent vector to γ_ϱ at $\omega = f(z)$, i.e.

$$\tau(z) = z'(t)f'(z(t)),$$

where $z = z(t)$ is given by (3.2). From (3.3) we get

$$\tau(z) = \frac{i(1-\overline{\xi}z)(z-\xi)f'(z)}{1-|\xi|^2}, \quad z \in C(\varrho).$$

Let $\varphi(z)$, $z \in C(\varrho)$, denote the directed angle from the vector $i(f(z)-w)$ to $\tau(z)$, i.e.

$$\begin{aligned}\varphi(z) &= \arg\{\tau(z)\} - \arg\{i(f(z)-w)\} \\ &= \arg\left\{\frac{i(1-\overline{\xi}z)(z-\xi)f'(z)}{(1-|\xi|^2)i(f(z)-f(\xi))}\right\} \\ &= \arg\left\{\frac{(1-\overline{\xi}z)(z-\xi)f'(z)}{f(z)-f(\xi)}\right\} = \arg Q(z,\xi).\end{aligned} \tag{3.9}$$

Let $z \in C(\varrho)$ and $\omega = f(z)$. By Theorem 2.7 the domain $\Omega(\xi,\varrho)$ is in $\mathcal{Z}(k;w)$. Therefore by a limit argument $E_k(w,\omega) \subset \overline{\Omega(\xi,\varrho)}$.

As was mentioned in Section 2, the boundary of the set $E_k(w,\omega)$ is positively oriented. Let s_1 and s_2 be one-sided tangent vectors to the arcs $\Gamma_k^+(w,\omega)$ and $\Gamma_k^-(w,\omega)$ at ω, respectively, and let p_1 and p_2 be the half-lines starting from ω with directional vectors s_1 and s_2, respectively. We denote by V the closed sector bounded by p_1 and p_2 with vertex ω for which $\operatorname{Int} V \cap \operatorname{Int} E_k(w,\omega) = \emptyset$. The normal line to the vector joining w and ω and going through ω divides the plane into two closed half-planes, one of them containing $E_k(w,\omega)$. Consequently, one of the two closed half-lines starting from ω and normal to the vector joining w and ω lies in V; denote it by p. Then p divides V into two closed sectors with vertex at ω: V_1 bounded by p_1 and p, and V_2 bounded by p_2 and p. Since $E_k(w,\omega)$ is symmetric w.r.t. the straight line going through w and ω which is normal to p, we see that p

bisects V. From the assumption that $\Omega(\xi,\varrho)$ is k-starlike w.r.t. w it follows that the tangent line to γ_ϱ at ω cannot intersect the interior of $E_k(w,\omega)$. Therefore the tangent vector $\tau(z)$ lies in V.

If $\tau(z)$ lies in V_1, then $\varphi(z)$ is nonnegative and in view of (3.9) we obtain

$$\varphi(z) \le \arg\{s_1\} - \arg\{i(f(z) - f(\xi))\} = \theta(1) = \arccos\frac{k}{2} = \alpha\frac{\pi}{2}. \tag{3.10}$$

If $\tau(z)$ lies in V_2, then $\varphi(z)$ is nonpositive and using again (3.9) we have

$$\varphi(z) \ge \arg\{s_2\} - \arg\{i(f(z) - f(\xi))\} = \vartheta(1) = -\arccos\frac{k}{2} = -\alpha\frac{\pi}{2}. \tag{3.11}$$

In consequence, the inequalities (3.10) and (3.11) are true for every point in $C(\varrho)$. As ϱ was arbitrary, they are satisfied in U.

Suppose that equality holds in (3.10). Then by the maximum principle for harmonic functions it holds for every point in U. But this is impossible since $Q(\xi,\xi)$ is a real number. Therefore the inequality (3.10) is strict, and similarly for (3.11).

(ii) Conversely, let $\alpha \in (0,1)$ and assume that (3.1) is satisfied for f regular in U, i.e. (3.8) holds. As in Part 1(ii) we can prove that $w = f(\xi)$ and therefore $f \in \mathcal{A}(\xi, w)$.

The inequality (3.8) is clearly true for $\alpha = 1$ also. But, as was shown in Part 1(ii), this implies that $f \in S^{\mathrm{g}}(\xi, w)$ and therefore f is univalent in U. Thus we need to prove that $f(U) \in \mathcal{Z}(k; w)$ for $k = 2\cos(\alpha\pi/2)$.

Suppose, on the contrary, that $f(U)$ is not k-starlike w.r.t. w for $k = 2\cos(\alpha\pi/2)$. By Theorem 2.7 there exists $\varrho > 0$ such that $\Omega(\xi,\varrho)$ is not k-starlike w.r.t. w. This means that there exists $w_0 \in \Omega(\xi,\varrho)$ such that $E_k(w, w_0)$ is not contained in $\Omega(\xi,\varrho)$.

Suppose that

$$\Gamma_k^+(w, w_0) \cap \gamma_\varrho \ne \emptyset.$$

Thus there exists $w_1 \in (\Gamma_k^+(w, w_0) \setminus \{w, w_0\}) \cap \gamma_\varrho$ such that the subarc of $\Gamma_k^+(w, w_0)$ joining w_1 and w_0 without the endpoint w_1 is contained in $\Omega(\xi,\varrho)$. Since $w_1 \in \gamma_\varrho$, there exists $z_1 \in C(\varrho)$ such that $w_1 = f(z_1)$. Let $\varphi(z_1)$ denote the directed angle defined by (3.9), where z is replaced by z_1. The tangent line to the convex set $E_k(w, w_0)$ at w_1 is the boundary of two closed half-planes denoted by H_1 and H_2. One of them, say H_1, supports the set $E_k(w, w_0)$, the other H_2 contains it. Since γ_ϱ is positively oriented, from the definition of w_1 it follows that the tangent vector $\tau(z_1)$ lies in H_2, and the vector $i(w_1 - w)$ lies in H_1. Hence the angle $\varphi(z_1)$ is positive. Further, using Observation 2.1, the fact that $w_1 \ne w_0$ and (3.9) we have

$$\arg Q(z_1,\xi) = \varphi(z_1) \ge \theta(z_1) = \theta\left(\frac{w_1 - w}{w_0 - w}\right) > \theta(1) = \arccos\frac{k}{2} = \alpha\frac{\pi}{2},$$

contrary to (3.8).

Suppose now that

$$\Gamma_k^-(w, w_0) \cap \gamma_\varrho \neq \emptyset.$$

Thus there exists $w_2 \in (\Gamma_k^-(w, w_0) \setminus \{w, w_0\}) \cap \gamma_\varrho$ such that the subarc of $\Gamma_k^-(w, w_0)$ joining w_2 and w_0 without the endpoint w_2 is contained in $\Omega(\xi, \varrho)$. Let $z_2 \in C(\varrho)$ be such that $w_2 = f(z_2)$. Since γ_ϱ is positively oriented, we see that the tangent vector $\tau(z_2)$ lies in the closed half-plane supporting $E_k(w, w_0)$ at w_2, and $i(f(z_2) - w)$ lies in the complementary closed half-plane. In consequence, the angle $\varphi(z_2)$ is negative. Moreover, by Observation 2.1, the fact that $w_2 \neq w$ and (3.9) we have

$$\arg Q(z_2, \xi) = \varphi(z_2) \le \vartheta(z_2) = \vartheta\left(\frac{w_2 - w}{w_0 - w}\right) < \vartheta(1) = -\arccos\frac{k}{2} = -\alpha\frac{\pi}{2},$$

which contradicts (3.8).

So $f \in S^{\mathrm{g}}(k; \xi, w)$ with $k = 2\cos(\alpha\pi/2)$, which ends the proof of the theorem.

4. Remarks. Taking into account (3.1) we can introduce the following

DEFINITION 4.1. For each $\alpha \in (0, 1]$ and $\xi \in U$ we denote by $S^*(\alpha; \xi)$ the class of all functions f regular in U satisfying the condition

$$\left|\arg\left\{\frac{(1 - \overline{\xi}z)(z - \xi)f'(z)}{f(z) - f(\xi)}\right\}\right| < \alpha\frac{\pi}{2}, \quad z \in U. \tag{4.1}$$

From Theorem 3.1 it follows that every function in $S^*(\alpha; \xi)$ is univalent and belongs to the unique class $S^{\mathrm{g}}(k; \xi, w)$ for $k = 2\cos(\alpha\pi/2)$.

Theorem 3.1 gives an equivalence between k-starlikeness with respect to a fixed point $w \in \mathbb{C}$, a property which defines the class $S^{\mathrm{g}}(k; \xi, w)$, and an analytic condition (4.1) which describes the class $S^*(\alpha; \xi)$, where $\alpha = (2/\pi)\arccos(k/2)$. For $\xi = 0$ and $w = f(\xi) = 0$ we get the results of Ma and Minda [3]. Then the inequality (4.1) reduces to (1.1) and with the normalization $f'(0) = 1$ defines the class $S^*(\alpha)$ of strongly starlike functions, which coincides with the subclass of $S^{\mathrm{g}}(k; 0, 0)$, $k = 2\cos(\alpha\pi/2)$, with standard normalization.

The subclass of $S^*(1; \xi)$ with normalization $f(0) = 0$ is known. For details about this class see [2, Vol. I, pp. 155–164]. But this normalization seems to be unnatural. It excludes situations like $\xi = 0$ and $w = f(\xi) \neq 0$ or $\xi \neq 0$ and $w = f(\xi) = 0$.

It is also natural to consider the subclass of $S^*(\alpha; \xi)$ with normalization $f'(\xi) = 1$.

It is worth noticing that the condition (3.7) was obtained in 1978 by Wald [6], who transformed the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)-f(\xi)}\right\} > 0, \qquad z \in U, \tag{4.2}$$

into the form (3.7). The inequality (4.2) says geometrically that the domains $f(U_r)$ are starlike with respect to $w = f(\xi)$ for all r such that $|\xi| < r < 1$. Since at $z = \xi$ the expression on the left side of (4.1) has a pole, this condition fails to characterize the class $\mathcal{Z}(w)$.

Looking at Theorem 2.7 it is clear that starlikeness of f with respect to $w = f(\xi)$ is not connected with the family U_r, $r \in (0,1)$, of Euclidean disks but rather with the family of hyperbolic disks $B(\xi, \varrho)$ where $\varrho > 0$. This last family is transformed by every function f in $S^*(1;\xi)$ onto a family of starlike domains with respect to $f(\xi)$.

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