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Univalent harmonic mappings II

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Abstract. Let $a < 0 < b$ and $\Omega(a,b) = \mathbb{C} - ((-\infty, a] \cup [b, +\infty))$ and $U = \{z : |z| < 1\}$. We consider the class $S_H(U, \Omega(a,b))$ of functions f which are univalent, harmonic and sense-preserving with $f(U) = \Omega$ and satisfying $f(0) = 0$, $f_z(0) > 0$ and $f_{\bar{z}}(0) = 0$.

1. Introduction. Let S_H be the class of functions f which are univalent, sense-preserving, harmonic mappings of the unit disk $U = \{z : |z| < 1\}$ and satisfy $f(0) = 0$ and $f_z(0) > 0$. Let F and G be analytic in U with $F(0) = G(0) = 0$ and $\operatorname{Re} f(z) = \operatorname{Re} F(z)$ and $\operatorname{Im} f(z) = \operatorname{Re} G(z)$ for z in U. Then $h = (F + iG)/2$ and $g = (F - iG)/2$ are analytic in U and $f = h + \overline{g}$. f is locally one-to-one and sense-preserving if and only if $|g'(z)| < |h'(z)|$ for z in U (cf. [4]). If $h(z) = a_1 z + a_2 z^2 + \ldots,\ a_1 > 0$, and $g(z) = b_1 z + b_2 z^2 + \ldots$ for z in U, it follows that $|b_1| < a_1$ and hence $a_1 f - \overline{b_1 f}$ also belongs to S_H. Thus consideration is often restricted to the subclass S_H^0 of S_H consisting of those functions in S_H with $f_{\bar{z}}(0) = 0$.

Various authors have studied subclasses of S_H^0 consisting of functions mapping U onto a specific simply connected domain. See for example Hengartner and Schober [5], Abu-Muhanna and Schober [1], and Cima and the author [2], [3]. Recently the author [7] studied the subclass of S_H^0 consisting of functions mapping U onto the plane with the interval $(-\infty, a]$, $a < 0$, removed. See also Hengartner and Schober [6]. In the present paper we consider the case when $f(U)$ is $C - ((-\infty, a] \cup [b, +\infty))$, $a < 0 < b$.

Let $a < 0 < b$ and $\Omega(a,b) = \mathbb{C} - ((-\infty, a] \cup [b, +\infty))$. Then $S_H(U, \Omega(a,b))$ is the class of functions f in S_H^0 with $f(U) = \Omega(a,b)$. Without loss of generality, we assume that $a + b \geq 0$.

In the sequel F and G will be functions analytic in U with $F(0) = G(0) = 0$, $\operatorname{Re} f(z) = \operatorname{Re} F(z)$ and $\operatorname{Im} f(z) = \operatorname{Re} G(z)$ for z in U. If $h = (F + iG)/2$ and $g = (F - iG)/2$, then $f = h + \overline{g}$ and $|g'(z)| < |h'(z)|$ for z in U.

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2. Preliminary lemmas. Let $\mathcal{P}$ be the class of functions $P(z)$ which are analytic in U with $P(0)=1$ and $\operatorname{Re} P(z)>0$ for z in U. To get an integral representation of functions in $S_H(U,\Omega(a,b))$ we require a few lemmas.

Lemma 1. *Let*

$$T(x)=\int_0^1\left(\frac{a(1+t)^2}{(1+xt+t^2)^2}+\frac{b(1-t)^2}{(1-xt+t^2)^2}\right)dt \tag{2.1}$$

and

$$S(x)=\int_0^1\left(\frac{a(1-t)^2}{(1+xt+t^2)^2}+\frac{b(1+t)^2}{(1-xt+t^2)^2}\right)dt \tag{2.2}$$

where $a<0<b$, $a+b\ge 0$ *and* $-2<x<2$. *There exist unique numbers* c_1 *and* c_2 *with* $-2<c_1<0<c_2<2$ *so that* $S(c_1)=T(c_2)=0$. *Moreover*, $T(x)\le 0\le S(x)$ *if and only if* $c_1\le x\le c_2$.

Proof. We note that

$$S(x)-T(x)=\int_0^1\left(\frac{-2at}{(1+xt+t^2)^2}+\frac{2bt}{(1-xt+t^2)^2}\right)dt\ge 0.$$

Thus $T(x)\le S(x)$ for $-2<x<2$. Also, it is easily checked that $T'(x)>0$ and $S'(x)>0$ for $-2<x<2$. Thus $T(x)$ and $S(x)$ are both strictly increasing. Since $\lim_{x\to-2}T(x)=\lim_{x\to-2}S(x)=-\infty$ and $\lim_{x\to 2}T(x)=\lim_{x\to 2}S(x)=+\infty$, it follows that there exist unique c_1 and c_2 so that $S(c_1)=T(c_2)=0$ and that $c_1<c_2$. Moreover, $S(0)>0$, thus $c_1<0$ and $T(x)\le 0\le S(x)$ if and only if $c_1\le x\le c_2$.

Lemma 2. *Let* $P(z)$ *be in* $\mathcal{P}$ *and*

$$Q(x)=a\int_0^1\frac{1-t^2}{(1+xt+t^2)^2}\operatorname{Re}P(t)\,dt+b\int_0^1\frac{1-t^2}{(1-xt+t^2)^2}\operatorname{Re}P(-t)\,dt \tag{2.3}$$

where $a<0<b$, $a+b\ge 0$ *and* $-2<x<2$. *There exists a unique* c, $-2<c<2$, *so that* $Q(c)=0$.

Proof. It is easily checked that $Q'(x)>0$ for $-2<x<2$, $\lim_{x\to-2}Q(x)=-\infty$ and $\lim_{x\to 2}Q(x)=+\infty$. The lemma then follows.

Lemma 3. *With the same hypotheses as in Lemma* 2 *and with* a *and* b *fixed we have* $c_1\le c\le c_2$ *where* c_1 *and* c_2 *are given in Lemma* 1. *The range for* c *is sharp in the sense that for each* c, $c_1\le c\le c_2$, *there exists* $P(z)$ *in* $\mathcal{P}$ *such that the corresponding* Q *given by* (2.3) *satisfies* $Q(c)=0$.

Proof. Let $P(z)$ be in $\mathcal{P}$ and the corresponding Q in (2.3) satisfy $Q(c) = 0$. Using the inequalities $(1-|z|)/(1+|z|) \le \operatorname{Re} P(z) \le (1+|z|)/(1-|z|)$ for z in U, we obtain

$$\frac{(1-t)^2}{(1+ct+t^2)^2} \le \frac{(1-t^2)\operatorname{Re} P(t)}{(1+ct+t^2)^2} \le \frac{(1+t)^2}{(1+ct+t^2)^2},$$

$$\frac{(1-t)^2}{(1-ct+t^2)^2} \le \frac{(1-t^2)\operatorname{Re} P(-t)}{(1-ct+t^2)^2} \le \frac{(1+t)^2}{(1-ct+t^2)^2}.$$

Since $a < 0 < b$, this gives

$$\int_0^1 \left(\frac{a(1+t)^2}{(1+ct+t^2)^2} + \frac{b(1-t)^2}{(1-ct+t^2)^2} \right) dt \le Q(c) \le \int_0^1 \left(\frac{a(1-t)^2}{(1+ct+t^2)^2} + \frac{b(1+t)^2}{(1-ct+t^2)^2} \right) dt.$$

Thus $T(c) \le 0 \le S(c)$ where T and S are given in Lemma 1. From Lemma 2 we have $c_1 \le c \le c_2$.

To see that the range of c is sharp, we note that $Q(c_1) = 0$ when $P(z) = (1-z)/(1+z)$ and $Q(c_2) = 0$ when $P(z) = (1+z)/(1-z)$. If $c_1 < c < c_2$ then $T(c) < 0 < S(c)$. That is,

$$\int_0^1 \left(\frac{a(1+t)^2}{(1+ct+t^2)^2} + \frac{b(1-t^2)}{(1-ct+t^2)^2} \right) dt < 0 < \int_0^1 \left(\frac{a(1-t)^2}{(1+ct+t^2)^2} + \frac{b(1+t)^2}{(1-ct+t^2)^2} \right) dt. \tag{2.4}$$

With c fixed, let

$$\phi(P) = a\int_0^1 \frac{(1-t^2)\operatorname{Re} P(t)}{(1+ct+t^2)^2}\, dt + b\int_0^1 \frac{(1-t^2)\operatorname{Re} P(-t)}{(1-ct+t^2)^2}\, dt;$$

then ϕ is a real-valued continuous functional on the convex space $\mathcal{P}$. From (2.4) it follows that

$$\phi\left(\frac{1+z}{1-z}\right) < 0 < \phi\left(\frac{1-z}{1+z}\right)$$

For $0 \le \lambda \le 1$,

$$\phi\left(\lambda\frac{1-z}{1+z} + (1-\lambda)\frac{1+z}{1-z}\right)$$

is a real-valued continuous function of λ for $0 \le \lambda \le 1$, with $\phi(0) < 0 < \phi(1)$. Then there is λ_1 so that $\phi(\lambda_1) = 0$. The function $P_1(z) = \lambda_1(1-z)/(1+z) + (1-\lambda_1)(1+z)/(1-z)$ is a member of $\mathcal{P}$ and the corresponding Q defined by (2.3) satisfies $Q(c) = 0$.

3. The class $S_H(U,\Omega(a,b))$. In the sequel the numbers c, c_1 and c_2 are those given by Lemmas 1–3.

Let $\mathcal{F}(a,b)$ be the class of functions which have the form

$$f(z)=A\left[\operatorname{Re}\int_0^z \frac{(1-\zeta^2)P(\zeta)}{2(1+c\zeta+\zeta^2)^2}\,d\zeta+i\operatorname{Im}\frac{z}{(1+cz+z^2)^2}\right] \tag{3.1}$$

where

$$A=b\Big/\int_0^1\frac{(1-t^2)\operatorname{Re}P(t)}{(1+ct+t^2)^2}\,dt$$

with $P(z)$ in $\mathcal{P}$ and c is chosen so that $c_1\le c\le c_2$ and

$$b\Big/\int_0^1\frac{(1-t^2)\operatorname{Re}P(t)}{(1+ct+t^2)^2}\,dt=a\Big/\int_0^{-1}\frac{(1-t^2)\operatorname{Re}P(t)}{(1+ct+t^2)^2}\,dt. \tag{3.2}$$

We note that by Lemmas 1–3, for each P in $\mathcal{P}$ there is a unique c, $c_1\le c\le c_2$, for which (3.2) is satisfied.

THEOREM 1. *If f is a member of $\mathcal{F}(a,b)$, then f is harmonic, sense-preserving and univalent in U. Moreover, $f(U)$ is convex in the direction of the real axis and $f(U)\subset\Omega(a,b)$.*

Proof. Let $f=h+\overline{g}=\operatorname{Re}F+i\operatorname{Re}G$; then

$$F(z)=A\int_0^z\frac{(1-\zeta)^2P(\zeta)}{(1+c\zeta+\zeta^2)^2}\,d\zeta\quad\text{and}\quad G(z)=\frac{-iAz}{1+cz+t^2}.$$

Since

$$\frac{g'(z)}{h'(z)}=\frac{F'(z)-iG'(z)}{F'(z)+iG'(z)}=\frac{P(z)-1}{P(z)+1},$$

it follows that $|g'(z)|<|h'(z)|$ for z in U. Thus f is locally one-to-one and sense preserving in U.

Also,

$$h(z)-g(z)=iG(z)=\frac{Az}{1+cz+z^2}$$

maps U onto a domain which is convex in the direction of the real axis. By a theorem of Clunie and Sheil-Small [4], f is univalent and $f(U)$ is convex in the direction of the real axis. Also, $f(z)$ is real if and only if z is real. Since $A>0$ and $\operatorname{Re}P(z)>0$, it follows that $f(r)=\operatorname{Re}F(r)$ is increasing in $[-1,1]$ and by (3.2), $\lim_{r\to-1^+}f(r)=a$ and $\lim_{r\to1^-}f(r)=b$. Thus $f(U)$ omits $(-\infty,a]$ and $[b,+\infty)$. Hence $f(U)\subset\Omega(a,b)$.

THEOREM 2. $S_H(U,\Omega(a,b))\subset\mathcal{F}(a,b)$.

Proof. Let f be a member of $S_H(U,\Omega(a,b))$ and $f=h+\overline{g}$. Since $\Omega(a,b)$ is convex in the direction of the real axis, by a result of Clunie and

Sheil-Small [4], $h - g = iG$ is univalent and convex in the direction of the real axis. Thus G is convex in the direction of the imaginary axis.

Let $h(z) = a_1 z + a_2 z^2 + \ldots,\ a_1 > 0$, and $g(z) = b_2 z^2 + b_3 z^3 + \ldots$; then $G = -i(h-g) = -a_1 iz + \ldots$ Since $f(U) = \Omega(a,b)$, it follows that $\operatorname{Re} G(z) = \operatorname{Im} f(z)$ is 0 on the boundary of U. Since G is convex in the direction of the imaginary axis, it follows that $G(U)$ is $\mathbb{C}$ slit along one or two infinite rays along the imaginary axis. Thus $G(z)/(-a_1 i)$ maps U into $\mathbb{C}$ slit along one or two infinite rays along the real axis. However, $G(z)/(-a_1 i)$ is a member of the class S of functions $q(z)$ analytic and univalent in U and normalized by $q(0) = q'(0) - 1 = 0$. Making use of subordination arguments, it follows that $G(z)/(-a_1 i) = z/(1 + cz + z^2)$, $-2 \le c \le 2$. Hence, $\operatorname{Im} f(r) = \operatorname{Re} G(r) = 0$ for $-1 < r < 1$. Since f is one-to-one and $f_z(0) > 0$, the function $f(r)$ is increasing on $(-1,1)$. Thus $\lim_{r\to -1^+} f(r) = a$ and $\lim_{r\to 1^-} f(r) = b$.

Since $|g'(z)/h'(z)| < 1$, it follows that

$$P(z) = (h'(z) + g'(z))/(h'(z) - g'(z))$$

is in $\mathcal{P}$. Thus, $h'(z) + g'(z) = (h'(z) - g'(z))P(z) = iG'(z)P(z)$.

Hence,

$$F(z) = h(z) + g(z) = \int_0^z iG'(\zeta)P(\zeta)\,d\zeta = a_1 \int_0^z \frac{(1-\zeta^2)P(\zeta)}{(1+c\zeta+\zeta^2)^2}\,d\zeta.$$

Therefore,

$$f(z) = a_1\left[\operatorname{Re}\int_0^z \frac{(1-\zeta^2)P(\zeta)}{(1+c\zeta+\zeta^2)^2}\,d\zeta + i\operatorname{Im}\frac{z}{1+cz+z^2}\right]$$

for some c, $-2 \le c \le 1$.

Since $a = \lim_{r\to -1^+} f(r)$ and $b = \lim_{r\to 1^-} f(r)$, we have

$$a_1 \int_0^{-1} \frac{(1-t^2)\operatorname{Re} P(t)}{(1+ct+t^2)^2}\,dt = a \quad \text{and} \quad a_1 \int_0^{1} \frac{(1-t^2)\operatorname{Re} P(t)}{(1+ct+t^2)^2}\,dt = b.$$

Thus c must be such that

$$a\int_0^1 \frac{(1-t^2)\operatorname{Re} P(t)}{(1+ct+t^2)^2}\,dt + b\int_0^1 \frac{(1-t^2)\operatorname{Re} P(-t)}{(1-ct+t^2)^2}\,dt = 0. \tag{3.3}$$

By Lemmas 2 and 3 there is a unique c, $c_1 \le c \le c_2$, satisfying (3.3). Thus f is a member of $\mathcal{F}(a,b)$.

LEMMA 4. *$\mathcal{F}(a,b)$ is closed.*

Proof. Let f_n be a sequence in $\mathcal{F}(a,b)$ with f_n converging to f uniformly on compact subsets of U. Suppose

$$f_n(z) = b\left(\int_0^1 \frac{(1-t)^2 \operatorname{Re} P_n(t)}{(1+d_n t+t^2)^2}\,dt\right)^{-1} \times \left[\operatorname{Re}\int_0^z \frac{(1-\zeta^2)P_n(\zeta)}{(1+d_n\zeta+\zeta^2)^2}\,d\zeta + i\operatorname{Im}\frac{z}{1+d_n z+z^2}\right],$$

where P_n is in $\mathcal{P}$ and d_n satisfies (3.2) with $c_1 \le d_n \le c_2$. Since $\mathcal{P}$ is normal and $c_1 \le d_n \le c_2$ we may assume that P_n converges uniformly on compact subsets of U to $P(z)$ in $\mathcal{P}$ and d_n converges to some c. It follows that (3.2) is satisfied for this c and $P(z)$ and that f has the form (3.1) and hence is a member of $\mathcal{F}(a,b)$.

THEOREM 3. $\overline{S_H(U, \Omega(a,b))} = \mathcal{F}(a,b)$.

Proof. Let $f(z)$ have the form (3.1) where (3.2) is satisfied and let r_n be a sequence with $0 < r_n < 1$ and $\lim r_n = 1$. Let $P_n(z) = P(r_n z)$ and denote by $f_n(z)$ the function obtained from (3.1) and (3.2) by replacing $P(z)$ with $P_n(z)$. Let c_n be the value of c satisfying (3.2) when P is replaced by P_n. We claim that f_n is a member of $S_H(U, \Omega(a,b))$. To see this let

$$A_n = b/\operatorname{Re}\int_0^1 \frac{(1-\zeta^2)P_n(\zeta)}{(1+c_n\zeta+\zeta^2)^2}\,d\zeta, \qquad F_n(z) = A_n\int_0^z \frac{(1-\zeta^2)P_n(\zeta)}{(1+c_n\zeta+\zeta^2)^2}\,d\zeta.$$

Let $s_n = [-c_n + i\sqrt{4-c_n^2}\,]/2$; then $(1+c_n\zeta+\zeta^2) = (\zeta - s_n)(\zeta - \overline{s}_n)$. Since P_n is analytic for $|z| \le 1$, there exists $\delta > 0$ so that for $|z - s_n| < \delta$,

$$P_n(z) = P_n(s_n) + P_n'(s_n)(z-s_n) + \frac{P_n''(s_n)}{2}(z-s_n)^2 + \ldots$$

Thus, for $0 < |z - s_n| < \delta$,

$$\begin{aligned} F_n'(z) &= \frac{A_n(1-z^2)P_n(z)}{(z-\overline{s}_n)^2(z-s_n)^2} \\ &= A_n\left[\frac{B_{-2}}{(z-s_n)^2} + \frac{B_{-1}}{(z-s_n)} + B_0 + B_1(z-s_n) + \ldots\right]. \end{aligned}$$

Let $D = \{z : |z - s_n| < \delta\} - \{z : z = s_n + te^{i\arg s_n},\ 0 \le t \le \delta\}$. If $z_0 = s_n + te^{i\arg s_n}$, $-\delta < t < 0$, z_0 fixed, then for $z \in D$,

$$F_n(z) - F_n(z_0) = \int_{z_0}^{z} F_n'(\zeta)\,d\zeta$$

where the path of integration is in D. Thus for z in D,

$$F_n(z) = A_n\left[\frac{d_{-1}}{z-s_n} + d\log(z-s_n) + q(z)\right]$$

where $q(z)$ is analytic at $z = s_n$, and

$$d_{-1} = \frac{1 - s_n^2}{4 - c_n^2} P_n(s_n).$$

Thus $\operatorname{Re} d_{-1} > 0$. We take the branch of log such that for z in D,

$$\log(z - s_n) = \ln|z - s_n| + i \arg(z - s_n)$$

where $\arg s_n < \arg(z - s_n) < \arg s_n + 2\pi$. Thus for z in D,

$$\begin{aligned}\operatorname{Re} f_n(z) = \operatorname{Re} F_n(z) = A_n \bigg[&\operatorname{Re} \frac{d_1}{z - s_n} + (\operatorname{Re} d) \ln|z - s_n| \\ &- (\operatorname{Im} d) \arg(z - s_n) + \operatorname{Re} q(z)\bigg].\end{aligned}$$

We want to prove that $f_n(z)$ cannot have a finite cluster point at $z = s_n$.

Let $z_j = s_n + t_j e^{i\theta_j}$ be in $U \cap D$ with $t_j > 0$ and $\lim t_j = 0$ and such that

$$\lim_{j\to\infty} \operatorname{Im} \left(\frac{z_j}{(1 + c_n z_j + z_j^2)} \right) = l. \tag{3.4}$$

Straightforward computation gives

$$\operatorname{Im} \left[\frac{z_j}{1 + c_n z_j + z_j^2} \right] = \frac{-2(\operatorname{Im} s_n) \operatorname{Re}(s_n e^{-i\theta_j}) + t_j T_j}{t_j |2i \operatorname{Im} s_n + t_j e^{i\theta_j}|^2}$$

where T_j is bounded. Because of (3.4), we must have

$$\lim_{j\to\infty} \operatorname{Re}(s_n e^{-i\theta_j}) = 0.$$

We now note that

$$\begin{aligned} d_{-1} e^{-i\theta_j} &= \frac{(1 - s_n^2) e^{-i\theta_j} P_n(s_n)}{4 - c_n^2} = \frac{(1/s_n - s_n) s_n e^{-i\theta_j} P_n(s_n)}{4 - c_n^2} \\ &= \frac{(\bar{s}_n - s_n) s_n e^{-i\theta_j} P_n(s_n)}{4 - c_n^2} = \frac{-2i(\operatorname{Im} s_n) s_n e^{-i\theta_j} P_n(s_n)}{4 - c_n^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \operatorname{Re}(d_{-1} e^{-i\theta_j}) &= \frac{2(\operatorname{Im} s_n) \operatorname{Im}(s_n e^{-i\theta_j} P_n(s_n))}{4 - c_n^2} \\ &= \frac{\operatorname{Im}(s_n e^{-i\theta_j} P_n(s_n))}{\sqrt{4 - c_n^2}}. \end{aligned}$$

Since $\lim_{j\to\infty} \operatorname{Re}(s_n e^{-i\theta_j}) = 0$, it follows that the only possible accumulation points of $\{s_n e^{-i\theta_j}\}$ are $\pm i$. Thus the only possible accumulation points of $\{s_n e^{-i\theta_j} P_n(s_n)\}$ are $\pm i P_n(s_n)$. Moreover, $\operatorname{Im}(\pm i P_n(s_n)) = \pm \operatorname{Re} P_n(s_n) \neq 0$. Thus $\operatorname{Re}(d_{-1} e^{-i\theta_j})$ is bounded away from 0.

It now follows that

$$
\begin{aligned}
&|\operatorname{Re} f_n(z_j)| \\
&\quad = |\operatorname{Re} F_n(z_j)| \\
&\quad = A_n \left| \frac{\operatorname{Re}(d_{-1}e^{-i\theta_j})}{t_j} + (\operatorname{Re} d)\ln(t_j) - (\operatorname{Im} d)\arg(t_j e^{-i\theta_j}) + \operatorname{Re} q(z_j) \right| \\
&\quad = A_n \left| \frac{\operatorname{Re}(d_{-1}e^{-i\theta_j}) + (\operatorname{Re} d) t_j \ln(t_j) - t_j(\operatorname{Im} d)\arg(t_j e^{-i\theta_j})}{t_j} + \operatorname{Re} q(z_j) \right|
\end{aligned}
$$

approaches ∞ as $j \to \infty$. Thus f_n has no finite cluster points at $z = s_n$.

Similarly, f_n has no finite cluster points at $z = \overline{s}_n$. At all other points of $|z| = 1$, the finite cluster points of $f_n(z)$ are real. Since $f_n(U) \subset \Omega(a,b)$ and $\lim_{r\to -1^+} f_n(r) = a$ and $\lim_{r\to 1^-} f_n(r) = b$, it follows that $f_n(U) = \Omega(a,b)$.

Thus for each n, f_n is a member of $S_H(U,\Omega(a,b))$. We know that the P_n converge to P uniformly on compact subsets of U. There exists a subsequence c_{n_k} convergent to some s. But then (3.2) will be satisfied with c replaced by s. Since the solution to (3.2) is unique, we must have $s = c$. Thus f_{n_k} converges to f uniformly on compact subsets of U. Therefore, f is a member of $\overline{S_H(U,\Omega(a,b))}$ and $\mathcal{F}(a,b) \subset \overline{S_H(U,\Omega(a,b))}$. Since $\mathcal{F}(a,b)$ is closed and $S_H(U,\Omega(a,b)) \subset \mathcal{F}(a,b)$, we have $\overline{S_H(U,\Omega(a,b))} \subset \mathcal{F}(a,b)$. Thus $\mathcal{F}(a,b) = \overline{S_H(U,\Omega(a,b))}$.

4. The case $a=-b$. Referring to the proof of Lemma 1, if $a = -b$ then

$$T(0) = \int_0^1 \frac{-4bt}{(1+t^2)^2}\,dt < 0.$$

Thus $c_2 > 0$. Moreover, since $S(-x) = -T(x)$, we have $c_1 = -c_2$.

Since $S_H(U,\Omega(-b,b))$ are the only classes that contain odd functions, we will be interested in f in $\mathcal{F}(-b,b)$ and f odd.

LEMMA 5. *Let $f \in \mathcal{F}(-b,b)$ and be odd. If $f(z) = h(z) + \overline{g(z)}$, then both h and g are odd.*

Proof. Since $f(-z) = -f(z)$, we have $h(z) + \overline{g(z)} = -(h(-z) + \overline{g(-z)})$. Thus $h(z) + h(-z) = -\overline{g(z) + g(-z)}$. It follows that $h(z) + h(-z)$ and $\overline{h(z) + h(-z)}$ are both analytic in U. Thus $h(z) + h(-z)$ is constant. Since its value is 0 at $z = 0$, we have $h(z) = -h(-z)$. Similarly, $g(z)$ is odd.

LEMMA 6. *If $f \in \mathcal{F}(-b,b)$ and f is odd then in the representation* (3.1), *$P(z)$ is even and $c = 0$.*

P r o o f. Let $h(z) = a_1 z + a_2 z^2 + \ldots$; then

$$h(z) = \frac{F(z) + iG(z)}{2} = \frac{a_1}{2}\left[\int_0^z \frac{(1-\zeta^2)P(\zeta)}{(1+c\zeta+\zeta^2)^2}\,d\zeta + \frac{z}{1+cz+z^2}\right]$$

where c and P satisfy (3.2). Since $(1-z^2)/(1+cz+z^2)^2 = (z/(1+cz+z^2))'$, this can be written as

$$h(z) = \frac{a_1}{2}\int_0^z \frac{(1-\zeta^2)(P(\zeta)+1)}{(1+c\zeta+\zeta^2)^2}\,d\zeta.$$

By Lemma 5, $h(z) = -h(-z)$. Thus,

$$\int_0^z \frac{(1-\zeta^2)(P(\zeta)+1)}{(1+c\zeta+\zeta^2)^2}\,d\zeta = -\int_0^{-z} \frac{(1-\zeta^2)(P(\zeta)+1)}{(1+c\zeta+\zeta^2)^2}\,d\zeta.$$

Let $z = r,\ 0 < r < 1$; then

$$\int_0^r \frac{(1-t^2)(P(t)+1)}{(1+ct+t^2)^2}\,dt = \int_0^r \frac{(1-t^2)(P(-t)+1)}{(1-ct+t^2)^2}\,dt.$$

Taking real parts, we get

$$\int_0^r \frac{(1-t^2)(\operatorname{Re} P(t)+1)}{(1+ct+t^2)^2}\,dt = \int_0^1 \frac{(1-t^2)(\operatorname{Re} P(-t)+1)}{(1-ct+t^2)^2}\,dt.$$

Letting $r \to 1$, since $-2 < -c_2 \le c \le c_2 < 2$, we obtain

$$\int_0^1 \frac{(1-t^2)(\operatorname{Re} P(t)+1)}{(1+ct+t^2)^2}\,dt = \int_0^1 \frac{(1-t^2)(\operatorname{Re} P(-t)+1)}{(1-ct+t^2)^2}\,dt. \tag{4.1}$$

But (3.2) with $a = -b$ gives

$$\int_0^1 \frac{(1-t^2)\operatorname{Re} P(-t)}{(1-ct+t^2)^2}\,dt = \int_0^1 \frac{(1-t^2)\operatorname{Re} P(t)}{(1+ct+t^2)^2}\,dt. \tag{4.2}$$

Equalities (4.1) and (4.2) imply

$$\int_0^1 \frac{1-t^2}{(1+ct+t^2)^2}\,dt = \int_0^1 \frac{1-t^2}{(1-ct+t^2)^2}\,dt.$$

Thus $1/(2+c) = 1/(2-c)$. Hence $c = 0$.

We now have

$$h(z) = \frac{a_1}{2}\left[\int_0^z \frac{(1-\zeta^2)P(\zeta)}{(1+\zeta^2)^2}\,d\zeta + \frac{z}{1+z^2}\right]$$

and $h(z)$ is odd. Thus

$$q(z)=\int_0^z \frac{(1-\zeta^2)P(\zeta)}{(1+\zeta^2)^2}\,d\zeta$$

is odd. Hence $q'(z)=(1-z^2)P(z)/(1+z^2)^2$ is even and thus $P(z)$ is even.

Lemma 7. *Let $f\in\mathcal{F}(-b,b)$ with representation* (3.1). *If $P(z)$ is even, then $c=0$ and f is odd.*

Proof. If $P(z)$ is even, then $Q(x)$ defined by (2.3), with $a=-b$, satisfies

$$Q(0)=-\int_0^1 \frac{(1-t^2)\operatorname{Re}P(t)}{(1+t^2)^2}\,dt+\int_0^1 \frac{(1-t^2)\operatorname{Re}P(-t)}{(1+t^2)^2}\,dt=0.$$

But the c given in Lemma 2 is unique. Thus $c=0$. Therefore

$$f(z)=a_1\biggl[\operatorname{Re}\int_0^z \frac{(1-\zeta^2)P(\zeta)}{(1+\zeta^2)^2}\,d\zeta+i\operatorname{Im}\frac{z}{(1+z^2)}\biggr], \tag{4.3}$$

and since $P(z)$ is even, it is easily checked that $f(-z)=-f(z)$.

We now let

$$G(-b,b)=\{f\in\mathcal{F}(-b,b): f \text{ is odd}\}.$$

If $f\in G(-b,b)$, then f has the representation (4.3) with $P(z)$ in $\mathcal{P}$ and $P(z)$ even. Also,

$$a_1=b\Big/\int_0^1 \frac{(1-t^2)\operatorname{Re}P(t)}{(1+t^2)^2}\,dt. \tag{4.4}$$

We now easily obtain

Theorem 4. *If $f\in G(-b,b)$, then*

$$\frac{4b}{\pi}\le a_1\le\frac{8b}{\pi} \tag{4.5}$$

and the inequalities are sharp.

Proof. Since $P\in\mathcal{P}$ and P is even, $(1-|z|^2)/(1+|z|^2)\le\operatorname{Re}P(z)\le(1+|z|^2)/(1-|z|^2)$. Thus

$$\frac{\pi}{8}=\int_0^1\frac{(1-t^2)^2}{(1+t^2)^3}\,dt\le\int_0^1\frac{(1-t^2)\operatorname{Re}P(t)}{(1+t^2)^2}\,dt\le\int_0^1\frac{dt}{1+t^2}=\frac{\pi}{4}$$

and the result follows from (4.4). Equality is attained on the right side of (4.5) when $P(z)=(1-z^2)/(1+z^2)$ and on the left side when $P(z)=(1+z^2)/(1-z^2)$. The corresponding extremal functions are

$$f_1(z)=\frac{8b}{\pi}\biggl[\operatorname{Re}\biggl[\frac{z(1-z^2)}{2(1+z^2)^2}+\frac12\arctan z\biggr]+i\operatorname{Im}\frac{z}{1+z^2}\biggr] \tag{4.6}$$

and

$$f_2(z) = \frac{4b}{\pi}\left[\operatorname{Re}(\arctan(z)) + i\operatorname{Im}\frac{z}{1+z^2}\right]. \tag{4.7}$$

We find in Section 5 that $f_1(z)$ is actually a member of $S_H(U, \Omega(-b,b))$. Thus the right side of (4.5) is sharp for odd functions in $S_H(U, \Omega(-b,b))$.

THEOREM 5. *Let $f(z) = h(z) + \overline{g(z)}$ be in $G(-b,b)$ and suppose*

$$h(z) = \sum_{n=0}^{\infty} a_{2n+1} z^{2n+1} \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_{2n+1} z^{2n+1}.$$

Then

$$|a_{2n+1}| \le \frac{(n+1)^2}{2n+1}|a_1|, \qquad n = 0,1,2,\ldots, \tag{4.8}$$

$$|b_{2n+1}| \le \frac{n^2}{2n+1}|a_1|, \qquad n = 1,2,\ldots, \tag{4.9}$$

and

$$|a_{2n+1} - b_{2n+1}| = |a_1| \tag{4.10}$$

and the inequalities are sharp in $S_H(U, \Omega(-b,b))$.

Proof. We have

$$h(z) = \frac{a_1}{2}\left[\int_0^z \frac{(1-\zeta^2)P(\zeta)}{(1+\zeta^2)^2}\,d\zeta + \frac{z}{1+z^2}\right] \tag{4.11}$$

where $P(z)$ is in $\mathcal{P}$ and is even. Let $P(z) = 1 + \sum_{n=1}^{\infty} p_{2n} z^{2n}$; then for $|z| < 1$,

$$\frac{1-z^2}{(1+z^2)^2}P(z) = 1 + \sum_{n=1}^{\infty} d_{2n} z^{2n}$$

where

$$d_{2n} = \sum_{k=0}^{n} (-1)^k (2k+1) p_{2(n-k)} \quad \text{and} \quad p_0 = 1.$$

Then (4.11) gives

$$\begin{aligned} \frac{2a_{2n+1}}{a_1} &= \frac{1}{2n+1}\sum_{k=0}^{n}(-1)^k(2k+1)p_{2(n-k)} + (-1)^n \\ &= \frac{1}{2n+1}\sum_{k=0}^{n-1}(-1)^k(2k+1)p_{2(n-k)} + 2(-1)^n. \end{aligned} \tag{4.12}$$

Since $|p_n| \le 2$ for all n, we have

$$\frac{2|a_{2n+1}|}{|a_1|} \le \frac{2}{2n+1}\sum_{k=0}^{n-1}(2k+1) + 2 = \frac{2n^2}{2n+1} + 2 = \frac{2(n+1)^2}{2n+1},$$

giving (4.8).

To see the sharpness, let $P(z) = (1-z^2)/(1+z^2)$. With this choice of P, we have $p_{2n} = 2(-1)^n$ and from (4.12),

$$\begin{aligned}\frac{2a_{2n+1}}{a_1} &= \frac{1}{2n+1}\sum_{k=0}^{n-1}(-1)^k(2k+1)(-1)^{n-k}\cdot 2 + 2(-1)^n \\ &= (-1)^n\left[\frac{2}{(2n+1)}\sum_{k=0}^{n-1}(2k+1) + 2\right] = \frac{2(-1)^n(n+1)^2}{2n+1},\end{aligned}$$

giving equality in (4.8). The extremal function is the $f_1(z)$ given in (4.6).

Next we have

$$g(z) = \frac{a_1}{2}\left[\int_0^z \frac{(1-\zeta^2)P(\zeta)}{(1+\zeta^2)^2}\,d\zeta - \frac{z}{1+z^2}\right].$$

If $g(z) = \sum_{n=1}^{\infty} b_{2n+1}z^{2n+1}$, then

$$\frac{2b_{2n+2}}{a_1} = \frac{1}{2n+1}\sum_{k=0}^{n-1}(-1)^k(2k+1)p_{2(n-k)}. \tag{4.13}$$

Thus

$$\frac{2|b_{2n+1}|}{|a_1|} \le \frac{2}{2n+1}\sum_{k=0}^{n-1}(2k+1) = \frac{2n^2}{2n+1},$$

giving (4.9). Equality again occurs when $P(z) = (1-z^2)/(1+z^2)$ and $f_1(z)$ is given in (4.6).

Finally, from (4.10) and (4.11),

$$|a_{2n+1} - b_{2n+1}| = |(-1)^n a_1| = |a_1|.$$

We remark that the inequalities involved are actually sharp for odd functions in $S_H(U, \Omega(-b,b))$ since $f_1 \in S_H(U, \Omega(-b,b))$.

THEOREM 6. *Let $f(z) = h(z) + \overline{g(z)}$ be a member of $G(-b,b)$. Then for $|z| = r < 1$,*

$$\frac{|a_1|(1-r^2)}{(1+r^2)^3} \le |f_z(z)| \le \frac{|a_1|(1+r^2)}{(1-r^2)^3} \tag{4.14}$$

and the inequalities are sharp.

Proof. We have

$$h(z)=\frac{a_1}{2}\left[\int_0^z\frac{(1-\zeta^2)P(\zeta)}{(1+\zeta^2)^2}\,d\zeta+\frac{z}{1+z^2}\right].$$

Thus,

$$f_z=h'(z)=\frac{a_1(1-z^2)}{2(1+z^2)^2}(P(z)+1). \tag{4.15}$$

Since $P(z)$ is in $\mathcal{P}$ and is even, we can write $P(z)=(1-w(z))/(1+w(z))$ where $w(z)=d_2z^2+\ldots$ is analytic in U and $|w(z)|\le|z|^2$ for z in U. Thus $P(z)+1=2/(1+w(z))$. Hence

$$\frac{2}{1+r^2}\le\frac{2}{1+|w(z)|}\le|P(z)+1|\le\frac{2}{1-|w(z)|}\le\frac{2}{1-r^2}. \tag{4.16}$$

Using (4.10) and (4.15) we obtain the inequalities (4.14). Equality on the right side of (4.14) is attained by $f_1(z)$ at $z=\pm ir$ and equality on the left side of (4.14) is attained by $f_1(z)$ when $z=\pm r$.

5. The extremal functions. We now verify that the extremal function $f_1(z)$ given by (4.6) is actually a member of $S_H(U,\Omega(-b,b))$, while the function $f_2(z)$ given by (4.7) maps U into the strip $\{z:-b<\operatorname{Re}z<b\}$ and hence is a member of $G(-b,b)-S_H(U,\Omega(-b,b))$.

To see this we first prove that $f_1(z)$ has no non-real finite cluster points at $z=i$. Let $z_j=i+t_je^{i\theta_j}$ be such that $0<t_j$, $\pi<\theta_j<2\pi$, $|z_j|<1$, and $\lim_{j\to\infty}\operatorname{Im}(z_j/(1+z_j^2))=l\ne0$. Necessarily $l>0$. A brief computation gives

$$A_j=\operatorname{Im}\left(\frac{z_j}{1+z_j^2}\right)=\frac{-(t_j+2\sin\theta_j)(1+t_j\sin\theta_j)}{t_j|z_j+i|^2}.$$

Thus $-(t_j+2\sin\theta_j)(1+t_j\sin\theta_j)=t_j|z_j+i|^2A_j=t_jB_j$ where $\lim B_j=4l>0$. Hence

$$-2\sin\theta_j[1+t_j\sin\theta_j]=t_jB_j+t_j[1+t_j\sin\theta_j]=t_jc_j,$$

where $\lim c_j=4l+1$. Therefore

$$\sin\theta_j=\frac{t_jc_j}{-2(1+t_j\sin\theta_j)}=t_jD_j \tag{5.1}$$

where $\lim D_j=-(4l+1)/2$. In particular, $\lim\sin\theta_j=0$, so $\lim|\cos\theta_j|=1$. Let

$$T(z)=\frac{z(1-z^2)}{(z-i)^2(z+i)^2};$$

then in a neighborhood of $z=i$,

$$T(z)=\frac{-i}{2(z-i)^2}-\frac{1}{2(z-i)}+q(z)$$

where $q(z)$ is analytic at $z = i$. Further,

$$T(z_j) = \frac{-ie^{-i2\theta_j}}{2t_j^2} - \frac{e^{-i\theta_j}}{2t_j} + q(z_j).$$

Using (5.1), we can write

$$\begin{aligned}\operatorname{Re} T(z_j) &= \frac{\sin\theta_j \cos\theta_j}{t_j^2} - \frac{\cos\theta_j}{2t_j} + \operatorname{Re} q(z_j)\\ &= \frac{-D_j\cos\theta_j}{t_j} - \frac{\cos\theta_j}{2t_j} + \operatorname{Re} q(z_j) = \frac{-\cos\theta_j(2D_j+1)}{2t_j} + \operatorname{Re} q(z_j).\end{aligned}$$

Since $\lim(2D_j+1) = -4l \neq 0$ and $\lim|\cos\theta_j| = 1$ it follows that $\lim|\operatorname{Re} T(z_j)| = \infty$ and hence $\lim|\operatorname{Re} f_1(z_j)| = \infty$. Thus f_1 has only real cluster points at $z = i$. Since $f_1(z)$ is odd, it has only real cluster points at $z = -i$ as well. If $z_0 \neq \pm i$ and $|z_0| = 1$, then $\lim_{z\to z_0} f_1(z) = \pm b$. Since $f_1(U) \subset \Omega(-b,b)$ and since the interval $(-b,b)$ is covered by $f_1(U)$, it follows that $f_1(U) = \Omega(-b,b)$. Thus f_1 is a member of $S_H(U,\Omega(-b,b))$.

We now prove that $f_2(U) = \{z : -b < \operatorname{Re} z < b\}$ where $f_2(z)$ is given by (4.7). We have

$$\operatorname{Re} f_2(z) = \frac{4b}{\pi}\operatorname{Re}(\arctan z) = \frac{4b}{\pi}\operatorname{Re}\left(\frac{i}{2}\log\frac{1-iz}{1+iz}\right) = \frac{-2b}{\pi}\arg\left(\frac{1-iz}{1+iz}\right).$$

Since $\operatorname{Re}[(1-iz)/(1+iz)] > 0$, it follows that

$$|\operatorname{Re} f_2(z)| = \frac{2b}{\pi}\left|\arg\frac{1-iz}{1+iz}\right| < \frac{2b}{\pi}\cdot\frac{\pi}{2} = b.$$

We claim that the cluster points of $f_1(z)$ at $z = \pm i$ form the two lines $\operatorname{Re} z = \pm b$. To see this, let $l > 0$. We can choose a sequence $z_j = i + t_j e^{-i\theta_j}$ with $\pi < \theta_j < 2\pi$, $t_j > 0$ and $\lim t_j = 0$, such that

$$\lim_{j\to\infty} \operatorname{Im}\frac{z_j}{1+z_j^2} = l.$$

As in the previous example, $\lim\sin\theta_j = 0$ and $\lim|\cos\theta_j| = 1$. We have

$$\operatorname{Re} f_2(z_j) = -\frac{2b}{\pi}\arg\left(\frac{1-iz_j}{1+iz_j}\right).$$

Moreover,

$$\tan\left[\arg\left(\frac{1-iz_j}{1+iz_j}\right)\right] = \frac{-2\operatorname{Re} z_j}{1-|z_j|^2} = \frac{-2t_j\cos\theta_j}{-2t_j\sin\theta_j - t_j^2} = \frac{2\cos\theta_j}{2\sin\theta_j + t_j}.$$

Making use of computations from the last example, we get

$$\tan\left[\arg\left(\frac{1-iz_j}{1+iz_j}\right)\right] = \frac{2\cos\theta_j}{2t_jD_j + t_j} = \frac{2\cos\theta_j}{t_j(2D_j+1)}$$

where $\lim 2D_j + 1 = -4l < 0$.

If θ_j is chosen so that $\lim \theta_j = \pi$ then $\tan(\arg((1 - iz_j)/(1 + iz_j)))$ tends to ∞ and $\arg((1 - iz_j)/(1 + iz_j))$ tends to $\pi/2$, and thus $\operatorname{Re} f_2(z_j)$ tends to $-b$. Hence $-b + il$, $l > 0$, is a cluster point. If θ_j is chosen so that $\lim \theta_j = 2\pi$, then we see that $b + il$, $l > 0$, is a cluster point. Since f_2 is odd, it follows that $\pm b + il$, $l < 0$, are cluster points at $z = -i$. It now follows that $f_2(U) = \{z : -b < \operatorname{Re}(z) < b\}$.

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