ANNALES
POLONICI MATHEMATICI
LXVII.2 (1997)

On a method of determining supports of Thoma's characters of discrete groups

by Ernest Płonka (Gliwice)

Abstract. We present a new approach to determining supports of extreme, normed by 1, positive definite class functions of discrete groups, i.e. characters in the sense of E. Thoma [8]. Any character of a group produces a unitary representation and thus a von Neumann algebra of linear operators with finite normal trace. We use a theorem of H. Umegaki [9] on the uniqueness of conditional expectation in finite von Neumann algebras. Some applications and examples are given.

I. It is well known that any positive definite central function α of a group G yields, via the Gelfand–Segal construction, a unitary representation U^α of G in a separable Hilbert space and α can be extended to a finite, normal trace on the von Neumann algebra $\{U_g^\alpha : g \in G\}''$. Such functions form a compact convex set in the topology of pointwise convergence and correspond to the finite traces of the C^*-algebra of the group G. The extreme points of the sphere $\{\alpha : \alpha(1) = 1\}$ are called *characters in the sense of E. Thoma*. If α is a character, the algebra $\{U_g^\alpha : g \in G\}''$ is a factor (cf. [8]). If $\{g \in G : \alpha(g) = 1\} = 1$, the representation U^α is faithful and so α is called *faithful*. It follows from [1] that if α is 0 off some subgroup H of G, then the representation U^α is the induced representation from the restriction $\alpha|H$ of α to H.

The problem of determining the supports of characters has been studied in many papers e.g. [2], [4], [5] and [6]. It has been proved in [5] that under some restrictions on G each faithful character of a nilpotent group G of class 3 is supported on the normal subgroup G_{f} of G consisting of all elements with finite conjugacy classes. Such groups are called *centrally inductive* [2]. R. Howe [4] has shown that all finitely generated torsion free nilpotent groups are centrally inductive. In 1985 A. Carey and W. Moran [2] have established the same for countable nilpotent groups G such that there exists an integer n with the property that every finitely generated subgroup

1991 *Mathematics Subject Classification*: Primary 43A35; Secondary 43A40.
Key words and phrases: positive definite functions, characters, traces.

of G/G_{f} is contained in a subgroup of G/G_{f} with n generators. They have also proved this result for nilpotent complete groups, i.e. groups which contain all nth roots of their elements. The groups of unipotent upper triangular matrices with coefficients from a field of characteristic 0 are examples of such groups. Carey and Moran also gave an example of non-centrally inductive nilpotent groups. All the above papers do not use von Neumann algebras. We show how from a theorem of H. Umegaki on conditional expectation in von Neumann algebras one can obtain a result which seems to be a useful tool in investigating the supports of characters.

Let $\mathbf{B}$ be a subalgebra of a von Neumann algebra $\mathbf{A}$ with a finite normal trace τ. In [9], H. Umegaki proves that there is precisely one linear mapping $E : \mathbf{A} \to \mathbf{B}$ preserving the trace τ and the involution $*$ such that the equality

$$E(b_1 a b_2) = b_1 E(a) b_2$$

holds for all $a \in \mathbf{A}$ and $b_1, b_2 \in \mathbf{B}$. The mapping E is said to be the *conditional expectation* of $\mathbf{A}$ with respect to $\mathbf{B}$.

II. We start with a simple

LEMMA. *Let α be a $*$-preserving automorphism of a von Neumann algebra $\mathbf{A}$ with a finite, normal trace τ such that $\tau(\alpha(a)) = \tau(a)$ for $a \in \mathbf{A}$. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}$, with $\alpha(b) = b$ for b in $\mathbf{B}$. Then for the conditional expectation E of $\mathbf{A}$ with respect to $\mathbf{B}$ the mapping E^α defined by*

$$E^\alpha(a) = E(a^\alpha)$$

is again a conditional expectation of $\mathbf{A}$ to $\mathbf{B}$ and consequently $E^\alpha = E$.

Proof. Straightforward.

THEOREM 1. *Let H be a normal subgroup of a group G of unitaries in a separable Hilbert space such that the von Neumann algebra $\mathbf{A} = G''$ has a finite, normal trace τ. Let x be an element of G such that $[x, g] = x^{-1}g^{-1}xg$ commutes with all h in H for $g \in G$. Assume also that $[x, h_0] \neq 1$ for some $h_0 \in H$. If there is no non-trivial G-invariant projection in the von Neumann subalgebra $\mathbf{B} = \{1 - [x, h] : h \in H\}''$ of the algebra $\mathbf{A}$, then $E(x) = 0$.*

Proof. The implementation mapping

$$\mathbf{A} \ni a \to g^{-1} a g = a^g \in \mathbf{A}$$

is an automorphism of $\mathbf{A}$ preserving τ and the involution $*$. Since the element $[x, g]$ commutes with all $h \in H$ for $g \in G$, we have, by the Lemma, $E^h = E$. Hence

$$E^h(x) = E(h^{-1}xh) = E(x[x, h]) = E(x)[x, h],$$

which implies

$$E(x)(1 - [x, h]) = 0 \quad \text{for all } h \in H.$$

This yields $E(x)P_{\mathrm{im}(1-[x,h])} = 0$, where $P_{\mathrm{im}(b)}$ is the projection operator onto the image of the operator $b \in \mathbf{B}$. Since $\mathbf{B}$ is commutative, the projection $P_{\mathrm{im}(1-[x,h])}$ is the central support $c(1-[x,h])$ of the element $1-[x,h]$ (cf. [7]). Thus we have

$$E(x)c(1-[x,h]) = 0 \quad \text{ for all } h \in H,$$

and consequently $E(x)P = 0$, where $P = \mathrm{LUB}\{c(1-[x,h]) : h \in H\}$.

Now the identity $[ab, c] = [a, c][[a, c], b][b, c]$ (cf. [3]) implies

$$[x, h^g] = [x^g, h^g] = [x[x, g], h^g] = [x, h^g][[x, h^g], [x, g]][[x, g], h^g].$$

Since h^g, $[x, h^g] \in H$ and $[x, g]$ commutes with $h \in H$, we obtain the equality

$$[x, h]^g = [x, h^g] \quad \text{ for all } h \in H \text{ and } g \in G.$$

Hence

$$\begin{aligned} P^g &= (\mathrm{LUB}\{c(1-[x,h]) : h \in H\})^g = \mathrm{LUB}(\{c(1-[x,h] : h \in H\})^g \\ &= \mathrm{LUB}\{(c(1-[x,h]))^g : h \in H\} = \mathrm{LUB}\{c(1-[x,h])^g : h \in H\} \\ &= \mathrm{LUB}\{c(1-[x,h^g]) : h \in H\} = P \end{aligned}$$

for all $g \in G$. From our hypothesis it follows that the projection P is not 0, because there is an element h_0 in H such that $1 - [x, h_0] \neq 0$. Since P is G-invariant, it has to be 1 projection. Hence $E(x) = 0$, as required.

THEOREM 2. *Let α be a positive definite function on a group G such that $\alpha(x^g) = \alpha(x)$ for all $x, g \in G$ and $\alpha(1) = 1$. Suppose that the restriction $\alpha|H$ of α to a normal subgroup H of G is a faithful, extreme point of the set $\{\beta : \beta(1) = 1,\ \beta(h^g) = \beta(h),\ h \in H,\ g \in G\}$. Let $x \in G$ be an element such that $[x, g]$ and h commute for all $h \in H$, $g \in G$ and $[x, h_0] \neq 1$ for some $h_0 \in H$. Then $\alpha(x) = 0$.*

Proof. Let U^α be the representation of G corresponding to α and $\mathbf{A} = \{U^\alpha_g : g \in G\}''$ be the von Neumann algebra generated by U^α_g, $g \in G$. Then α is (can be extended to) a finite, normal trace on $\mathbf{A}$. Let $\mathbf{B} = \{1 - U^\alpha_{[x,h]} : h \in H\}''$. Since α is faithful and $1 \neq [x, h_0] \in H$, we have $U^\alpha_{[x,h_0]} \neq 1$ and, consequently, $\mathbf{B}$ is a non-trivial commutative subalgebra of $\mathbf{A}$. Moreover, since the restriction $\alpha|H$ is an extreme point of a G-invariant, positive definite function on H, there is no non-trivial G-invariant projection in $\mathbf{B}$ (Lemma 2 of [8]). Let E be the conditional expectation of $\mathbf{A}$ with respect to $\mathbf{B}$. We see that all assumptions of the previous theorem are satisfied and therefore $E(U^\alpha_x) = 0$. Since the mapping E preserves the trace α, it follows that $\alpha(x) = \alpha(U^\alpha_x) = \alpha(E(U^\alpha_x)) = \alpha(0) = 0$, which completes the proof.

III. The following examples show how Theorem 2 can be used.

EXAMPLE 1. Let R be a commutative, associative ring with identity e. The set $\{T_{a,b} : a, b \in R,\ a \text{ invertible in } R\}$ of all transformations $T_{a,b}$:

$R \to R$ defined by $T_{a,b}(x) = ax + b$ forms a group G with multiplication $T_{a,b}T_{a',b'} = T_{aa',ab'+b}$. Obviously, $H = \{T_{e,b} : b \in R\}$ is a normal subgroup of G with $[G, H] = H$. Thus the support of any faithful character of G is contained in H.

EXAMPLE 2. Let $\triangle_k$, $0 \le k \le n$, be the set of all $n \times n$ matrices A with entries from a commutative, associative ring with identity such that $A(i, j) = 0$ for $i > j - k$. Let $\square_k$, $0 \le k \le n$, be the set of all matrices $A \in \triangle_1$ with $A(i, j) = 0$ for $i > n - k$ or $j \le k$. It is clear that $\triangle_k \square_{n-k} = 0$ for $k = 0, 1, \ldots, n$ and $\triangle_k \triangle_l \subseteq \triangle_s$, where $s = k + l \pmod{n}$. Hence $(\triangle_k)^n = 0$ and therefore $G_k = \{I + x : x \in \triangle_k\}$, $1 \le k \le n$, form a group with $(I + x)^{-1} = I - x + x^2 - \ldots + (-x)^{n-1}$. Thus $[G_k, G_1] \subseteq G_{k+1}$ for $k = 1, \ldots, n - 1$. Observe that the element $I + x$ belongs to the centralizer $C_{G_1}(G_k)$ of G_k, $1 \le k \le n$, in G_1 iff $xE_{r,s} = E_{r,s}x$ for all units $E_{r,s} \in \triangle_k$, which yields $C_{G_1}(G_k) = I + \square_{n-k}$. Similarly, we get $C_{G_1}(I + \square_k) = I + \square_{n-k}$, $1 \le k \le n$. Now it follows from Theorem 2 that any faithful character of G_1 vanishes on the set $\bigcup_{k=1}^{n} ((I + \triangle_k) - (I + \square_{n-k}))$.

References

[1] R. J. Blattner, *On induced representations*, Amer. J. Math. 83 (1961), 79–98.

[2] A. L. Carey and W. Moran, *Characters of nilpotent groups*, Math. Proc. Cambridge Philos. Soc. 96 (1984), 123–137.

[3] M. Hall, *The Theory of Groups*, New York, Macmillan, 1969.

[4] R. E. Howe, *On representation of finitely generated discrete torsion free nilpotent groups*, Pacific J. Math. 73 (1977), 281–305.

[5] E. Kaniuth und R. Lasser, *Zum verallgemeinerten Wienerschen Satz über diskrete nilpotente Gruppen der Klasse 3*, Math. Z. 163 (1978), 39–55.

[6] E. Płonka, *Remarks on characters of discrete nilpotent groups*, Math. Ann. 240 (1979), 97–102.

[7] S. Sakai, *C^*-algebras and W^*-algebras*, Springer, 1971.

[8] E. Thoma, *Über unitäre Darstellungen abzählbarer Gruppen*, Math. Ann. 153 (1964), 111–139.

[9] H. Umegaki, *Condional expectation in operator algebras*, Tôhoku Math. J. 6 (1954), 1977–1981.

Institute of Mathematics
Silesian Technical University
Kaszubska 23
44-100 Gliwice, Poland
E-mail: eplonka@zeus.polsl.gliwice.pl

Reçu par la Rédaction le 20.9.1996
Révisé le 5.11.1996