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Continuous mappings with an infinite number of topologically critical points

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Abstract. We prove that the *topological φ-category* of a pair (M, N) of topological manifolds is infinite if the *algebraic φ-category* of the pair of fundamental groups $(\pi_1(M), \pi_1(N))$ is infinite. Some immediate consequences of this fact are also pointed out.

1. Introduction. In this section we recall the notions of topologically regular point and topologically critical point of a continuous mapping and the topological φ-category of a pair of topological manifolds.

Let M^m, N^n be topological manifolds and let $f : M \to N$ be a continuous map. For a given point $x_0 \in M$ consider a pair (U, φ), (V, ψ) of charts at x_0 and $f(x_0)$ respectively, satisfying the relation $f(U) \subseteq V$. Recall that the map $f_{\varphi\psi} : \varphi(U) \to \psi(V)$ defined by $f_{\varphi\psi} = \psi \circ f \circ \varphi^{-1}$ is the *local representation* of f at x_0 with respect to the charts (U, φ), (V, ψ).

Definition. The point $x_0 \in M$ is called a *topologically regular point* of f if there exists a local representation $f_{\varphi\psi}$ of f at x_0 such that for any $z = (z^1, \ldots z^m) \in \varphi(U) \subseteq \mathbb{R}^m$,

$$f_{\varphi\psi}(z) = \begin{cases} (z^1, \ldots, z^m, \underbrace{0, \ldots, 0}_{n-m}) & \text{if } m \leq n, \\ (z^1, \ldots, z^n) & \text{if } m \geq n. \end{cases} \tag{1}$$

Otherwise x_0 is called a *topologically critical point* of the map f.

Recall the following notations:

1) $R_{\mathrm{top}}(f)$ is the set of all topologically regular points,
2) $C_{\mathrm{top}}(f)$ is the set of all topologically critical points,
3) $B_{\mathrm{top}}(f) = f(C_{\mathrm{top}}(f))$ is the set of all topologically critical values of f.

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Define also the *topological φ-category* of the pair (M,N) as follows:

$$\varphi_{\mathrm{top}}(M,N)=\min\{|C_{\mathrm{top}}(f)| : f\in C(M,N)\}$$

where $|A|$ denotes the cardinality of the set A. If $|C_{\mathrm{top}}(f)|$ is infinite for all $f\in C(M,N)$, we write $\varphi_{\mathrm{top}}(M,N)=\infty$.

If M, N are differentiable manifolds and $f:M\to N$ is a differentiable mapping, then $R(f)$ and $C(f)$ denotes the set of all regular points of f and the set of all critical points of f respectively. (Regular and critical points are considered here in the usual sense, that is, they are defined by means of the rank of the tangent map.)

The *φ-category* of the pair (M,N) is given by

$$\varphi(M,N)=\min\{|C(f)| : f\in C^\infty(M,N)\}.$$

Again, $\varphi(M,N)=\infty$ if $|C(f)|$ is infinite for all $f\in C^\infty(M,N)$. A remarkable inequality which involves the φ-category of the pair $(M,\mathbb{R})$ is the following:

$$\varphi(M,\mathbb{R})\geq \mathrm{cat}(M)\geq \mathrm{cuplong}(M),$$

where $\mathrm{cat}(M)$ denotes the Lusternik–Schnirelmann category of the manifold M and $\mathrm{cuplong}(M)$ denotes the cup-length of the manifold M (see for instance [5, pp. 190–191]). Other results concerning the φ-category of the pair $(M,\mathbb{R})$ are obtained in [6]. For the equivariant (invariant) situation see also [2].

Remarks. 1) Let M^m, N^n be topological manifolds such that $m\geq n$ and $f:M\to N$ be a continuous mapping. If a point $x_0\in M$ is topologically regular, then there is an open neighbourhood U of x_0 such that the restriction $f|_U : U\to N$ is open, that is, f is locally open at x_0. If $m=n$, then $x_0\in M$ is a topologically regular point if and only if f is a local homeomorphism at x_0 (see [1, Proposition 1.3]).

2) Obviously $R_{\mathrm{top}}(f)$ is an open subset of M, while $C_{\mathrm{top}}(f)$ is closed, the two subsets being complementary to each other. A similar statement is true for $R(f)$ and $C(f)$ in the differentiable case.

3) If M, N are differentiable manifolds and $f:M\to N$ is a differentiable mapping, then, according to the well-known Rank Theorem, the relation $R(f)\subseteq R_{\mathrm{top}}(f)$ holds, or equivalently $C_{\mathrm{top}}(f)\subseteq C(f)$. Therefore

$$\varphi_{\mathrm{top}}(M,N)\leq \varphi(M,N). \tag{2}$$

2. Preliminary results. We start by proving the following theorem:

THEOREM 2.1. *Let M^m, N^n be two connected topological manifolds such that $m\geq n\geq 2$. If $f:M\to N$ is a non-surjective closed and continuous mapping, then f has infinitely many topologically critical points. In particular, if M is compact and N non-compact then $\varphi_{\mathrm{top}}(M,N)=\infty$.*

P r o o f. Let us first prove that $f^{-1}(\partial \operatorname{Im} f) \subseteq C_{\mathrm{top}}(f)$. Indeed, otherwise there exists $x_0 \in f^{-1}(\partial \operatorname{Im} f)$ such that $x_0 \in R_{\mathrm{top}}(f)$. This means that f is locally open around x_0 and therefore x_0 has an open neighbourhood U such that $f_U : U \to N$ is open, namely $f(U)$ is open. But this is a contradiction with the fact that $f(x_0) \in \partial \operatorname{Im} f$. From the inclusion $f^{-1}(\partial \operatorname{Im} f) \subseteq C_{\mathrm{top}}(f)$ it follows that

$$\partial \operatorname{Im} f \subseteq B_{\mathrm{top}}(f). \tag{3}$$

Further on, we consider the following two cases:

Case I. $B_{\mathrm{top}}(f) = \operatorname{Im} f$. If the image of f is finite, then the mapping f is constant. This means that $C_{\mathrm{top}}(f) = M$ and therefore $C_{\mathrm{top}}(f)$ is infinite. Otherwise $B_{\mathrm{top}}(f)$ is infinite, hence $C_{\mathrm{top}}(f)$ is also infinite.

Case II. $\operatorname{Im} f \backslash B_{\mathrm{top}}(f) \neq \emptyset$. In this case we show that $N \backslash B_{\mathrm{top}}(f)$ is not connected and therefore $B_{\mathrm{top}}(f)$ is infinite. Because $\operatorname{Im} f \backslash B_{\mathrm{top}}(f) \neq \emptyset$ and f is non-surjective we can consider $y \in \operatorname{Im} f \backslash B_{\mathrm{top}}(f)$ and $y' \in N \backslash \operatorname{Im} f$. Because $y \in \operatorname{Im} f$ and $y' \in N \backslash \operatorname{Im} f$ it follows that any continuous path joining y to y' intersects $\partial \operatorname{Im} f$ and consequently the set $B_{\mathrm{top}}(f)$. But since $y, y' \in N \backslash B_{\mathrm{top}}(f)$, it follows that $N \backslash B_{\mathrm{top}}(f)$ is not connected. ∎

Further on, the equivariant case will be briefly studied.

Let G be a Lie group, M a manifold and $\varphi : G \times M \to M$, $(g, x) \mapsto gx$, be a smooth action of G on M. The triple (G, M, φ) is called a *G-manifold*. The orbit of a point $x \in M$ will be denoted by Gx. If the action of G on M is free, recall that M/G can be endowed with a differential structure such that the canonical projection $\pi_M : M \to M/G$ is a smooth G-bundle (see [3, Theorem 4.11, p. 186]). A function $f : M \to N$ between G-manifolds M and N is said to be *G-equivariant* if $f(gx) = gf(x)$ for all $g \in G$ and all $x \in M$. If M and N are two G-manifolds and $f : M \to N$ is G-equivariant, denote by $\widetilde{f} : M/G \to N/G$ the function which makes the following diagram commutative:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi_M"'] & N \arrow[d, "\pi_N"] \\ M/G \arrow[r, "\widetilde{f}"] & N/G \end{tikzcd}$$

Let X be a differentiable manifold, $Y \subseteq X$ be a submanifold of X and $l : Y \hookrightarrow X$ be the inclusion mapping. The subspace $(dl)_y(T_yY)$ of the tangent space T_yX will be simply denoted by T_yY.

Definition. Let $f : M \to N$ be a differentiable mapping and P be a submanifold of N. We say that f *intersects transversally* the submanifold P at $x \in M$ if either $f(x) \notin P$ or $(df)_x(T_xM) + T_{f(x)}P = T_{f(x)}N$.

We close this section with the following result:

Theorem 2.2. *Let G be a Lie group and M, N be two G-manifolds such that the action of G on M and N is free and* $\dim M \geq \dim N$. *Consider a G-equivariant map $f : M \to N$ and let $\widetilde{f} : M/G \to N/G$ be its associated map defined above. For $x \in M$, the following assertions are equivalent*:

(i) *x is a regular point of the function f;*
(ii) *$\pi_M(x)$ is a regular point of the function $\widetilde{f}$;*
(iii) *f intersects transversally the orbit $Gf(x)$ at x.*

The proof of Theorem 2.2 is left to the reader.

3. The main result. In the first part of this section, the algebraic φ-category of a pair of groups is defined and studied. In the second part we prove the principal result of the paper.

For an abelian group G, the subset $t(G)$ of all elements of finite order forms a subgroup of G called the *torsion subgroup.*

If G, H are groups, then the algebraic φ-category of the pair (G, H) is defined as follows

$$\varphi_{\mathrm{alg}}(G, H) = \min\{[H : \operatorname{Im} f] \mid f \in \operatorname{Hom}(G, H)\}.$$

If $[H : \operatorname{Im} f]$ is infinite for all $f \in \operatorname{Hom}(G, H)$ we write $\varphi_{\mathrm{alg}}(G, H) = \infty$.

Proposition 3.1. *If G, H are finitely generated abelian groups such that*

$$\operatorname{rank}[G/t(G)] < \operatorname{rank}[H/t(H)]$$

then $\varphi_{\mathrm{alg}}(G, H) = \infty$.

Proof. Let $f : G \to H$ be a group homomorphism. Because $f(t(G)) \subseteq t(H)$ there exists a group homomorphism $\widetilde{f} : G/t(G) \to H/t(H)$ which makes the following diagram commutative:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p_G"'] & H \arrow[d, "p_H"] \\ G/t(G) \arrow[r, "\widetilde{f}"] & H/t(H) \end{tikzcd}$$

p_G and p_H being the canonical projections. Because $(H/t(H))/\operatorname{Im}\widetilde{f}$ is a finitely generated abelian group it follows, by the structure theorem, that

$$\frac{H/t(H)}{\operatorname{Im}\widetilde{f}} \cong \mathbb{Z}^{n-m} \oplus t\left(\frac{H/t(H)}{\operatorname{Im}\widetilde{f}}\right)$$

where $n = \operatorname{rank}[H/t(H)]$ and $m = \operatorname{rank}(\operatorname{Im}\widetilde{f}) \leq \operatorname{rank}[G/t(G)]$. The remainder of the proof is obvious. ■

Corollary 3.2. *If G, H are free abelian groups such that* $\operatorname{rank} G < \operatorname{rank} H < \infty$, *then $\varphi_{\mathrm{alg}}(G, H) = \infty$.*

The next theorem is the principal result of the paper.

THEOREM 3.3. *Let M^m, N^n be compact connected topological manifolds such that $m \ge n \ge 2$. If $\varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)) = \infty$ then $\varphi_{\mathrm{top}}(M, N) = \infty$.*

Proof. Let $f : M \to N$ be a continuous mapping and $f_* : \pi_1(M) \to \pi_1(N)$ be the induced homomorphism. Because $\varphi_{\mathrm{alg}}(\pi_1(M), \pi_1(N)) = \infty$ it follows that $[\pi_1(N) : \mathrm{Im}\, f_*] = \infty$. On the other hand, using the theory of covering maps, there exists a covering map $p : \widetilde{N} \to N$ such that $p_*(\pi_1(\widetilde{N})) = \mathrm{Im}\, f_*$. Because the number of sheets of the covering $p : \widetilde{N} \to N$ is the index $[\pi_1(N) : \mathrm{Im}\, f_*]$, it follows that $p : \widetilde{N} \to N$ has an infinite number of sheets, that is, $\widetilde{N}$ is a non-compact manifold. From the equality $p_*(\pi_1(\widetilde{N})) = \mathrm{Im}\, f_*$ it follows, using the lifting criterion, that there exists a mapping $\widetilde{f} : M \to \widetilde{N}$ such that $p \circ \widetilde{f} = f$. But since p is locally a homeomorphism it implies that $C_{\mathrm{top}}(f) = C_{\mathrm{top}}(\widetilde{f})$, which together with the second part of Theorem 2.1 leads to the conclusion that $C_{\mathrm{top}}(f)$ is infinite. ∎

COROLLARY 3.4. *Let M^m, N^n be compact connected topological manifolds such that $m \ge n \ge 2$. If $\pi_1(M)$ is finite and $\pi_1(N)$ is infinite, then $\varphi_{\mathrm{top}}(M, N) = \infty$.*

4. Applications. In this section some applications of Theorem 3.3 will be given.

PROPOSITION 4.1. (i) *If m, n, k are natural numbers such that $1 < k < m$ and $k + n \ge m \ge 2$, then $\varphi_{\mathrm{top}}(T^k \times S^n, T^m) = \infty$.*

(ii) *If T_g is the connected sum of g tori and $g < g'$, then $\varphi_{\mathrm{top}}(T_g, T_{g'}) = \infty$.*

(iii) *If P_g is the connected sum of g projective planes and $g < g'$, then $\varphi_{\mathrm{top}}(P_g, P_{g'}) = \infty$.*

Proof. (i) follows easily from Theorem 3.3 by taking into account the fact that $\pi_1(T^k \times S^n) = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{k \text{ times}}$ and $\pi_1(T^m) = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{m \text{ times}}$.

(ii) We show that $\varphi_{\mathrm{alg}}(\pi_1(T_g), \pi_1(T_{g'})) = \infty$. Let $f : \pi_1(T_g) \to \pi_1(T_{g'})$ be a group homomorphism. Because $f([\pi_1(T_g), \pi_1(T_g)]) \subseteq [\pi_1(T_{g'}), \pi_1(T_{g'})]$, f induces a group homomorphism

$$[f] : \pi_1(T_g)/[\pi_1(T_g), \pi_1(T_g)] \to \pi_1(T_{g'})/[\pi_1(T_{g'}), \pi_1(T_{g'})]$$

which makes the following diagram commutative:

$$\begin{tikzcd}
\pi_1(T_g) \arrow[r, "f"] \arrow[d, "p_g"'] & \pi_1(T_{g'}) \arrow[d, "p_{g'}"] \\
\pi_1(T_g)/[\pi_1(T_g), \pi_1(T_g)] \arrow[r, "{[f]}"] & \pi_1(T_{g'})/[\pi_1(T_{g'}), \pi_1(T_{g'})]
\end{tikzcd}$$

where p_g, $p_{g'}$ are the canonical projections. Taking into account the fact that the groups $\pi_1(T_g)/[\pi_1(T_g),\pi_1(T_g)]$ and $\pi_1(T_{g'})/[\pi_1(T_{g'}),\pi_1(T_{g'})]$ are free abelian groups of rank $2g$ and $2g'$ respectively (see [4, p. 135]), by Corollary 3.2, we see that

$$\frac{\pi_1(T_{g'})/[\pi_1(T_{g'}),\pi_1(T_{g'})]}{\operatorname{Im}[f]}$$

is an infinite group. The remainder of the proof is obvious.

(iii) The proof is similar to that of (ii). ∎

PROPOSITION 4.2. *Let M^m, N^n be compact connected differentiable manifolds such that $m \geq n \geq 3$ and G be a compact connected Lie group acting freely on both manifolds. If $\pi_1(M)$ is finite and $\varphi_{\mathrm{alg}}(\pi_1(G),\pi_1(N))=\infty$, then any equivariant mapping $f : M \to N$ has an infinite number of critical orbits.*

Proof. Because $f : M \to N$ is a G-equivariant mapping, it induces a differentiable mapping $\widetilde{f} : M/G \to N/G$ which makes the following diagram commutative:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "p_M"'] & N \arrow[d, "p_N"] \\ M/G \arrow[r, "\widetilde{f}"] & N/G \end{tikzcd}$$

It is enough to show that $\widetilde{f}$ has an infinite number of critical points. For this purpose it is enough to show $\varphi_{\mathrm{alg}}(\pi_1(M/G),\pi_1(N/G)) = \infty$. Consider the exact homotopy sequences

$$\ldots \to \pi_q(G) \xrightarrow{i_q} \pi_q(M) \xrightarrow{(p_M)_q} \pi_q(M/G) \to \pi_{q-1}(G) \to \ldots$$
$$\ldots \to \pi_q(G) \xrightarrow{j_q} \pi_q(N) \xrightarrow{(p_N)_q} \pi_q(N/G) \to \pi_{q-1}(G) \to \ldots$$

of the fibrations $G \overset{i}{\hookrightarrow} M \to M/G$ and $G \overset{j}{\hookrightarrow} N \to N/G$. Taking $q = 1$ it follows, using the connectedness of G, that

$$\pi_1(M/G) \cong \pi_1(M)/\operatorname{Im} i_1, \qquad \pi_1(N/G) \cong \pi_1(N)/\operatorname{Im} j_1.$$

Because $\pi_1(M)$ is finite, $\pi_1(M)/\operatorname{Im} i_1 \cong \pi_1(M/G)$ is finite. The hypothesis $\varphi_{\mathrm{alg}}(\pi_1(G),\pi_1(N)) = \infty$ implies that $\pi_1(N)/\operatorname{Im} j_1 \cong \pi_1(N/G)$ is infinite. Therefore, by Corollary 3.4, $\varphi_{\mathrm{alg}}(\pi_1(M/G),\pi_1(N/G)) = \infty$. ∎

EXAMPLE. Let m, n, $a_1,\ldots,a_m$ be natural numbers such that $2n \geq m \geq 3$ and $(a_1,\ldots,a_m) = 1$. Consider the actions of S^1 on S^{2n+1} and T^m given by

$$S^1 \times S^{2n+1} \to S^{2n+1}, \ \ (z,(z_1,\ldots,z_n)) \mapsto (zz_1,\ldots,zz_m),$$
$$S^1 \times T^m \to T^m, \ \ (z,(z_1,\ldots,z_n)) \mapsto (z^{a_1}z_1,\ldots,z^{a_m}z_m).$$

The above two actions are obviously free and the conditions of Proposition 4.2 are satisfied. Therefore, any S^1-equivariant mapping $f : S^{2n+1} \to T^m$ has an infinite number of critical orbits.

References

[1] D. Andrica and C. Pintea, *Critical points of vector-valued functions*, in: Proceedings of the 24^{th} National Conference on Geometry and Topology, Timişoara 1993.
[2] D. Rozpłoch-Nowakowska, *Equivariant maps of joins of finite G-sets and an application to critical point theory*, Ann. Polon. Math. 56 (1992) 195–211.
[3] K. Kawakubo, *The Theory of Transformation Groups*, Oxford University Press, Oxford, 1991.
[4] W. S. Massey, *Algebraic Topology: An Introduction*, Harcourt, Brace & World, New York, 1967.
[5] R. S. Palais and C. L. Terng, *Critical Point Theory and Submanifold Geometry*, Lecture Notes in Math. 1353, Springer, 1988.
[6] F. Takens, *The minimal number of critical points of a function on a compact manifold and the Lusternik–Schnirelmann category*, Invent. Math. 6 (1968), 197–244.

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