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On operators with unitary ϱ-dilations

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To the memory of Professor Włodzimierz Mlak

Abstract. We show a polynomially boundend operator T is similar to a unitary operator if there is a singular unitary operator W and an injection X such that $XT = WX$. If, in addition, T is of class C_ϱ, then T itself is unitary.

According to Sz.-Nagy and Foiaş [5], a (bounded linear) operator T on a separable Hilbert space $\mathcal{H}$ is said to be of class C_ϱ with $\varrho > 0$ if there exists a unitary operator U on a Hilbert space $\mathcal{K}$ ($\supset \mathcal{H}$) such that $T^n = \varrho P_{\mathcal{H}} U^n | \mathcal{H}$ for $n = 1, 2, \ldots$, where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. For $\varrho = 2$ it is known (see [5, Chapter I, Proposition 11.2]) that T is of class C_2 if and only if its numerical radius $w(T)$ ($:= \sup\{|(Tx, x)| : \|x\| \le 1\}$) is not greater than one. In this paper we show that if T is of class C_ϱ and there exist a singular unitary operator W and an injection X such that $XT = WX$, then T is unitary. Here a unitary operator is *singular* by definition if its spectral measure is singular with respect to the (linear) Lebesgue measure on the unit circle $\mathbb{T}$. Such a situation occurs in connection with a compact operator A, as observed by Watanabe [6], which satisfies $|(Ax, x)| \le (|A|x, x)$ for all x. Our result gives an affirmative answer to a conjecture that such an operator A is normal. Clearly, if T is of class C_ϱ, then T is polynomially bounded, i.e., there exists a constant M such that $\|p(T)\| \le M \max\{|p(z)| : |z| = 1\}$ for every polynomial p. In our main result (Theorem 1) an assertion for the case of a polynomially bounded operator T is also included. Though this part can be derived from a result of Mlak [2] (see also [3]), our proof is quite different from Mlak's.

Let $A(\mathbb{T})$ be the disk algebra, that is, $A(\mathbb{T})$ is the norm closure of polynomials in the algebra $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$ with norm

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$\|f\| = \sup\{|f(z)| : z \in \mathbb{T}\}$ for $f \in C(\mathbb{T})$. If T is a polynomially bounded operator, then there exists a bounded algebra homomorphism, $f \mapsto f(T)$, from $A(\mathbb{T})$ to the uniformly closed algebra generated by T and I which maps each polynomial p to $p(T)$.

THEOREM 1. *Let T be an operator on $\mathcal{H}$, and suppose that there exist a singular unitary operator W on $\mathcal{G}$ and an injection $X : \mathcal{H} \to \mathcal{G}$ such that $XT = WX$.*

(i) *If T is polynomially bounded, then T is similar to a unitary operator.*
(ii) *If T is of class C_ϱ, then T is unitary.*

Proof. (i) Let $W = \int_{\mathbb{T}} z\, dE(z)$ be the spectral decomposition of W. Let us first show that for every closed set δ ($\subset \mathbb{T}$) of Lebesgue measure zero, there exists an idempotent Q on $\mathcal{H}$ such that

$$XQ = E(\delta)X \quad \text{and} \quad \|Q\| \le M,$$

where $M = \sup\{\|h(T)\| : h \in A(\mathbb{T}) \text{ and } \|h\| \le 1\}$. For such a set δ, we can take a function g in $A(\mathbb{T})$ such that $g(z) = 1$ for $z \in \delta$ and $|g(z)| < 1$ for $z \in \mathbb{T} \setminus \delta$ (see [1, p. 81]). Then, since T is polynomially bounded, $g^n(T)$ is well defined and it follows from the identity $XT = WX$ that $Xg^n(T) = g^n(W)X$. Clearly, $g^n(W)$ converges strongly to $E(\delta)$, so $\lim_{n\to\infty}(g^n(T)x, X^*y) = (E(\delta)Xx, y)$ for $x \in \mathcal{H}$ and $y \in \mathcal{G}$. Hence, since $\|g^n(T)\| \le M\|g^n\| = M$ for $n = 1, 2, \ldots$ and X^* has dense range, $g^n(T)$ converges weakly to an operator Q such that $XQ = E(\delta)X$ and $\|Q\| \le M$. Since X is injective, the identity $XQ = E(\delta)X$ shows that Q is idempotent.

Now we prove that T is invertible and $\|T^k\| \le M^2$ for $k = 0, \pm1, \pm2, \ldots$ Then it follows from the theorem of Sz.-Nagy [4] that T is similar to a unitary operator. Since W is singular, there exists a sequence $\{\delta_n\}$ of closed sets with Lebesgue measure zero such that $E(\delta_n)$ converges to the identity I as $n \to \infty$. Applying the fact shown above to $\delta = \delta_n$, we obtain an idempotent Q_n such that $XQ_n = E(\delta_n)X$ and $\|Q_n\| \le M$. For each k and $n = 1, 2, \ldots$, take an $h_{k,n} \in A(\mathbb{T})$ such that $h_{k,n}(z) = z^{-k}$ for $z \in \delta_n$ and $\|h_{k,n}\| \le 1$ (see [1, p. 81]). Then we have

$$Xh_{k,n}(T)Q_n = h_{k,n}(W)E(\delta_n)X = W^{*k}E(\delta_n)X.$$

For each k, $W^{*k}E(\delta_n)$ converges strongly to W^{*k} as $n \to \infty$ and, for $n = 1, 2, \ldots$, $\|h_{k,n}(T)Q_n\| \le M^2$. Therefore we can conclude that $h_{k,n}(T)Q_n$ converges weakly to an operator S_k such that $XS_k = W^{*k}X$ and $\|S_k\| \le M^2$. Since

$$XS_kT^k = W^{*k}XT^k = W^{*k}W^kX = X \quad \text{and} \quad XT^kS_k = W^kW^{*k}X = X,$$

the injectivity of X shows that T is invertible and $T^{-k} = S_k$ for $k = 1, 2, \ldots$, so that $\|T^k\| \le M^2$ for all $k = 0, \pm1, \pm2, \ldots$

(ii) Let U be a unitary ϱ-dilation of T on $\mathcal{K}$, i.e., a unitary operator such that $T^n = \varrho P_{\mathcal{H}} U^n|\mathcal{H}$ for $n = 1, 2, \ldots$, and let $U = \int_{\mathbb{T}} z\, dF(z)$ be the spectral decomposition of U. For any closed set δ with Lebesgue measure zero, let g and Q be as in the proof of (i). Since the idempotent Q is a weak limit of $g^n(T)$ and

$$g^n(T) = P_{\mathcal{H}}[\varrho g^n(U) + (1-\varrho)g^n(0)I_{\mathcal{K}}]|\mathcal{H},$$

we have $Q = \varrho P_{\mathcal{H}} F(\delta)|\mathcal{H}$ because $g^n(0) \to 0$ as $n \to \infty$, so that Q is selfadjoint. Hence it follows from $XQ = E(\delta)X$ that $E(\delta)XX^* = XQX^*$ and so XX^* commutes with $E(\delta)$. Then, since W is singular, XX^* commutes with $E(\alpha)$ for any Borel set α and so commutes with W. Thus $W|(\operatorname{ran} X)^-$ is unitary and, using the polar decomposition of X^*, we can conclude that T is unitarily equivalent to $W|(\operatorname{ran} X)^-$, so T itself is unitary. This completes the proof.

Clearly, a polynomially bounded operator T is *power-bounded*, that is, $\sup\{\|T^n\| : n = 1, 2, \ldots\} < \infty$. When an operator T is power-bounded, by requiring compactness of the intertwining operator X in Theorem 1(i) we can obtain a similar conclusion.

THEOREM 2. *Let T be a power-bounded operator on $\mathcal{H}$ and let V be an isometry on $\mathcal{G}$. If there exists a compact injection $K : \mathcal{H} \to \mathcal{G}$ having dense range such that $KT = VK$, then V is a singular unitary operator, and T is similar to a unitary operator.*

Proof. Since T is power-bounded, we can take a subsequence $\{T^{n(k)}\}$ of $\{T^n\}$ which converges weakly to an operator S as $k \to \infty$. Then, since $KT^{n(k)} = V^{n(k)}K$ for $k = 1, 2, \ldots$ and K is compact, $V^{n(k)}K$ converges strongly to KS. But K has dense range, so it follows that $V^{n(k)}$ converges strongly to an isometry W. Considering the Wold decomposition of V (see [5, Chapter I, Theorem 1.1]) and the decomposition of its unitary part into the sum of the singular and absolutely continuous summands, we see that V is singular unitary because $U^n \to 0$ weakly as $n \to \infty$ for an isometry U whose unitary part is absolutely continuous. Next, for integers j, l and k with $n(k) > n(l) + j$, we have

$$KT^{n(k)-n(l)-j} = V^{*(j+n(l))}V^{n(k)}K$$

and $V^{*(j+n(l))}V^{n(k)}$ converges weakly to $V^{*(j+n(l))}W$ as $k \to \infty$. Hence, since T is power-bounded and K^* has dense range by the injectivity of K, $T^{n(k)-n(l)-j}$ converges weakly to an operator $S_{j,l}$ as $k \to \infty$, which satisfies

$$KS_{j,l} = V^{*(j+n(l))}WK \quad \text{and} \quad \|S_{j,l}\| \le M,$$

where $M = \sup\{\|T^n\| : n = 0, 1, 2, \ldots\}$ Also, for each j, $V^{*(j+n(l))}W$ converges weakly to V^{*j} as $l \to \infty$ (because W is isometric). So, letting $l \to \infty$ in the identity $KS_{j,l} = V^{*(j+n(l))}WK$, we get an operator S_j such that

$KS_j = V^{*j}K$ and $\|S_j\| \le M$. Thus, as in the proof of Theorem 1(i), it follows that T is invertible and $\|T^{-j}\| \le M$ for $j = 1, 2, \ldots$, and by the theorem of Sz.-Nagy [4], T is similar to a unitary operator.

Theorem 1(ii) and Theorem 2 can give an affirmative answer to the question posed in [6]:

COROLLARY 3. *If A is a compact operator and satisfies $|(Ax, x)| \le (|A|x, x)$ for all x, then A is normal.*

Proof. Let $A = V|A|$ be the polar decomposition. By [6, Theorem 2.1] there exists an operator T with $w(T) \le 1$ such that $V|A|^{1/2} = |A|^{1/2}T$. Let $\mathcal{M} = (\operatorname{ran}|A|)^{-}$. The identity $V|A|^{1/2} = |A|^{1/2}T$ implies that $\mathcal{M}$ is invariant for V. Let $V_1 = V|\mathcal{M}$ and $T_1 = P_{\mathcal{M}}T|\mathcal{M}$. Then V_1 is isometric and T_1 belongs to the class C_2. Also, the operator $X = |A|^{1/2}|\mathcal{M} : \mathcal{M} \to \mathcal{M}$ is a compact injection with dense range and satisfies $XT_1 = V_1X$. So, by Theorem 2 and the proof of Theorem 1(ii), V_1 is a unitary operator which commutes with $|A||\mathcal{M}$. Hence it follows that A is normal.

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