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Existence of the fundamental solution of a second order evolution equation

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Abstract. We give sufficient conditions for the existence of the fundamental solution of a second order evolution equation. The proof is based on stable approximations of an operator $A(t)$ by a sequence $\{A_n(t)\}$ of bounded operators.

1. Introduction. Let X be a real Banach space. Suppose that for each $t \in [0,T]$ a linear (in general unbounded) operator $A(t) : X \to X$ is defined.

We make the assumption

(Z_1) $D(A) := D(A(t))$ is independent of $t, D(A)$ is dense in X and for each $t \in [0,T]$ the operator $A(t)$ has a bounded inverse $A^{-1}(t)$.

Suppose that the operator $A(t)$ is strongly continuous on $D(A)$, i.e. for every $x \in D(A)$ the function $t \to A(t)x$ is continuous. It follows that the operator $A(t)A^{-1}(0)$ is bounded and continuous in t on $[0,T]$, so in view of the Banach–Steinhaus theorem it is uniformly bounded in t, i.e.

$$\|A(t)A^{-1}(0)\| \le c \quad \text{for } t \in [0,T] \tag{1}$$

(cf. [7, p. 9]).

If we assume that $A^{-1}(t)$ is uniformly bounded on $[0,T]$ then the mapping

$$[0,T] \ni t \to A^{-1}(t) \tag{2}$$

is strongly continuous. Analogously, if we assume that

$$\|A(0)A^{-1}(s)\| \le M \quad \text{for } 0 \le s \le T, \tag{3}$$

then it will follow from the foregoing that the mapping

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$$(4) \qquad [0,T]\times[0,T]\ni(t,s)\to A(t)A^{-1}(s)$$

is strongly continuous (cf. [7, p. 177]).

We shall need the following.

LEMMA 1 ([7, Lemma 1.5, p. 179]). *Suppose that $A(t)$ is strongly continuously differentiable on its domain $D(A)$ and has a bounded inverse $A^{-1}(t)$. Then*

1° *The operator $A(t)A^{-1}(s)$ is continuous in the operator norm in the variables s and t taken together, where $0\le s,t\le T$, and satisfies a Lipschitz condition in each of them. In particular*, (3) *holds.*

2° *The operator $A(t)A^{-1}(s)$ is strongly differentiable relative to t and s, and the derivatives*

$$(5) \qquad [A(t)A^{-1}(s)]'_t = A'(t)A^{-1}(s)$$

and

$$(6) \qquad [A(t)A^{-1}(s)]'_s = -A(t)A^{-1}(s)A'(s)A^{-1}(s)$$

are strongly continuous as functions of two variables s and t.

The aim of this paper is to investigate the abstract second order linear initial value problem

$$(7) \qquad \frac{d^2u}{dt^2}=A(t)u+f(t),\quad t\in[0,T],\qquad u(0)=u_0,\qquad u'(0)=u_1,$$

where $A(t)$, for $t\in[0,T]$, is the operator defined above, u and f are functions from $\mathbb{R}$ into X, and $u_0,u_1\in X$.

First we shall consider the first order problem.

2. First order initial value problem. In this section we consider the homogeneous differential equation

$$(8) \qquad \frac{dx}{dt}=A(t)x,\qquad 0\le t\le T,$$

where $A(t)$ is the operator defined in the introduction and satisfying assumption (Z_1).

DEFINITION 1. By a *Cauchy problem* for the equation (8) in the triangle $\Delta_T := \{(s,t): 0\le s\le t\le T\}$ we mean the problem of finding for each fixed $s\in[0,T)$ a solution $x(t,s)$ of (8) on the segment $[s,T]$, satisfying a given initial condition

$$(9) \qquad x(s,s)=x_0\in D(A).$$

DEFINITION 2 ([7, p. 193]). The Cauchy problem (8), (9) is said to be *uniformly correct* if:

1° For each $s \in [0,T]$ and any $x_0 \in D(A)$ there exists a unique solution $x(t,s)$ of (8) on the segment $[s,T]$ satisfying condition (9).

2° The function $\varDelta_T \ni (t,s) \to x(t,s)$ and its derivative $\varDelta_T \ni (t,s) \to x'_t(t,s)$ are continuous.

3° The solution depends continuously on the initial data in the sense that if the $x_{0n} \in D(A)$ converge to zero then the corresponding solutions $x_n(t,s)$ converge to zero uniformly relative to $(t,s) \in \varDelta_T$.

It is known (see [7, p. 195]) that if the mapping $[0,T] \ni t \to A(t)$ is strongly continuous on $D(A)$ and the Cauchy problem (8), (9) is uniformly correct, then it is possible to introduce a linear operator $U(t,s) : X \to X$ for $(t,s) \in \varDelta_T$ which has the following properties:

1° The operator $U(t,s)$ is bounded in X relative to $(t,s) \in \varDelta_T$, i.e.,

$$\|U(t,s)\| \le M \quad (0 \le s \le t \le T). \tag{10}$$

2° The mapping $\varDelta_T \ni (t,s) \to U(t,s)$ is strongly continuous.

3° The following identities hold:

$$U(t,s) = U(t,r)U(r,s), \quad U(t,t) = I \quad (0 \le s \le r \le t \le T). \tag{11}$$

4° On $D(A)$ the mapping $\varDelta_T \ni (t,s) \to U(t,s)$ is strongly differentiable relative to t and s, and

$$\partial U(t,s)/\partial t = A(t)U(t,s), \quad \partial U(t,s)/\partial s = -U(t,s)A(s). \tag{12}$$

5° If, additionally, the operator $A(t)$ has a bounded inverse such that condition (3) is satisfied, then the operator

$$V(t,s) = A^{-1}(t)U(t,s)A^{-1}(s) \tag{13}$$

is bounded and strongly continuous in the triangle $\varDelta_T$.

Definition 3. The operator $U(t,s)$, with $(t,s) \in \varDelta_T$, having the properties 1°–4° is called the *evolution operator* (or *fundamental solution*) corresponding to equation (8).

Consider the family of equations

$$\frac{dx}{dt} = A_n(t)x, \quad n = 1,2,\dots,\ t \in [0,T], \tag{14}$$

with bounded strongly continuous operators $A_n(t)$.

Definition 4 ([7, p. 199]). If there exists a sequence of bounded strongly continuous operators $A_n(t)$ for which

$$\lim_{n\to\infty} \{ \sup_{0 \le t \le T} \|[A(t) - A_n(t)]A^{-1}(t)x\|\} = 0, \quad x \in X, \tag{15}$$

$A(t)$ satisfies (3), and the condition of uniform boundedness of the evolution operators corresponding to equation (14) is satisfied, i.e.

$$\|U_n(t,s)\| \le M \tag{16}$$

(M does not depend on n, t and s), then we will say that the operator $A(t)$ is *stably approximated* by the operators $A_n(t)$.

THEOREM 1. *Suppose that the operator $A(t)$ is strongly continuously differentiable on $D(A)$ for $t \in [0,T]$, has a bounded inverse and is stably approximated by operators $A_n(t)$ such that*:

1° *each $A_n(t)$ is strongly continuously differentiable for $t \in [0,T]$ and has a bounded inverse such that $\|A_n^{-1}(t)\| \le C$ (C does not depend on n and t)*,

2° *the sequence $\{A_n'(t)A_n^{-1}(t)\}$ converges strongly and uniformly on $[0,T]$ to a bounded operator $B(t)$.*

If the evolution operators $U_n(t,s)$ converge, as $n \to \infty$, strongly and uniformly in t and s to an operator $U(t,s)$, then the Cauchy problem for equation (8) is uniformly correct and $U(t,s)$ is the evolution operator corresponding to it.

Proof. Consider the equation

$$\frac{dy}{dt} = A_n(t)y + A_n'(t)A_n^{-1}(t)y. \tag{17}$$

It follows from the continuous differentiability of $A_n(t)$ that the operator $A_n'(t)A_n^{-1}(t)$ is strongly continuous (cf. [7, Lemma 1.5]). Denote by $V_n(t,s)$ the evolution operator corresponding to equation (17). According to Remark 2.1 in [7, p. 192] the operators $V_n(t,s)$ converge strongly and uniformly relative to t and s to a limit which we shall denote by $V(t,s)$. The operator $V(t,s)$ is strongly continuous for t and s in Δ_T.

We make the substitution $A_n^{-1}(t)y(t,s) = x(t,s)$ in (17). Then

$$\begin{aligned}\frac{dx}{dt} &= A_n^{-1}(t)\frac{dy}{dt} - A_n^{-1}(t)A_n'(t)A_n^{-1}(t)y\\ &= A_n^{-1}(t)A_n(t)y = A_n(t)A_n^{-1}(t)y = A_n(t)x.\end{aligned}$$

In view of the uniqueness of the solution of equation (14) we have

$$A_n^{-1}(t)y(t,s) = x(t,s) = U_n(t,s)x(s,s),$$

so that

$$y(t,s) = A_n(t)U_n(t,s)x(s,s) = A_n(t)U_n(t,s)A_n^{-1}(s)y(s,s)$$

or, in another form,

$$V_n(t,s) = A_n(t)U_n(t,s)A_n^{-1}(s). \tag{18}$$

From (15) and the assumption $\|A_n^{-1}(t)\| \le C$ we obtain

$$\begin{aligned}\|[A_n^{-1}(t) - A^{-1}(t)]x\| &\le \|A_n^{-1}(t)\| \cdot \|[I - A_n(t)A^{-1}(t)]x\|\\ &\le C\|[A(t) - A_n(t)]A^{-1}(t)x\| \to 0\end{aligned}$$

for each $x \in X$ uniformly in $t \in [0,T]$ as $n \to \infty$. On the other hand, by (18) we get

$$A_n^{-1}(t)V_n(t,s)x = U_n(t,s)A_n^{-1}(s)x \quad \text{for } x \in X. \tag{19}$$

Since the $U_n(t,s)$ converge to $U(t,s)$ and $V_n(t,s)$ converge to $V(t,s)$ and $A_n^{-1}(t)$ converge strongly and uniformly to $A^{-1}(t)$, we have, from (19),

$$A^{-1}(t)V(t,s)x = U(t,s)A^{-1}(s)x \quad \text{for } x \in X,$$

so that

$$V(t,s) = A(t)U(t,s)A^{-1}(s). \tag{20}$$

Let $x_0 \in D(A)$ and $A(s)x_0 = y_s$. We have

$$\begin{aligned} dU_n(t,s)x_0/dt &= A_n(t)U_n(t,s)x_0 = A_n(t)A_n^{-1}(t)V_n(t,s)A_n(s)A^{-1}(s)y_s \\ &= V_n(t,s)A_n(s)A^{-1}(s)y_s. \end{aligned}$$

In view of (15) the operators $A_n(s)A^{-1}(s)$ tend strongly, and uniformly in $s \in [0,T]$, to the identity operator, so that the derivatives $dU_n(t,s)x_0/dt$ converge uniformly in t to the function $V(t,s)y_s = A(t)U(t,s)x_0$. Since the differentiation operator is closed we have

$$\frac{dU(t,s)x_0}{dt} = A(t)U(t,s)x_0,$$

i.e. the function $x(t,s) := U(t,s)x_0$ is a solution of equation (8) on $[s,T]$. Since $U_n(s,s) = I$ for each $n \in \mathbb{N}$, we have

$$x(s,s) = U(s,s)x_0 = \lim_{n\to\infty} U_n(s,s)x_0 = x_0.$$

This means that the function $x = x(t,s)$ is a solution of the Cauchy problem (8), (9).

The uniqueness of solution of the problem (8), (9) follows from Lemma 3.1 in [7, p. 199], because each solution of this problem is given by

$$x(t,s) = \lim_{n\to\infty} U_n(t,s)x_0. \tag{21}$$

Properties 2° and 3° of Definition 2 follow from the boundedness and strong continuity of the operators $U(t,s)$ and $V(t,s)$, $(t,s) \in \mathit{\Delta}_T$. Theorem 1 is proved.

Remark 1. Theorem 1 and its proof are a slight modification of Theorem 3.6 of the monograph [7, pp. 200–201]. This modification consists in omitting the assumption "the operators $A_n(t)$ for each $t \in [0,T]$ commute with $A(t)$ on $D(A)$".

3. Fractional powers of operators. In this section we shall need the following assumptions:

(Z_2) For each $t \in [0,T]$, $A(t)$ is the infinitesimal generator of a strongly continuous cosine family $\{C_t(\xi) : \xi \in \mathbb{R}\}$ of bounded linear operators from X into itself.

(Z_3) For each $t \in [0,T]$ there exists a linear operator $B(t) : X \to X$ such that $B^2(t) = A(t)$, the domain $D(B(t)) := D(B)$ is independent of t and 0 belongs to the resolvent set of $B(t)$.

(Z_4) For each $x \in D(B)$ the mapping $t \to B(t)x$ is of class C^1 in $[0,T]$.

Remark 2. In (Z_3) the essential assumption is the condition "the domain $D(B(t)) = D(B)$ is independent of t". The existence of the operator $B(t)$ for each $t \in [0,T]$ may be obtained by translation of $A(t)$ without loss of generality (cf. [10]).

Let $A(t)$, $t \in [0,T]$, satisfy (Z_1)–(Z_3). Then the resolvent

$$R(\lambda; A(t)) := (\lambda - A(t))^{-1} \tag{22}$$

is defined for $\lambda > 0$ and satisfies the inequality

$$\|R(\lambda; A(t))\| \le \frac{M}{\lambda}, \qquad \lambda > 0,\ t \in [0,T], \tag{23}$$

where $M \ge 1$ is a constant independent of λ and t (cf. [2] or [4, p. 61]).

We define the operators

$$A_n(t) := nA(t)R(n; A(t)), \qquad t \in [0,T],\ n \in \mathbb{N}. \tag{24}$$

(see [7, p. 204]). Obviously, each $A_n(t)$ is bounded. It is proved in [7, pp. 204–205] that if $A(t)$ is strongly continuous on $D(A)$, then the operators $A_n(t)$ satisfy the condition (15).

By the identity (cf. [7, p. 205])

$$R(\lambda; A_n(t)) = \frac{1}{\lambda + n} I + \frac{n^2}{(n+\lambda)^2} R\left(\frac{n\lambda}{n+\lambda}; A(t)\right) \tag{25}$$

and inequality (23) we get

$$\|R(\lambda; A_n(t))\| \le \frac{M}{\lambda}, \qquad \lambda > 0,\ t \in [0,T], \tag{26}$$

where M is the constant from (23).

From (26) we deduce that it is possible to define positive fractional powers $(-A_n(t))^\alpha$ ($\alpha > 0$) of $-A_n(t)$, for each $n \in \mathbb{N}$ and $t \in [0,T]$, by

$$(-A_n(t))^\alpha := \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} R(\lambda; A_n(t))(-A_n(t))\, d\lambda \tag{27}$$

(cf. [4, p. 51], [7, p. 112] or [12, Chapter IX, 11]). In formula (27) the branch of $\lambda^{\alpha-1}$ is that which is real when $\lambda \geq 0$. We let

$$B_n(t) := [A_n(t)]^{1/2} := i[-A_n(t)]^{1/2}. \tag{28}$$

It is proved in [4, p. 59] that if $\mu^2 \in \varrho(-A_n(t))$ then $\mu \in \varrho(B_n(t))$ and

$$R(\mu; B_n(t)) = \frac{1}{\pi}\int_0^\infty \frac{\lambda^{1/2}}{\lambda+\mu^2} R(\lambda; A_n(t))\,d\lambda. \tag{29}$$

From (29) by the estimate (26) we get

$$\|R(\mu; B_n(t))\| \leq \frac{M}{|\mu|}, \qquad \mu \neq 0,\ t \in [0,T], \tag{30}$$

where M is the constant of (26).

LEMMA 2. *Assume* (Z_1) *and* (Z_2). *If the family of generators* $\{A(t)\}$, $t \in [0,T]$, *is* stable *with constants* $M \geq 1$ *and* $\omega = 0$ (*called the* stability constants), *i.e.*, *the following conditions are satisfied*:

$$\|(\lambda_k - A(t_k))^{-1}(\lambda_{k-1} - A(t_{k-1}))^{-1}\ldots(\lambda_1 - A(t_1))^{-1}\| \leq M\prod_{j=1}^{k}\lambda_j^{-1}, \qquad \lambda_j > 0, \tag{31}$$

and

$$\|(\lambda_1 - A(t_1))^{-1}(\lambda_2 - A(t_2))^{-1}\ldots(\lambda_k - A(t_k))^{-1}\| \leq M\prod_{j=1}^{k}(-\lambda_j)^{-1}, \qquad \lambda_j < 0, \tag{32}$$

then the family $\{B_n(t)\}$, $t \in [0,T]$, $n \in \mathbb{N}$, *is uniformly* twice-stable, *i.e.*, *the following conditions are satisfied*:

$$\|(\mu - B_n(t_k))^{-1}(\mu - B_n(t_{k-1}))^{-1}\ldots(\mu - B_n(t_1))^{-1}\| \leq M|\mu|^{-k} \tag{33}$$

and

$$\|(\mu - B_n(t_1))^{-1}(\mu - B_n(t_2))^{-1}\ldots(\mu - B_n(t_k))^{-1}\| \leq M|\mu|^{-k}, \tag{34}$$

for $\mu \neq 0$ *and any finite sequence* $0 \leq t_1 \leq \ldots \leq t_k \leq T$, $k, n \in \mathbb{N}$.

P r o o f. In [1, Lemma 1] we proved that if a family $\{A(t)\}$, $t \in [0,T]$, is stable with stability constants M and ω, then the family $\{A_n(t)\}$, $t \in [0,T]$, $n \in \mathbb{N}$, where $A_n(t)$ is defined by (24), is uniformly stable with stability constants M and 2ω. Since by assumption $\omega = 0$, the family $\{A_n(t)\}$, $t \in [0,T]$, $n \in \mathbb{N}$, is uniformly stable with constants M and $\omega = 0$.

As remarked above, if $\lambda \in \varrho(-A_n(t))$ and $\lambda > 0$ then $\mu \in \varrho(B_n(t))$, where $\mu^2 = \lambda$. From this, by (29), we have

$$(\mu - B_n(t_j))^{-1} = \frac{1}{\pi} \int_0^\infty \frac{\lambda_j^{1/2}}{\lambda_j + \mu^2} (\lambda_j - A_n(t_j))^{-1} \, d\lambda_j, \tag{35}$$
$$j = 1, \ldots, k, \ \lambda_j > 0, \ 0 \le t_1 \le t_2 \le \ldots \le t_k \le T.$$

Hence

$$(\mu - B_n(t_k))^{-1}(\mu - B_n(t_{k-1}))^{-1} \ldots (\mu - B_n(t_1))^{-1}$$
$$= \pi^{-k} \int_0^\infty \ldots \int_0^\infty \prod_{j=1}^k \frac{\lambda_j^{1/2}}{\lambda_j + \mu^2} (\lambda_k - A_n(t_k))^{-1} \ldots (\lambda_1 - A_n(t_1))^{-1} \, d\lambda_k \ldots d\lambda_1.$$

From this, by the uniform stability of the family $\{A_n(t)\}$, $t \in [0, T]$, $n \in \mathbb{N}$, we obtain

$$\|(\mu - B_n(t_k))^{-1}(\mu - B_n(t_{k-1}))^{-1} \ldots (\mu - B_n(t_1))^{-1}\|$$
$$\le M\pi^{-k} \int_0^\infty \ldots \int_0^\infty \prod_{j=1}^k \frac{\lambda_j^{-1/2} \, d\lambda_j}{\lambda_j + \mu^2}$$
$$= M \prod_{j=1}^k \left(\frac{1}{\pi} \int_0^\infty \frac{\lambda_j^{-1/2} \, d\lambda_j}{\lambda_j + \mu^2} \right) = M|\mu|^{-k}.$$

Inequality (33) is proved. The proof of (34) is similar and so we omit it.

LEMMA 3. *Let $A(t)$, $t \in [0, T]$, be the operator defined in the introduction. If $A(t)$ satisfies inequalities* (3) *and* (23), *then for each $n \in \mathbb{N}$ and any $t \in [0, T]$ the operator $B_n(t)$ defined by* (28) *is invertible and*

$$\|B_n^{-1}(t)\| \le C, \tag{36}$$

where C does not depend on n and t.

Proof. From (28) we get

$$B_n^{-1}(t) = [A_n(t)]^{-1/2} = \frac{1}{i} [-A_n(t)]^{-1/2}.$$

It follows that (see [7, p. 112])

$$B_n^{-1}(t) = \frac{1}{\pi i} \int_0^\infty \lambda^{-1/2} R(\lambda; A_n(t)) \, d\lambda, \tag{37}$$

where $A_n(t)$ is defined by (24).

In order to prove the existence of $B_n^{-1}(t)$ it suffices to show the convergence of the integral in (37). We have

$$\begin{aligned}\int_0^\infty &\lambda^{-1/2}R(\lambda;A_n(t))\,d\lambda\\ &= \int_0^1 \lambda^{-1/2}R(\lambda;A_n(t))\,d\lambda + \int_1^\infty \lambda^{-1/2}R(\lambda;A_n(t))\,d\lambda\\ &= \int_0^1 \lambda^{-1/2}A_n(t)R(\lambda;A_n(t))A_n^{-1}(t)\,d\lambda + \int_1^\infty \lambda^{-1/2}R(\lambda;A_n(t))\,d\lambda\\ &= \int_0^1 \lambda^{-1/2}[\lambda R(\lambda;A_n(t)) - I]A_n^{-1}(t)\,d\lambda + \int_1^\infty \lambda^{-1/2}R(\lambda;A_n(t))\,d\lambda.\end{aligned}$$

By (24) we obtain

$$A_n^{-1} = A^{-1}(t) - \frac{1}{n}I.$$

It follows from (3) that $\|A^{-1}(t)\| \le C_1$ for $t \in [0,T]$, and so $\|A_n^{-1}(t)\| \le C_1 + 1$. From this and from (23) we get

$$\Big\|\int_0^1 \lambda^{-1/2}R(\lambda;A_n(t))\,d\lambda\Big\| \le (M+1)(C_1+1)\int_0^1 \lambda^{-1/2}\,d\lambda = 2(M+1)(C_1+1)$$

and

$$\Big\|\int_1^\infty \lambda^{-1/2}R(\lambda;A_n(t))\,d\lambda\Big\| \le M\int_1^\infty \lambda^{-3/2}\,d\lambda = 2M,$$

which proves the existence of $B_n^{-1}(t)$, and also the estimate (36) with $C := \frac{2}{\pi}[(M+1)(C_1+1)+M]$.

LEMMA 4. *Under the assumptions of Lemma* 3, *if the mapping* $[0,T] \ni t \to A(t)x$ *is continuous for* $x \in D(A)$ *then the sequence* $\{B_n(t)\}$ *is strongly and uniformly convergent to the operator* $B(t)$ *on the domain* $D(A)$, *where* $B(t)$ *is taken from assumption* (Z_3).

Proof. From the definition of $B(t)$ and from inequality (23) it follows that for $x \in D(A)$ we have

$$B(t)x = [A(t)]^{1/2}x = \frac{i}{\pi}\int_0^\infty \lambda^{-1/2}R(\lambda;A(t))(-A(t)x)\,d\lambda, \qquad t \in [0,T]. \tag{38}$$

Combining (27) with (28) we obtain

$$B_n(t)x = \frac{i}{\pi}\int_0^\infty \lambda^{-1/2}R(\lambda;A_n(t))(-A_n(t)x)\,d\lambda, \qquad x \in X,\ t \in [0,T]. \tag{39}$$

Fix $x \in D(A)$. In this case, similarly to Lemma 3, one may prove that the improper integral in (39), as well in (38), is convergent absolutely and uniformly relative to t and n.

By the definition of $A_n(t)$ and by the strong continuity of the mapping $t \to A(t)$, we have, for $x \in D(A)$ and $t \in [0,T]$,

$$\begin{aligned}\|[A_n(t) - A(t)]A^{-1}(t)x\| &= \|[nR(n; A(t))A(t) - A(t)]A^{-1}(t)x\| \\ &= \|R(n; A(t))A(t)x\| \\ &\le \frac{M}{n}\max_{0\le t\le T}\|A(t)x\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

The norms of the operators $nR(n; A(t))$ are bounded by M, so that in view of the Banach–Steinhaus theorem the operators $A_n(t)A^{-1}(t)$ tend to the identity operator strongly and uniformly in $t \in [0,T]$. Therefore, the operators $A_n(t)$ tend strongly and uniformly on $D(A)$ to the operator $A(t)$. Letting $n \to \infty$ in (39) we get

$$\lim_{n\to\infty} B_n(t)x = \frac{i}{\pi}\int_0^\infty \lambda^{-1/2}R(\lambda; A(t))(-A(t))x\,d\lambda = B(t)x, \qquad x \in D(A).$$

Hence,

$$B_n(t)x \to B(t)x \quad \text{as } n \to \infty \qquad \text{for } x \in D(A), \tag{40}$$

uniformly in $t \in [0,T]$.

Formula (38) defines the operator $B(t)$ on $D(A)$.

The entire operator $B(t)$ may be obtained by closure from its restriction to $D(A)$ (cf. [7, p. 114]). As in the proof of Lemma 3, we may show that the operator $B(t)$ is invertible for each $t \in [0,T]$ and

$$\|B^{-1}(t)\| \le C, \tag{41}$$

where C is the same as in (36).

The domain $D(B(t)) = D(B)$, by assumption (Z_3), is dense in X and contains $D(A)$.

In the sequel we shall use a generalized convergence of sequences of operators (see [5, Section IV. 2]). We only use a sufficient condition for the generalized convergence (see [5, Theorem 2.29, p. 207]), and we make the following definition.

Definition 5. Let $T_n(t), T(t) \in \mathcal{C}(X)$ for $t \in [a,b]$. Consider a second Banach space Y and operators $U_n(t), U(t), V_n(t), V(t) \in B(Y,X)$ such that $U_n(t)$, $U(t)$ map Y one-to-one onto $D(T_n(t)), D(T(t))$, respectively, and $T_n(t)U_n(t) = V_n(t), T(t)U(t) = V(t)$. If, for each $x \in Y$, $U_n(t)x \to U(t)x$ and $V_n(t)x \to V(t)x$ as $n \to \infty$ for $t \in [a,b]$, then we shall say that the sequence $\{T_n(t)\}$ converges strongly in $[a,b]$ to the operator $T(t)$ *in the generalized sense*.

LEMMA 5. *Let assumptions* (Z_1)–(Z_4) *hold. If the mapping* $[0,T] \ni t \to A(t)x$, $x \in D(A)$, *is continuous and inequality* (3) *is satisfied*, *then the operators* $B_n(t)B^{-1}(s)$ *are uniformly bounded in the square* $K_T := [0,T] \times [0,T]$ *and*

$$B_n(t)B^{-1}(s)x \to B(t)B^{-1}(s)x \quad \textit{for } x \in X,$$

uniformly in K_T.

Proof. By (3) and the strong continuity of the operator $A(t)$ we get the uniform boundedness of the operator $A(t)A^{-1}(s)$ in K_T. From this, by (38), we obtain the uniform boundedness of $B(t)A^{-1}(s)$ in K_T. Analogously, from (39) we get the uniform boundedness of the sequence $\{B_n(t)A^{-1}(s)\}$ in K_T. Next, from (40), it follows that

$$B_n(t)A^{-1}(s)x \to B(t)A^{-1}(s)x \quad \text{as } n \to \infty \quad \text{for } x \in X, \tag{42}$$

uniformly in the square K_T.

From (42), by Definition 5, we deduce that the sequence $\{B_n(t)\}$, where the $B_n(t)$ are defined in the subspace $D(A)$, converges strongly and uniformly to the operator $B(t)$ on $D(A)$ for $t \in [0,T]$ in the generalized sense.

Consider now the sequence $\{B_n(t)B^{-1}(t)\}$, defined on $D(B)$, where $D(B)$ is taken from assumption (Z_3). We proceed to show that

$$B_n(t)B^{-1}(s) \to B(t)B^{-1}(s) \tag{43}$$

uniformly in K_T, in the sense of Definition 5. Indeed, we take

$$Y := X, \quad U_n(s) := B^{-1}(s), \quad U(s) := B^{-1}(s) \quad \text{for } s \in [0,T].$$

It follows that $V_n(t,s) = B_n(t)A^{-1}(s)$ and $V(t,s) = B(t)A^{-1}(s)$, $(t,s) \in K_T$. From this, by (42), we obtain (43).

On the other hand, by assumption (Z_3), the operator $B(t)B^{-1}(s) : X \to X$ is bounded uniformly in K_T. Therefore, by Theorem 2.23 in [5, p. 206], the operators $B_n(t)B^{-1}(s)$ are uniformly bounded in K_T for sufficiently large n and

$$\|B_n(t)B^{-1}(s)x - B(t)B^{-1}(s)x\| \to 0, \quad \text{for } x \in D(B),$$

uniformly in K_T. Using the Banach–Steinhaus theorem we obtain the assertion of Lemma 5.

LEMMA 6. *If assumption* (Z_1) *holds*, *the mapping* $[0,T] \ni t \to A(t)x$, $x \in D(A)$, *is of class* C^1 *and inequality* (23) *is satisfied*, *then the mapping* $[0,T] \ni t \to A_n(t)x$, $x \in D(A)$, *is of class* C^1, *where* $A_n(t)$ *is given by* (24), *and*:

$$(44)\qquad A_n'(t)A_n^{-1}(t)x \to A'(t)A^{-1}(t)x \quad \textit{as } n\to\infty, \textit{ for } x\in X,$$

$$(45)\qquad A_n'(t)A^{-1}(s)x \to A'(t)A^{-1}(s)x \quad \textit{as } n\to\infty, \textit{ for } x\in X,$$

$$(46)\qquad A_n'(t)x \to A'(t)x \quad \textit{as } n\to\infty, \textit{ for } x\in D(A),$$

uniformly in $t\in[0,T]$ *or* $(t,s)\in K_T$, *respectively.*

Proof. Let us remark that the operator $A_n(t)$ may be written in the form

$$(47)\qquad A_n(t) = n^2R(n;A(t)) - nI.$$

Formula (47) implies the existence of $A_n'(t)$ in the strong sense, and

$$(48)\qquad A_n'(t) = n^2R(n;A(t))A'(t)R(n;A(t)),$$

and so the mapping $[0,T]\ni t\to A_n'(t)x$, $x\in D(A)$, is continuous.

By (48) and (24) we get

$$A_n'(t)A_n^{-1}(t)x = nR(n;A(t))[A'(t)A^{-1}(t)]x.$$

Since the bounded operators $nR(n;A(t))$ tend to the identity operator strongly and uniformly in $t\in[0,T]$, we have (44). Further,

$$A_n'(t)A^{-1}(s)x = [A_n'(t)A_n^{-1}(t)][A_n(t)A^{-1}(t)][A(t)A^{-1}(s)x].$$

Because the operators $A_n'(t)A^{-1}(t)$ and $A_n(t)A^{-1}(t)$ are bounded and tend to $A'(t)A^{-1}(t)$ and the identity operator, respectively, by Lemmas 3.7 and 3.8 in [5, p. 151] we obtain (45); (46) follows immediately from (45).

Lemma 7. *Assume* (Z_1)–(Z_4). *If the operator* $A(t)$ *for* $t\in[0,T]$ *satisfies inequality* (23) *and the mapping*

$$(49)\qquad [0,T]\ni t\to A(t)x,\qquad x\in D(A),$$

is of class C^1, *then for every* $n\in\mathbb{N}$ *and* $x\in D(A)$ *the mapping*

$$[0,T]\ni t\to B_n(t)x$$

is of class C^1 *and*

$$(50)\qquad B_n'(t)B^{-1}(s)x\to B'(t)B^{-1}(s)x,\qquad x\in X,$$

uniformly in the square K_T.

Proof. From (24) we see that if the mapping (49) is of class C^1, then so is

$$[0,T]\ni t\to A_n(t)x,\qquad x\in D(A),\ n\in\mathbb{N}.$$

From (39) we get

$$(51)\qquad B_n(t)x = \frac{i}{\pi}\int_0^\infty \lambda^{-1/2}[I-\lambda R(\lambda;A_n(t))]\,d\lambda.$$

Fix $x \in D(A)$. We let

$$I(t) := \frac{i}{\pi} \int_0^\infty \frac{d}{dt}\{\lambda^{-1/2}[I - \lambda R(\lambda; A_n(t))]\} x \, d\lambda$$
$$= \frac{i}{\pi} \int_0^\infty \lambda^{1/2} R(\lambda; A_n(t))(-A_n'(t)) R(\lambda; A_n(t)) x \, d\lambda = I_1(t) + I_2(t),$$

where

$$I_1(t) := \frac{i}{\pi} \int_0^1 \lambda^{1/2} R(\lambda; A_n(t))(-A_n'(t)) R(\lambda; A_n(t)) x \, d\lambda$$

and

$$I_2(t) := \frac{i}{\pi} \int_1^\infty \lambda^{1/2} R(\lambda; A_n(t))(-A_n'(t)) R(\lambda; A_n(t)) x \, d\lambda.$$

From this we get

$$\begin{aligned} \|I_1(t)\| &\le \frac{1}{\pi} \int_0^1 \lambda^{-1/2} \|\lambda R(\lambda; A_n(t))\| \\ &\quad \times \|A_n'(t) A_n^{-1}(t)\| \cdot \|A_n(t) R(\lambda; A_n(t))\| \cdot \|x\| \, d\lambda \\ &\le \frac{1}{\pi} MC(M+1)\|x\| \int_0^1 \lambda^{-1/2} \, d\lambda = \frac{2}{\pi} M(M+1)C\|x\| \end{aligned} \tag{52}$$

and

$$\begin{aligned} \|I_2(t)\| &\le \frac{1}{\pi} \int_1^\infty \lambda^{-3/2} \|\lambda R(\lambda; A_n(t))\|^2 \| \\ &\quad \times \|A_n'(t) A_n^{-1}(t)\| \cdot \|A_n(t) A^{-1}(t)\| \cdot \|A(t)x\| \, d\lambda \\ &\le \frac{2}{\pi} CM^2 C_1 \sup_{0 \le t \le T} \|A(t)x\|, \end{aligned} \tag{53}$$

where

$$\|A_n'(t) A_n^{-1}(t)\| \le C, \quad \|A_n(t) A^{-1}(t)\| \le C_1 \quad \text{and} \quad \|\lambda R(\lambda; A_n(t))\| \le M.$$

From (52) and (53) it follows that for $x \in D(A)$ and $n \in \mathbb{N}$ we have

$$B_n'(t)x = \frac{i}{\pi} \int_0^\infty \lambda^{1/2} R(\lambda; A_n(t))(-A_n'(t)) R(\lambda; A_n(t)) x \, d\lambda. \tag{54}$$

Using (52) and (53) once more, and in view of the well-known theorem on the passing to the limit under the improper integral in (54), for $x \in D(A)$

we obtain

$$B'(t)x = \frac{i}{\pi} \int_0^{\infty} \lambda^{1/2} R(\lambda; A(t))(-A'(t))R(\lambda; A(t))x \, d\lambda. \tag{55}$$

We conclude from (55) that

$$B_n'(t)x \to B'(t)x \quad \text{as } n \to \infty \text{ for } x \in D(A).$$

The same reasoning as in the proof of Lemma 5 shows that (50) holds. Lemma 7 is proved.

COROLLARY 1. *Under the assumptions of Lemma* 7, *formula* (50) *implies that there exists a constant* $L > 0$ *such that*

$$\|B_n'(t)B^{-1}(s)\| \le L, \tag{56}$$

where L *does not depend on* $(t, s) \in K_T$ *and on* $n \in \mathbb{N}$.

4. Second order initial value problem. Similarly to Section 2, we now consider the homogeneous differential equation corresponding to problem (7), i.e.

$$\frac{d^2u}{dt^2} = A(t)u, \quad t \in [0, T], \tag{57}$$

where $A(t)$ is the operator defined in the introduction and satisfying assumption (Z_1) and inequality (23). It follows that, for each $t \in [0, T]$, there is a linear operator $B(t) : X \to X$ such that

$$B^2(t)x = A(t)x \quad \text{for } x \in D(A),\ t \in [0, T]. \tag{58}$$

We assume that the operator $B(t)$ satisfies (Z_3) and (Z_4).

DEFINITION 6 (cf. [6]). A family S of bounded operators $S(t, s) : X \to X$, $t, s \in [0, T]$, is said to be the *fundamental solution* for equation (57) if:

(D_1) For each $x \in X$ the mapping $K_T \ni (t, s) \to S(t, s)x \in X$ is of class C^1 and
(a) for each $t \in [0, T]$, $S(t, t) = 0$,
(b) for all $t, s \in [0, T]$ and each $x \in X$,

$$\frac{\partial}{\partial t} S(t, s)\Big|_{t=s} x = x, \qquad \frac{\partial}{\partial s} S(t, s)\Big|_{t=s} x = -x.$$

(D_2) For all $t, s \in [0, T]$, if $x \in D(B)$ then $S(t, s)x \in D(A)$ and the mapping $[0, T] \ni t \to S(t, s)x \in X$ is of class C^2; moreover,

(a) $$\frac{\partial^2}{\partial t^2} S(t, s)x = A(t)S(t, s)x,$$

(b) $$\frac{\partial^2}{\partial s \partial t} S(t, s)\Big|_{t=s} x = 0,$$

while for $x \in D(A)$ the mapping $K_T \ni (t,s) \to S(t,s)x \in X$ is of class C^2 and

(c) $$\frac{\partial^2}{\partial s^2} S(t,s)x = S(t,s)A(s)x.$$

(D$_3$) For all $t,s \in [0,T]$, if $x \in D(A)$ then $\frac{\partial}{\partial s} S(t,s)x \in D(A)$, the derivatives $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x$ exist and

(a) $$\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x = A(t)\frac{\partial}{\partial s} S(t,s)x,$$

(b) $$\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x = \frac{\partial}{\partial t} S(t,s)A(s)x,$$

and the mapping $K_T \ni (t,s) \to A(t)\frac{\partial}{\partial s} S(t,s)x$ is continuous.

(D$_4$) For all $t,s,r \in [0,T]$,

$$S(t,s) = S(t,r)\frac{\partial}{\partial r} S(r,s) - \frac{\partial}{\partial r} S(t,r)S(r,s).$$

Under assumptions (Z$_1$)–(Z$_4$) we denote by Y the linear space $D(B)$ with the norm $\|\cdot\|_Y$ given by

(59) $$\|y\|_Y := \|y\| + \|B(0)y\|, \quad y \in D(B).$$

Let $\mathcal{A}(t)$, $t \in [0,T]$, be a linear operator such that $\mathcal{A}(t) : Y \times X \to Y \times X$, where

(60) $$\mathcal{A}(t) := \begin{bmatrix} 0 & I \\ A(t) & 0 \end{bmatrix}, \quad t \in [0,T],$$

with domain $D(A) \times D(B)$. We define the sequence $\{\mathcal{A}_n(t)\}$ of linear operators acting from $Y \times X$ into itself by

(61) $$\mathcal{A}_n(t) := \begin{bmatrix} 0 & I \\ A_n(t) & 0 \end{bmatrix} \quad \text{for } t \in [0,T],\ n \in \mathbb{N},$$

where $A_n(t)$ is defined by formula (24).

THEOREM 2. *Under assumptions* (Z$_1$) *and* (Z$_3$), *if the mapping* $[0,T] \ni t \to A(t)x$, *for* $x \in D(A)$ *is of class* C^1 *and inequality* (23) *holds*, *then the sequence* $\{\mathcal{A}_n(t)\}$ *satisfies the following conditions*:

1° *for each* $n \in \mathbb{N}$, *the operator* $\mathcal{A}_n(t)$, $t \in [0,T]$, *is bounded*;

2° *the mapping* $[0,T] \ni t \to \mathcal{A}_n(t)$ *is strongly continuously differentiable*;

3° *the operator* $\mathcal{A}_n(t)$ *is invertible for each* $t \in [0,T]$ *and*

(62) $$\|\mathcal{A}_n^{-1}(t)\| \le C \quad (C \text{ does not depend on } n \in \mathbb{N} \text{ and } t);$$

4° *the sequence* $\{\mathcal{A}_n'(t)\mathcal{A}_n^{-1}(t)\}$ *is strongly and uniformly convergent to a bounded operator* $\mathcal{G}(t)$;

5° $\lim_{n\to\infty} \left\{ \sup_{0\le t\le T} \left\| [\mathcal{A}_n(t) - \mathcal{A}(t)]\mathcal{A}^{-1}(t) \left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right] \right\| \right\} = 0$, $(y,x) \in Y \times X$.

P r o o f. 1° Let us remark that for each $t \in [0,T]$, we have

$$\mathcal{A}_n(t) := \begin{bmatrix} B_n^{-1}(t) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & B_n(t) \\ B_n(t) & 0 \end{bmatrix} \begin{bmatrix} B_n(t) & 0 \\ 0 & I \end{bmatrix}, \tag{63}$$

where

$$\begin{bmatrix} B_n^{-1}(t) & 0 \\ 0 & I \end{bmatrix} : X \times X \to Y \times X, \qquad \begin{bmatrix} B_n(t) & 0 \\ 0 & I \end{bmatrix} : Y \times X \to X \times X,$$
$$\begin{bmatrix} 0 & B_n(t) \\ B_n(t) & 0 \end{bmatrix} : X \times X \to X \times X$$

(cf. [10]). It is easy to prove that each operator on the right-hand side of (63) is bounded. Indeed,

$$\begin{aligned} \left\| \begin{bmatrix} B_n^{-1}(t) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} \right\| &= \left\| \begin{bmatrix} B_n^{-1}(t)y \\ x \end{bmatrix} \right\| \\ &= \|B_n^{-1}(t)y\| + \|B(0)B_n^{-1}(t)y\| + \|x\| \\ &\le \alpha\|y\| + \beta\|y\| + \|x\| \\ &\le \gamma(\|y\| + \|x\|) = \gamma \left\| \begin{bmatrix} y \\ x \end{bmatrix} \right\|, \end{aligned}$$

where $\gamma = \max(\alpha + \beta, 1)$ and the existence of the constants α and β follows from Lemma 5. Thus

$$\left\| \begin{bmatrix} B_n^{-1}(t) & 0 \\ 0 & I \end{bmatrix} \right\| \le \gamma. \tag{64}$$

Further,

$$\begin{aligned} \left\| \begin{bmatrix} B_n(t) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} \right\| &= \left\| \begin{bmatrix} B_n(t)y \\ x \end{bmatrix} \right\| = \|B_n(t)y\| + \|x\| \\ &= \|B_n(t)B^{-1}(0)B(0)y\| + \|x\| \\ &\le \|B_n(t)B^{-1}(0)\| \cdot \|B(0)y\| + \|x\| \\ &\le a\|B(0)y\| + \|x\| \le a(\|y\| + \|B(0)y\|) + \|x\| \\ &\le b(\|y\| + \|B(0)y\| + \|x\|) = b \left\| \begin{bmatrix} y \\ x \end{bmatrix} \right\|, \end{aligned}$$

where $b = \max(a, 1)$ and the existence of the constant a follows from Lemma 5. Thus

$$\left\| \begin{bmatrix} B_n(t) & 0 \\ 0 & I \end{bmatrix} \right\| \le b. \tag{65}$$

Since

$$\left\| \begin{bmatrix} 0 & B_n(t) \\ B_n(t) & 0 \end{bmatrix} \right\| = 2\|B_n(t)\|, \tag{66}$$

part 1° is proved.

2° follows imediately from the assumptions.

3° From (63) we get

$$\mathcal{A}_n^{-1}(t) = \begin{bmatrix} B_n^{-1}(t) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & B_n^{-1}(t) \\ B_n^{-1}(t) & 0 \end{bmatrix} \begin{bmatrix} B_n(t) & 0 \\ 0 & I \end{bmatrix}. \tag{67}$$

Hence

$$\begin{aligned} \|\mathcal{A}_n^{-1}(t)\| &\le \left\| \begin{bmatrix} B_n^{-1}(t) & 0 \\ 0 & I \end{bmatrix} \right\| \left\| \begin{bmatrix} 0 & B_n^{-1}(t) \\ B_n^{-1}(t) & 0 \end{bmatrix} \right\| \left\| \begin{bmatrix} B_n(t) & 0 \\ 0 & I \end{bmatrix} \right\| \\ &\le \gamma 2\|B_n^{-1}(t)\| b \le 2\alpha\gamma b := C, \end{aligned}$$

where C does not dependend on $t \in [0,T]$ and $n \in \mathbb{N}$.

4° An easy computation shows that

$$\mathcal{A}_n'(t)\mathcal{A}_n^{-1}(t) = \begin{bmatrix} 0 & 0 \\ 0 & A_n'(t)A_n^{-1}(t) \end{bmatrix}.$$

From this, by formula (44) of Lemma 6, we get 4°, where the operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t) = \begin{bmatrix} 0 & 0 \\ 0 & A'(t)A^{-1}(t) \end{bmatrix}. \tag{68}$$

It is evident that $\mathcal{G}(t)$ is bounded and

$$\|\mathcal{G}(t)\| \le \|A'(t)A^{-1}(t)\|. \tag{69}$$

5° It is easy to check that

$$\left\| [\mathcal{A}_n(t) - \mathcal{A}(t)]\mathcal{A}^{-1}(t) \begin{bmatrix} y \\ x \end{bmatrix} \right\| = \| [A_n(t) - A(t)]A^{-1}(t)x\|, \quad x \in X,\ y \in Y.$$

This and (15) imply 5°, and the proof is complete.

For each $t \in [0,T]$ and $n \in \mathbb{N}$ let us denote by $\mathcal{B}_n(t)$ the linear operator from $X \times X$ into itself given by

$$\mathcal{B}_n(t) = \begin{bmatrix} 0 & B_n(t) \\ B_n(t) & 0 \end{bmatrix}. \tag{70}$$

Using Lemma 2.4 of [6] and our Lemma 2 we get

LEMMA 8. *Under the assumptions of Lemma* 2, *the family* $\{\mathcal{B}_n(t)\}$, $t \in [0,T]$, $n \in \mathbb{N}$, *is uniformly stable with stability constants* $\overline{M} = 2M$ *and* $\overline{\omega} = 0$, *i.e., the following inequalities hold*:

$$\|(\lambda - \mathcal{B}_n(t_k))^{-1}(\lambda - \mathcal{B}_n(t_{k-1}))^{-1} \dots (\lambda - \mathcal{B}_n(t_1))^{-1}\| \le 2M|\lambda|^{-k} \tag{71}$$

and

$$\|(\lambda - \mathcal{B}_n(t_1))^{-1}(\lambda - \mathcal{B}_n(t_2))^{-1} \dots (\lambda - \mathcal{B}_n(t_k))^{-1}\| \le 2M|\lambda|^{-k}. \tag{72}$$

THEOREM 3. *Let assumptions* (Z_1)–(Z_4) *hold. If the family of generators* $\{A(t)\}$, $t \in [0,T]$, *is stable with stability constants* $M \ge 1$ *and* $\omega = 0$ (*cf.*

Lemma 2), *and the mapping* $[0,T] \ni t \to A(t)x$, $x \in D(A)$, *is continuously differentiable*, *then the family* $\{\mathcal{A}_n(t)\}$, $t \in [0,T]$, $n \in \mathbb{N}$, *is uniformly stable*, *i.e.*, *the following inequalities hold*:

$$\|(\lambda - \mathcal{A}_n(t_k))^{-1}(\lambda - \mathcal{A}_n(t_{k-1}))^{-1} \ldots (\lambda - \mathcal{A}_n(t_1))^{-1}\| \le \widetilde{M}\lambda^{-k} \tag{73}$$

for $\lambda > 0$, *and*

$$\|(\lambda - \mathcal{A}_n(t_1))^{-1}(\lambda - \mathcal{A}_n(t_2))^{-1} \ldots (\lambda - \mathcal{A}_n(t_k))^{-1}\| \le \widetilde{M}(-\lambda)^{-k} \tag{74}$$

for $\lambda < 0$, *for any finite sequence* $0 \le t_1 \le t_2 \le \ldots \le t_k \le T$, $k, n \in \mathbb{N}$, *where* $\widetilde{M} \ge 1$ *is a constant independent of* n.

P r o o f. The proof of this theorem is an analogue of that of Lemma 2.5 in [6] as well as Theorem 2.4 in [8, p. 133]. If λ belongs to the resolvent set of the operator $\mathcal{A}_n(t)$, then it is easily seen that

$$(\lambda - \mathcal{A}_n(t))^{-1} = \begin{bmatrix} B_n^{-1}(t) & 0 \\ 0 & I \end{bmatrix} (\lambda - \mathcal{B}_n(t))^{-1} \begin{bmatrix} B_n(t) & 0 \\ 0 & I \end{bmatrix} \tag{75}$$

for $t \in [0,T]$, $n \in \mathbb{N}$. By (70), for $\lambda > 0$ we obtain

$$\prod_{j=1}^{k} (\lambda - \mathcal{A}_n(t_j))^{-1} = \prod_{j=1}^{k} \begin{bmatrix} B_n^{-1}(t_j) & 0 \\ 0 & I \end{bmatrix} (\lambda - \mathcal{B}_n(t_j))^{-1} \begin{bmatrix} B_n(t_j) & 0 \\ 0 & I \end{bmatrix}. \tag{76}$$

Letting, for $j = 2, \ldots, k$,

$$P_{n,j} := \left(\begin{bmatrix} B_n(t_j) & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} B_n(t_{j-1}) & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} B_n^{-1}(t_{j-1}) & 0 \\ 0 & I \end{bmatrix},$$

we get

$$\begin{bmatrix} B_n(t_j) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_n^{-1}(t_{j-1}) & 0 \\ 0 & I \end{bmatrix} = \widetilde{I} + P_{n,j},$$

where

$$\widetilde{I} := \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

is the unit matrix in $X \times X$.

From this we get

$$\begin{aligned} \prod_{j=1}^{k} (\lambda - \mathcal{A}_n(t_j))^{-1} = \begin{bmatrix} B_n^{-1}(t_k) & 0 \\ 0 & I \end{bmatrix} \{(\lambda - \mathcal{B}_n(t_k))^{-1}(\widetilde{I} + P_{n,k}) \ldots \\ \ldots (\lambda - \mathcal{B}_n(t_2))^{-1}(\widetilde{I} + P_{n,2})(\lambda - \mathcal{B}_n(t_1))^{-1}\} \begin{bmatrix} B_n(t_1) & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{77}$$

Using the uniform stability of the family $\{\mathcal{B}_n(t)\}$ (cf. Lemma 8), the norm of the expression in curly brackets in formula (77) may be estimated by

$$2M\lambda^{-k} \prod_{j=2}^{k} (1 + 2M\|P_{n,j}\|). \tag{78}$$

Let us remark that

$$\|P_{n,j}\| \le \left\| \begin{bmatrix} B_n(t_j) & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} B_n(t_{j-1}) & 0 \\ 0 & I \end{bmatrix} \right\| \left\| \begin{bmatrix} B_n^{-1}(t_{j-1}) & 0 \\ 0 & I \end{bmatrix} \right\|. \tag{79}$$

Now we estimate the norms of the matrices in (79). We have

$$\begin{aligned} &\left\| \left(\begin{bmatrix} B_n(t_j) & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} B_n(t_{j-1}) & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} y \\ x \end{bmatrix} \right\| \\ &\qquad = \|[B_n(t_j) - B_n(t_{j-1})]y\| \\ &\qquad \le \|[B_n(t_j) - B_n(t_{j-1})]B^{-1}(0)\| \cdot \|B(0)y\| \\ &\qquad \le \sup_{0\le t\le T} [\|B_n'(t)B^{-1}(0)\|](t_j - t_{j-1})\|y\|_Y. \end{aligned}$$

By Lemma 5, from the above it follows that

$$\left\| \begin{bmatrix} B_n(t_j) & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} B_n(t_{j-1}) & 0 \\ 0 & I \end{bmatrix} \right\| \le L(t_j - t_{j-1}),$$

where $L := \sup\{\|B_n'(t)B^{-1}(0)\| : t \in [0,T],\ n \in \mathbb{N}\}$ is a constant independent of $t \in [0,T]$ and of $n \in \mathbb{N}$ (cf. (56)).

Since the second factor on the right-hand side of (79) is, by virtue of (64), estimated by γ, which does not depend on n either, we get

$$\|P_{n,j}\| \le K(t_j - t_{j-1}), \qquad j = 2, \dots, k, \tag{80}$$

where $K = L\gamma$.

Using the estimates (64), (65), (80) and the elementary inequality $1+x \le e^x$ for $x \ge 0$, from (77) we obtain

$$\begin{aligned} \left\| \prod_{j=1}^{k} (\lambda - \mathcal{A}_n(t_j))^{-1} \right\| &\le 2Mb\gamma|\lambda|^{-k} \prod_{j=2}^{k} \exp(2MK(t_j - t_{j-1})) \\ &\le 2Mb\gamma \exp(2MKT)|\lambda|^{-k}. \end{aligned}$$

Therefore

$$\left\| \prod_{j=1}^{k} (\lambda - \mathcal{A}_n(t_j))^{-1} \right\| \le \widetilde{M}|\lambda|^{-k}, \tag{81}$$

where $\widetilde{M} = 2Mb\gamma \exp(2MKT)$ is a constant independent of $n \in \mathbb{N}$.

Analogously we can prove the estimate (74).

Because the stability constants $\widetilde{M}$ and $\omega = 0$ do not depend on $n \in \mathbb{N}$, this implies that the sequence $\{\mathcal{A}_n(t)\}$ is uniformly stable (see [1]). Moreover, each operator $\mathcal{A}_n(t)$ for fixed $t \in [0,T]$ and $n \in \mathbb{N}$ is the infinitesimal generator of a strongly continuous group.

Now we can prove the following important theorem.

Theorem 4. *Let assumptions* (Z_1)–(Z_4) *hold. If* $\{A(t)\}$, $t \in [0,T]$, *is a stable family with stability constants* $M \geq 1$ *and* $\omega = 0$, *and the mapping* $[0,T] \ni t \to A(t)x$, $x \in D(A)$, *is continuously differentiable*, *then the conditions of Theorem* 1 *are satisfied for the operator* $\mathcal{A}(t)$ *given by* (60) *and the operators* $\mathcal{A}_n(t)$ *constructed according to formula* (61).

P r o o f. The stability of $\{A(t)\}$ yields the estimate (23), and so, by Theorem 2, we get assumptions 1° and 2° of Theorem 1. Let $\mathcal{V}_n(t,s)$ denote the evolution operator (fundamental solution) corresponding to the operator $\mathcal{A}_n(t)$. From Lemmas 2 and 8 and Theorem 3 it follows that the approximating sequence $\{\mathcal{A}_n(t)\}$ is uniformly stable with stability constants $\widetilde{M}$ and $\omega = 0$ (see (81)). By Theorem 1 of [1] we obtain

$$\|\mathcal{V}_n(t,s)\| \leq \widetilde{M} \quad \text{for } t,s \in [0,T],\ n \in \mathbb{N}. \tag{82}$$

Basing on the estimate (82), analogously to the proof of Theorem 3.11 in [7, p. 208] we can prove that $\{\mathcal{V}_n(t,s)\}$ is strongly and uniformly convergent to $\mathcal{V}(t,s)$ in $K_T = [0,T] \times [0,T]$. We omit the details.

From this we have, as a consequence of Theorem 1, the following corollary.

Corollary 2. *Under the assumptions of Theorem* 4 *the Cauchy problem for the equation*

$$\frac{dw}{dt} = \mathcal{A}(t)w, \qquad t \in [0,T], \tag{83}$$

is uniformly correct and $\mathcal{V}(t,s)$ *is the evolution operator corresponding to it.*

By definition, $\mathcal{A}(t)$ for each $t \in [0,T]$ is a linear operator acting from $Y \times X$ into itself, where Y is the space $D(B)$ with graph norm (see (59)). Therefore, the evolution operator $\mathcal{V}(t,s)$ may be written as

$$\mathcal{V}(t,s) = \begin{bmatrix} C(t,s) & S(t,s) \\ b(t,s) & a(t,s) \end{bmatrix}. \tag{84}$$

Let us define (cf. [10] and [6]) the operator

$$S(t,s)x := \Pi_1 \mathcal{V}(t,s) \begin{bmatrix} 0 \\ x \end{bmatrix} \qquad \text{for every } x \in X, \tag{85}$$

where $\Pi_1 \left[\begin{smallmatrix} y \\ x \end{smallmatrix} \right] := y$ for $y \in Y$, $x \in X$. Using Definition 3 and formula (85) one can prove that $S(t,s)$ is the evolution operator (fundamental solution) for equation (57). For details see [6].

Summing up, we can formulate the main result of this paper.

Theorem 5. *If the assumptions of Theorem* 4 *are satisfied*, *then the Cauchy problem for equation* (57) *is uniformly correct and the* $S(t,s)$ *defined by formula* (85) *is the evolution operator corresponding to it.*

If the fundamental solution $S(t,s)$ for (57) is known and if the function $f:[0,T]\to X$ satisfies certain conditions, and if $u_0,u_1\in D(A)$, then the problem (7) has the unique solution u given by

$$u(t)=-\frac{\partial}{\partial s}S(t,s)\Big|_{s=0}u_0+S(t,0)u_1+\int_0^t S(t,s)f(s)\,ds \tag{86}$$

(see for example [6] and [11]).

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