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Projectivity and lifting of Hilbert module maps

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Abstract. In a recent paper, Carlson, Foiaş, Williams and the author proved that isometric Hilbert modules are projective in the category of Hilbert modules similar to contractive ones. In this paper, a simple proof, based on a strengthened lifting theorem, is given. The proof also applies to an equivalent theorem of Foiaş and Williams on similarity to a contraction of a certain 2×2 operator matrix.

1. Introduction. One of the obstacles to understanding homological algebra invariants for Hilbert modules is lack of knowledge about projectives in the Hilbert module category (see [3], [1]).

In the work of Douglas, Foiaş and Paulsen, an analogous concept, *hypoprojectivity*, is defined and it is shown that lifting theorem techniques of operator theory, recast in the Hilbert module setting (see Lemma 0 below), are precisely what is needed to characterize the concept. A result is a proof that the hypoprojective Hilbert modules (over the disk algebra $\mathbb{A}(\mathbb{D})$, for example) are precisely the isometric ones; see [3, Chapter 4]. However, there is no indication that hypoprojectivity is as useful as projectivity in the study of cohomology and other invariants from homological algebra.

In [2], Carlson, Foiaş, Williams and the present author identified the projectives in the category of *cramped* Hilbert modules over $\mathbb{A}(\mathbb{D})$; the term *cramped* refers to Hilbert modules similar to contractive ones. In the present note, we obtain one of the main results of [2] (our Theorem 1) in a simpler and more straightforward manner, following the outline of the proof of the characterization of hypoprojectivity in [3, Chapter 4].

Following Douglas and Paulsen [3], we use the term *Hilbert module* to mean a Hilbert space H together with the action of a function algebra $\mathbb{A}$. If $\mathbb{A}$ is the disk algebra $\mathbb{A} = \mathbb{A}(\mathbb{D})$, properties of the Hilbert module H reflect properties of the operator $T : H \to H$ defined by $Tf = zf$.

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2. Operator matrices. Theorem 1 below yields a Hilbert-module-theoretic proof of the following result, known to Foiaş and Williams some years ago, and published in [2].

Let T_0 and T_1 be contraction operators on Hilbert spaces H_0 and H_1, T_1 similar to an isometry. Then the operator

$$T = \begin{bmatrix} T_0 & X \\ 0 & T_1 \end{bmatrix}$$

is similar to a contraction if and only if there is a bounded operator $L : H_1 \to H_0$ with $X = LT_1 - T_0L$.

The "if" implication is easy, using the similarity

$$\begin{bmatrix} I & -L \\ 0 & I \end{bmatrix} \begin{bmatrix} T_0 & LT_1 - T_0L \\ 0 & T_1 \end{bmatrix} \begin{bmatrix} I & L \\ 0 & I \end{bmatrix} = \begin{bmatrix} T_0 & 0 \\ 0 & T_1 \end{bmatrix}.$$

For the "only if" part, regard H_0 and H_1 as Hilbert modules over $\mathbb{A}(\mathbb{D})$, with multiplication defined by $zf = T_if$ $(f \in H_i)$. Make the direct sum $H_0 \mathbin{\widehat{\oplus}} H_1$ into a Hilbert module by defining

$$z(f, g) = (T_0f + Xg, T_1g), \qquad (f, g) \in H_0 \mathbin{\widehat{\oplus}} H_1.$$

Then $H_0 \mathbin{\widehat{\oplus}} H_1$ is cramped if and only if T is similar to a contraction and the exact sequence of Hilbert modules

$$0 \longrightarrow H_0 \xrightarrow{\alpha} H_0 \mathbin{\widehat{\oplus}} H_1 \xrightarrow{\beta} H_1 \longrightarrow 0 \tag{1}$$

$(\alpha f = (f, 0),\ \beta(f, g) = g)$ splits if and only if there exists $L : H_1 \to H_0$ such that $f \to (Lf, f)$ is a Hilbert-module map. The latter is equivalent to the relation $X = LT_1 - T_0L$.

But by Theorem 1 of the present note, H_1, being similar to an isometric module, is projective in the cramped category. It is a matter of elementary homological algebra to show that this implies that (1) splits.

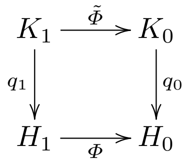
3. Hypoprojectivity. For a Hilbert module H over $\mathbb{A}(\mathbb{D})$, we have automatically

$$\|p(T)f\| = \|p(z)f\| \le K\|p\|_\infty \|f\| \tag{2}$$

for $p \in \mathbb{A}(\mathbb{D})$ and $f \in H$; that is, the operator T is polynomially bounded. We shall say that the Hilbert module H has a certain property (H is contractive, isometric, unitary, etc.) according as the operator T has that property (T is a contraction, an isometry, a unitary operator, etc.). By the same convention, a Hilbert module K is a (minimal, isometric) dilation of H if $K \supseteq H$, as Hilbert *spaces*, and the operator T' of multiplication by z on K is a (minimal, isometric) dilation of the corresponding operator T on H. In this case, the orthogonal projection $q : K \to H$ is a Hilbert-module map (i.e. q carries the action of $\mathbb{A}(\mathbb{D})$ on K into the action of $\mathbb{A}(\mathbb{D})$ on H). The term *isometric* *dilation Hilbert module* is a compromise on the term *Shilov dominant* of [3], where more general function algebras are considered.

In the terminology just adopted, the lifting theorem of Sz.-Nagy and Foiaş may be stated as follows.

LEMMA 0. *Let H_0 and H_1 be contractive Hilbert modules, with K_0 and K_1 their minimal isometric dilation modules, and let $\Phi : H_1 \to H_0$ be a Hilbert-module map. Then there is a Hilbert-module map $\widetilde{\Phi} : K_1 \to K_0$ with $\|\widetilde{\Phi}\| = \|\Phi\|$ and such that the diagram*

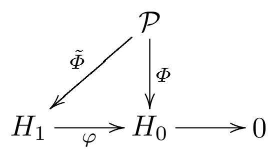


commutes, where q_0 and q_1 are the orthogonal projections.

See Douglas and Paulsen [3, Theorem 4.12]; we have, in this version, avoided the use of their term *Shilov resolution*.

The following theorem of Douglas and Foiaş can be stated as "Shilov modules are hypoprojective", and its proof can be based upon Lemma 0 (see [3, Theorems 4.11 and 4.16]).

THEOREM 0. *Suppose $\mathcal{P}, H_0$ and H_1 are Hilbert modules over $\mathbb{A}(\mathbb{D})$ with $\mathcal{P}$ isometric and H_i contractive, $i = 0, 1$. Suppose $\varphi : H_1 \to H_0$ and $\Phi : \mathcal{P} \to H_0$ are Hilbert-module maps with φ partially isometric and surjective. Then there exists a Hilbert-module map $\tilde{\Phi} : \mathcal{P} \to H_1$ with $\|\widetilde{\Phi}\| = \|\Phi\|$ and making*



commute.

The principal device in the proof of Theorem 0, besides Lemma 0, is Lemma 1 below, in the case where φ is a partial isometry; the two diagrams are put together as in the proof of Lemma 2 below.

4. Projectivity. Our first lemma is a modest generalization of the main step in the proof of Theorem 4.16 of Douglas and Paulsen [3].

LEMMA 1. *Suppose H_0 and H_1 are contractive Hilbert modules over $\mathbb{A}(\mathbb{D})$ and φ is a Hilbert-module map of H_1 onto H_0. Then there exists a Hilbert-module map $\eta : K_0 \to H_1$, where K_0 is the minimal isometric dilation* *module of* H_0, *such that*

$$\begin{array}{ccc} & & K_0 \\ & \swarrow^{\eta} & \downarrow{\scriptstyle q_0} \\ H_1 & \xrightarrow[\varphi]{} & H_0 \end{array}$$

commutes and $\|\eta\| \le \|[\varphi|_{(\ker\varphi)^\perp}]^{-1}\|$.

Proof. Write $H_1 = \ker\varphi \oplus H_1'$ and let $\psi : H_0 \to H_1'$ be the inverse of the restriction of φ to H_1'.

Give H_1' the compressed Hilbert module structure inherited from H_1. That is, let $\mathbb{A}(\mathbb{D})$ act on H_1' by

$$p(z')f = Pp(z)f$$

for $p \in \mathbb{A}(\mathbb{D})$ and $f \in H_1'$, where P is the projection of H_1 and H_1'. Then ψ is a Hilbert-module map because, for $g \in H_0$,

$$\psi zg = \psi z\varphi\psi g = \psi\varphi z\psi g = \psi\varphi Pz\psi g = z'\psi g.$$

Now let K_1' denote the minimal isometric dilation module of H_1'. We can arrange to have

$$K_1' \subset K_1,$$

where K_1 is the minimal isometric dilation module of H_1; indeed, K_1' is isomorphic to $\mathbb{A}(\mathbb{D}) \cdot H_1'$ (in the multiplication of K_1).

Now the lifting theorem (Lemma 0) implies the existence of a Hilbert-module map $\Psi : K_0 \to K_1'$ with $\|\Psi\| = \|\psi\|$ and with

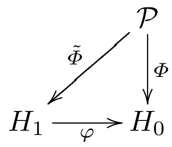
$$\begin{array}{ccc} K_0 & \xrightarrow{\Psi} & K_1' \\ {\scriptstyle q_0}\downarrow & & \downarrow{\scriptstyle q_1'} \\ H_0 & \xrightarrow[\psi]{} & H_1' \end{array}$$

commuting.

For $f \in K_0$, we therefore have $q_1'\Psi f = \psi q_0 f$, or $\varphi q_1'\Psi f = q_0 f$. If we set $\eta = q_1'\Psi$ and replace H_1' by $H_1 = H_1' \oplus \ker\varphi$, the lemma follows.

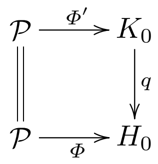
The following lemma shows how to put the diagrams in Lemmas 0 and 1 together. The result contains Theorem 0.

Lemma 2. *If* $\mathcal{P}$ *is an isometric Hilbert module*, H_0 *and* H_1 *are contractive Hilbert modules*, $\varphi : H_1 \to H_0$ *is a surjective Hilbert-module map and* $\Phi : \mathcal{P} \to H_0$ *is a Hilbert-module map*, *then there exists a Hilbert-module* *map $\widetilde{\Phi} : \mathcal{P} \to H_1$ with*

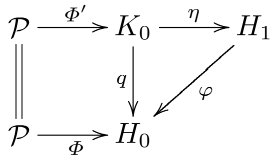


commuting and $\|\widetilde{\Phi}\| \le \|\Phi\| \cdot \|[\varphi|_{(\ker \varphi)^\perp}]^{-1}\|$.

P r o o f. Let K_0 be the minimal isometric dilation module of H_0. By the lifting theorem (Lemma 0), there exists $\Phi' : \mathcal{P} \to K_0$ making



commute and with $\|\Phi'\| = \|\Phi\|$. By Lemma 1, we see that



commutes and $\|\eta\| \le \|[\varphi|_{(\ker \varphi)^\perp}]^{-1}\|$. Setting $\widetilde{\Phi} = \eta\Phi'$ completes the proof.

The following theorem is equivalent to Corollary 3.3 of [2]: Hilbert modules similar to isometric Hilbert modules are projective in the category of cramped Hilbert modules. But since Theorem 1 comes from the lifting theorem, we can also give a bound on the operator constructed.

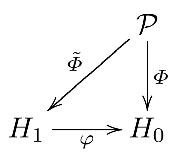
If L is a bounded, invertible operator on Hilbert space, we denote by $k(L)$ the positive constant

$$k(L) = \|L\| \cdot \|L^{-1}\|.$$

THEOREM 1. *Let $\mathcal{P}$, H_0, and H_1 be Hilbert modules with $\mathcal{P}$ similar to an isometric Hilbert module and H_0 and H_1 cramped. That is, let*

$$T = LSL^{-1} \quad \text{and} \quad T_i = L_i S_i L_i^{-1}, \quad i = 0, 1,$$

where T is multiplication by z on $\mathcal{P}$, T_i is multiplication by z on H_i, $i = 0, 1$, S is an isometry and S_i is a contraction, $i = 0, 1$. Suppose further that $\varphi : H_1 \to H_0$ and $\Phi : \mathcal{P} \to H_0$ are Hilbert-module maps with φ surjective. Then there exists a Hilbert-module map $\widetilde{\Phi} : \mathcal{P} \to H_1$ with



commuting and with

$$\|\widetilde{\Phi}\| \le k(L)k(L_0)k(L_1)\|[\varphi|_{(\ker\varphi)^\perp}]^{-1}\| \cdot \|\Phi\|.$$

Proof. Let $\mathcal{P}'$ denote the Hilbert space $\mathcal{P}$ with the $\mathbb{A}(\mathbb{D})$-action

$$p(z') \cdot f = p(S)f, \qquad p \in \mathbb{A}(\mathbb{D}).$$

Similarly, for $i = 0, 1$, let H_i' denote H_i with

$$p(z') \cdot f = p(S_i)f, \qquad p \in \mathbb{A}(\mathbb{D}).$$

Then we have the diagram

$$\begin{array}{ccc} & & \mathcal{P}' \\ & & \downarrow{\scriptstyle \Phi'} \\ H_1' & \xrightarrow[\varphi']{} & H_0' \end{array}$$

where $\varphi' = L_0^{-1}\varphi L_1$ and $\Phi' = L_0^{-1}\Phi L$. By Lemma 2, we can lift φ' in the above diagram to $\widetilde{\Phi}' : \mathcal{P}' \to H_1'$. Then, observing that $L : \mathcal{P}' \to \mathcal{P}$, $L_0 : H_0' \to H_0$ and $L_1 : H_1' \to H_1$ are Hilbert-module maps, and checking norms, the theorem follows.

The existence of isometric dilations for contractive Hilbert modules over $\mathbb{A}(\mathbb{D})$ provides projective resolutions in the category of cramped Hilbert modules. For details of their use, as well as connections between Theorem 1 and the famous problem of the existence of noncramped Hilbert modules, one may consult [2].

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