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Turbulent maps and their ω-limit sets

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Abstract. One-dimensional turbulent maps can be characterized via their ω-limit sets [1]. We give a direct proof of this characterization and get stronger results, which allows us to obtain some other results on ω-limit sets, which previously were difficult to prove.

Let $I = [0,1]$ and denote by $C(I)$ the set of continuous maps from I into I. A map $f \in C(I)$ is *turbulent* if there are closed intervals J and K, with at most one point in common, such that $f(J) \cap f(K) \supseteq J \cup K$ [3]. These maps were also called L-schemes [2].

It is not difficult to prove the more operative characterization of such maps [3]: "A map $f \in C(I)$ is turbulent if and only if there are points $a, b, c \in I$ such that: $a < c < b$; $f(a) = f(b) = a$; $f(c) = b$ and $f(x) > a$ for $a < x < b$; $x < f(x) < b$ for $a < x < c$, or the same with all inequalities reversed".

Let $f \in C(I)$. The *orbit* $\mathrm{Orb}_f(x)$ of a point $x \in I$ is the sequence $(f^n(x))_{n=0}^{\infty}$ where $f^0(x) = x$ and $f^n(x) = f \circ f^{n-1}(x)$ for every $n > 0$. We say that $\mathrm{Orb}_f(x)$ is *periodic* of order k if $f^k(x) = x$ and $f^i(x) \neq f^j(x)$ for all $0 \leq i \neq j \leq k-1$. If $f(x) = x$, then x is a *fixed point* of f. We define the *ω-limit set* $\omega(x, f)$ of a point $x \in I$ to be the set of limit points of $\mathrm{Orb}_f(x)$. For every $x \in I$ this is a non-empty, closed and invariant set ($f(\omega(x,f)) = \omega(x,f)$). On the other hand, it is well known that if $\omega(x,f) = A \cup B$ and A, B are closed and disjoint sets then $f(A) \cap B \neq \emptyset$.

Turbulent maps have complicated dynamics as far as periodic structure is concerned. They have periodic points of period 3 and according to Sharkovskiĭ's Theorem [2] have periodic points of all periods. But they can also produce nowhere dense ω-limit sets and in some cases ω-limit sets with non-empty interior.

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In [1] a characterization of turbulent maps is given in terms of the ω-limit sets produced by them: "The map f is turbulent if and only if there exists $x_0 \in I$ such that $\omega(x_0, f)$ is a unilaterally convergent sequence of points, that is, a monotone convergent sequence and its limit".

The aim of this note is to prove Theorem 1 which establishes a general condition on some ω-limit sets produced by turbulent maps. As a particular case we obtain the characterization given in [1] and some other results contained in [3].

THEOREM 1. *A map $f \in C(I)$ is turbulent if and only if for some $x_0 \in I$, $\omega(x_0, f)$ is an infinite set and there exist $\overline{x} < y \le z$ (resp. $\overline{x} > y \ge z$) such that $f(y) \ge z$ (resp. $f(y) \le z$), $\overline{x} \in [\min \omega(x_0, f), \max \omega(x_0, f)]$ being a fixed point of f and $z \in \omega(x_0, f)$.*

Proof. "If". Supposing $f(z) \ne z$ let

$$a = \max\{x \in [\overline{x}, y] : f(x) = x\} \quad \text{and} \quad b = \min\{x > y : f(x) = a\}.$$

There exists $c \in (a, b)$ such that $f(c) \ge b$. If not, $\omega(x_0, f) \cap (a, b) \ne \emptyset$ and $f(a, b) \subset (a, b)$, which is not possible. Therefore f is a turbulent map.

"Only if". We will prove the existence of uncountable and infinite countable ω-limit sets.

Let $[a, b] \subset I$ be the interval where f is turbulent and let $\{A_n\}$ be the sequence of interiors of the maximal closed intervals J included in $[a, b]$ and not equal to $[a, b]$, satisfying any of the following conditions:

(i) The image of J by an iterate of f is outside of $[a, b]$;

(ii) J is periodic or mapped to a periodic interval or a point by an iterate of f;

(iii) $f^n(J)$ converges to a point or $f^m(J) \cap f^n(J) = \emptyset$ if $m \ne n$.

Now, we consider the perfect and invariant set $K = [a, b] \setminus \bigcup_n A_n$. This set contains cycles of orders n or $2n$ for all n.

Since all the intervals in (i), (ii), or (iii) are disjoint and maximal, any open set with some of them in its interior contains two points whose images by iterates of f are a, b respectively, and thus contains a periodic point. Therefore, there exists a countable set $\{C_i\}$ of cycles such that $\overline{\bigcup C_i} = K$.

For C_i, a cycle of order m, we pick $p_i \in C_i$ and the decreasing sequences (x_n^i), (y_n^i) satisfying $x_1^i = p_i$, $f(x_n^i) = x_{n-1}^i$, $\lim(x_n^i) = a$; $y_1^i = b$, $f^m(y_n^i) = y_{n-1}^i$, $\lim(y_n^i) = p_i$.

Let (z_n^i) be an ordering of $\{(x_n^i), (y_n^i)\}$. Then the set

$$M = \bigcup_i \Big\{ \bigcup_{j=1}^{m} f^j(z_n^i) \Big\} \subset K$$

is countable and $\overline{M} = K$.

Let (x_n) be an ordering of M and $(V_n^j)_{n,j=1}^{\infty}$ a countable basis of neighborhoods of (x_n) whose diameters are respectively less than $1/j$, and $B_1, B_2, \ldots$ an ordering of them. Then given B_i and B_{i+1} it is not difficult to prove the existence of compact intervals $J_i \subset K_i \subset B_i$ and $m_i \in \mathbb{N}$ such that $f^{m_i}(J_i) \subset B_{i+1}$ and $(f^{m_i}(J_i)) \cap (x_n) \neq \emptyset$.

If (J_n), (K_n) are sequences of compact intervals recurrently defined in such a way that $K_{n+1} = f^{m_n}(J_n)$ and $J_{n+1} \subset K_{n+1} \subset B_{n+1}$, then the sequence of compact intervals $(f^{-\sum_{i=1}^{n} m_i}(J_{i+1}))$ is decreasing. If $x_0 \in \bigcap_n (f^{-\sum_{i=1}^{n} m_i}(J_{i+1}))$ it is easy to check that $\omega(x_0, f) = K$.

Now we will prove the existence of an infinite countable ω-limit set.

If f is a turbulent map with $a < c < b$, then the sequence (x_n) recurrently defined by: $x_1 = b$, $x_2 = c$, $x_{n+1} = \inf\{x \in (a, x_n) : f(x) = x_n\}$ is decreasing. Moreover, $f(x_n) = x_{n-1}$, $f(x_1) = a$. By similar arguments to those above we get the result.

COROLLARY 2. *If $f \in C(I)$ and there exists $x_0 \in I$ such that $\omega(x_0, f)$ is an infinite set with a one-sided fixed point $\overline{x}$ ($(x, \overline{x}) \cap \omega(x_0, f) = \emptyset$ or $(\overline{x}, x) \cap \omega(x_0, f) = \emptyset$), then f is turbulent.*

P r o o f. If $\overline{x}$ is an isolated fixed point from the left, then there exists $\overline{x} < z \in \omega(x_0, f)$ such that $f(z) > z$. Otherwise the closed sets $A = [\overline{x}, 1) \cap \omega(x_0, f)$ and $B = \omega(x_0, f) \setminus A$ satisfy $f(A) \cap B = \emptyset$.

THEOREM 3. *A map $f \in C(I)$ is turbulent if and only if for some $x_0 \in I$, $\omega(x_0, f)$ is an infinite set and there exist fixed points $\overline{x} < \overline{y}$ of f such that*

$$[\overline{x}, \overline{y}] \subseteq [\min \omega(x_0, f), \max \omega(x_0, f)] \quad \textit{and} \quad [\overline{x}, \overline{y}] \cap \omega(x_0, f) \neq \emptyset.$$

P r o o f. If f is a turbulent map, then there exists an infinite ω-limit set with a one-sided isolated limit point, which yields the existence of such an interval $[\overline{x}, \overline{y}]$.

If there exists an interval $[\overline{x}, \overline{y}] \subseteq [\min \omega(x_0, f), \max \omega(x_0, f)]$ with $[\overline{x}, \overline{y}] \cap \omega(x_0, f) \neq \emptyset$ and $\overline{x}$, $\overline{y}$ are not one-sided isolated points belonging to $\omega(x_0, f)$, then there exists $z \in [\overline{x}, \overline{y}] \cap \omega(x_0, f)$ such that $f(z) > z$ or $f(z) < z$. Hence the map f is turbulent.

COROLLARY 4. *If $f \in C(I)$ and there exists $x_0 \in I$ such that $\omega(x_0, f)$ is an infinite set with two fixed points $\overline{x} < \overline{y}$, then f is turbulent.*

COROLLARY 5. *Suppose $f \in C(I)$ is not a turbulent map. The map f^2 is turbulent if there exists a point $x_0 \in I$ such that $\omega(x_0, f)$ is an infinite set satisfying at least one of the conditions*: (i) *it possesses a fixed point*; (ii) *it possesses a two-periodic point $\overline{x}$ and there exist $\overline{x} < y \leq z$ (resp. $\overline{x} > y \geq z$) with $z \in \omega(x_0, f)$ such that $f^2(y) \geq z$ (resp. $f^2(y) \leq z$).*

P r o o f. If (ii) holds, then Theorem 1 ends the proof. If (i) holds, then $\omega(x_0, f)$ possesses only a fixed point $\overline{x}$ and two points $x_1 < \overline{x} < x_2$ such that $f(x_1) = \max \omega(x_0, f)$ and $f(x_2) = \min \omega(x_0, f)$. Therefore, there exists $y \in [\overline{x}, \max \omega(x_0, f^2)]$ such that $f^2(y) \geq z$ with $z \in \omega(x_0, f^2)$.

COROLLARY 6. *If* $\omega(x_0, f) = I$ *and* f *is not turbulent*, *then* (i) *it possesses a unique fixed point* $\overline{x} \in (0, 1)$ *and* f^2 *is a turbulent map*; (ii) *either* $\omega(x_0, f^2) = [0, \overline{x}]$, *or* $\omega(f(x_0), f^2) = [0, \overline{x}]$, *or* $\omega(x_0, f^2) = I$.

Proof. If $\omega(x_0, f) = I$ then f possesses a fixed point $\overline{x}$ and since it is not turbulent, $\overline{x}$ is unique, not an end-point and there exist $z_1 < \overline{x} < z_2$ such that $f^2(z_1) = 0$ and $f^2(z_2) = 1$. The sets $\omega(x_0, f^2)$ and $\omega(f(x_0), f^2)$ are intervals and if none of them is $[0, 1]$, then $\omega(x_0, f^2)$ or $\omega(f(x_0), f^2)$ must be $[0, \overline{x}]$.

The following result, proved in a difficult way in the literature, can now be proved in an easy way, using the characterization of turbulent maps.

THEOREM 7. *If* $f \in 2^\infty$ (*it has only periodic points of period* 2^n *for each* $n \in \mathbb{N}$), *then* $\omega(x_0, f)$ *is finite or an uncountable nowhere dense set for any* $x_0 \in I$.

Proof. If $\mathrm{Int}(\omega(x_0, f)) \neq \emptyset$ for some $x_0 \in I$, then there exist an interval J and $n \in \mathbb{N}$ such that $\omega(x_0, f^n|_J) = J$. If $\omega(x_0, f)$ were an infinite countable set then a fixed point would belong to $\omega(x_0, f^n)$ for some n.

In both cases there would exist $m \in \mathbb{N}$ such that f^m would be a turbulent map and f would not be a 2^∞-map.

References

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