

On n -circled \mathcal{H}^∞ -domains of holomorphy

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Abstract. We present various characterizations of n -circled domains of holomorphy $G \subset \mathbb{C}^n$ with respect to some subspaces of $\mathcal{H}^\infty(G)$.

Introduction. We say that a domain $G \subset \mathbb{C}^n$ is n -circled if $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in G$ for arbitrary $(z_1, \dots, z_n) \in G$ and $(\theta_1, \dots, \theta_n) \in \mathbb{R}^n$.

Put $\log G := \{(x_1, \dots, x_n) \in \mathbb{R}^n : (e^{x_1}, \dots, e^{x_n}) \in G\}$.

If $X \subset \mathbb{R}^n$ is a convex domain, then $\mathcal{E}(X)$ denotes the largest vector subspace $F \subset \mathbb{R}^n$ such that $X + F = X$.

A vector subspace $F \subset \mathbb{R}^n$ is said to be of *rational type* if F is spanned by $F \cap \mathbb{Z}^n$.

Let

$$L_h^2(G) := \mathcal{O}(G) \cap L^2(G)$$

and

$$\mathcal{A}^k(G) := \{f \in \mathcal{O}(G) : \forall \sigma \in (\mathbb{Z}_+)^n, |\sigma| \leq k \exists_{f_\sigma \in \mathcal{C}(\bar{G})} : f_\sigma = \partial^\sigma f \text{ in } G\},$$

$$k \in \mathbb{Z}_+ \cup \{\infty\},$$

where

$$\partial^\sigma := \frac{\partial^{|\sigma|}}{\partial z_1^{\sigma_1} \dots \partial z_n^{\sigma_n}}, \quad \sigma = (\sigma_1, \dots, \sigma_n) \in (\mathbb{Z}_+)^n.$$

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ put

$$\Omega(\alpha) := \{(z_1, \dots, z_n) \in \mathbb{C}^n : \forall_{j \in \{1, \dots, n\}} : \alpha_j < 0 \Rightarrow z_j \neq 0\}.$$

A domain $G \subset \mathbb{C}^n$ is said to be an $\mathcal{F}(G)$ -domain of holomorphy ($\mathcal{F}(G) \subset \mathcal{O}(G)$) if for any pair of domains $G_0, \tilde{G} \subset \mathbb{C}^n$ with $\emptyset \neq G_0 \subset \tilde{G} \cap G, \tilde{G} \not\subset G$, there exists a function $f \in \mathcal{F}(G)$ such that $f|_{G_0}$ is not the restriction of a function $\tilde{f} \in \mathcal{O}(\tilde{G})$.

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The following results are known.

PROPOSITION 1 ([Jar–Pfl 1]). *Let $G \subset \mathbb{C}^n$ be an n -circled domain of holomorphy. Then the following conditions are equivalent:*

- (i) G is fat (i.e. $G = \text{int } \overline{G}$) and the space $\mathcal{E}(\log G)$ is of rational type;
- (ii) there exist $A \subset \mathbb{Z}^n$ and a function $c : A \rightarrow \mathbb{R}_{>0}$ such that

$$G = \text{int } \bigcap_{\alpha \in A} \{z \in \Omega(\alpha) : |z^\alpha| < c(\alpha)\};$$

- (iii) G is an $\mathcal{H}^\infty(G)$ -domain of holomorphy.

PROPOSITION 2 ([Jar–Pfl 1]). *Let $G \subsetneq \mathbb{C}^n$ be a fat n -circled domain of holomorphy. Then the following conditions are equivalent:*

- (i) $\mathcal{E}(\log G) = \{0\}$;
- (ii) $L_h^2(G) \neq \{0\}$;
- (iii) G is an $L_h^2(G)$ -domain of holomorphy.

PROPOSITION 3 ([Sib]). *Let $G = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$ (the Hartogs triangle). Then:*

- (a) G is an $\mathcal{A}^k(G)$ -domain of holomorphy for arbitrary $k \in \mathbb{Z}_+$,
- (b) G is not an $\mathcal{A}^\infty(G)$ -domain of holomorphy.

The aim of this paper is to generalize Propositions 1, 2, 3. The starting point of these investigations was our attempt to understand the general situation behind Proposition 3.

PROPOSITION 4. *Let $G \subset \mathbb{C}^n$ be a fat n -circled domain of holomorphy. Then G is an $\mathcal{A}^k(G)$ -domain of holomorphy for arbitrary $k \in \mathbb{Z}_+$.*

Let

$$V_0 := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_1 \cdots z_n = 0\}$$

and

$$\mathcal{H}^\infty(G, \text{loc}) := \{f \in \mathcal{O}(G) : \text{for any bounded domain } D \subset \mathbb{C}^n, f \in \mathcal{H}^\infty(G \cap D)\}.$$

Remark 5. Let $G \subset \mathbb{C}^n$ be an n -circled domain of holomorphy. Then $(\text{int } \overline{G}) \setminus G \subset V_0$ (cf. [Jar–Pfl 1]). In particular, if G is an $\mathcal{H}^\infty(G, \text{loc})$ -domain of holomorphy (e.g. G is an $\mathcal{A}^0(G)$ -domain of holomorphy), then G is fat.

For $j = 1, \dots, n$ let

$$V_j := \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j = 0\},$$

$$\tilde{G}^{(j)} := \{(z_1, \dots, z_{j-1}, \lambda z_j, z_{j+1}, \dots, z_n) \in \mathbb{C}^n : (z_1, \dots, z_n) \in G, \lambda \in \overline{E}\},$$

where E denotes the unit disc. Define

$$\mathcal{H}^{\infty, \infty}(G, \text{loc}) := \{f \in \mathcal{O}(G) : \forall \sigma \in (\mathbb{Z}_+)^n : \partial^\sigma f \in \mathcal{H}^\infty(G, \text{loc})\}.$$

PROPOSITION 6. *Let $G \subset \mathbb{C}^n$ be an n -circled domain of holomorphy. Then the following conditions are equivalent:*

- (i) G is fat and
- (*) $\forall_{j \in \{1, \dots, n\}} : (\partial G) \cap V_j \neq \emptyset \Rightarrow \tilde{G}^{(j)} \subset G;$
- (ii) G is an $\mathcal{H}^{\infty, \infty}(G, \text{loc})$ -domain of holomorphy;
- (iii) G is an $\mathcal{A}^\infty(G)$ -domain of holomorphy;
- (iv) G is an $\mathcal{O}(\bar{G})$ -domain of holomorphy.

Moreover, if G is an $\mathcal{H}^\infty(G)$ -domain of holomorphy, then each of the above conditions is equivalent to the following one:

- (v) G is an $\mathcal{H}^\infty(G) \cap \mathcal{O}(\bar{G})$ -domain of holomorphy.

REMARK 7. (a) The Hartogs triangle does not satisfy (*) and therefore Proposition 3 follows from Propositions 4 and 6.

- (b) It is clear that if G is complete, then (*) is automatically satisfied.
- (c) One can prove (cf. [Fu]) that (*) is satisfied whenever ∂G is \mathcal{C}^1 .

For $p \in [1, \infty]$, $k \in \mathbb{Z}_+$ let

$$L_h^{p,k}(G) := \{f \in \mathcal{O}(G) : \forall_{|\sigma| \leq k} : \partial^\sigma f \in L^p(G)\},$$

$$L_h^p(G) := L_h^{p,0}(G), \quad \mathcal{H}^{\infty,k}(G) := L_h^{\infty,k}(G),$$

$$L_h^{\diamond,k}(G) := \bigcap_{p \in [1, \infty]} L_h^{p,k}(G).$$

REMARK 8. (a) We have $L_h^{\diamond,k} \subset \mathcal{H}^{\infty,k}(G)$, $k \in \mathbb{Z}_+$. Moreover, equality holds for one k (and then for all k) iff G has finite volume.

- (b) If G is bounded, then $\mathcal{A}^k(G) \subset \mathcal{H}^{\infty,k}(G)$.

(c) We will show (Lemma 18) that if G is n -circled, then $\mathcal{H}^{\infty,k}(G) \subset \mathcal{A}^{k-1}(G)$. Observe that for $G = \{(z_1, z_2) \in E^2 : |z_1| < |z_2|\}$ the function $f(z) := z_1^{2k}/z_2^k$ belongs to $\mathcal{H}^{\infty,k}(G)$, but not to $\mathcal{A}^k(G)$.

PROPOSITION 9. *Let $G \subset \mathbb{C}^n$ be an n -circled domain of holomorphy. Then the following conditions are equivalent:*

- (i) G is fat and $\mathcal{E}(\log G) = \{0\}$;
- (ii) G is fat and there exists $p \in [1, \infty)$ such that $L_h^p(G) \neq \{0\}$;
- (iii) $G \subsetneq \mathbb{C}^n$ and for each $k \in \mathbb{Z}_+$ the domain G is an $L_h^{\diamond,k}(G)$ -domain of holomorphy.

REMARK 10. Condition (iii) is equivalent (cf. Remark 8(c)) to the following one:

- (iv) $G \subsetneq \mathbb{C}^n$ and for each $k \in \mathbb{Z}_+$ the domain G is an $L_h^{\diamond,k}(G) \cap \mathcal{A}^k(G)$ -domain of holomorphy.

In particular, if G is bounded we get another proof of Proposition 4.

PROPOSITION 11. *Let $G \subset \mathbb{C}^n$ be an n -circled domain of holomorphy. Then the following conditions are equivalent:*

- (i) *G is fat and there exist $0 \leq m \leq n$ and a permutation of coordinates such that $G = D \times \mathbb{C}^{n-m}$ with $\mathcal{E}(\log D) = \{0\}$;*
- (ii) *G is an $\mathcal{H}^{\infty,1}(G)$ -domain of holomorphy;*
- (iii) *G is an $\mathcal{H}^{\infty,k}(G)$ -domain of holomorphy for any $k \in \mathbb{Z}_+$.*

Let

$$\mathcal{H}^{\infty,\Sigma} := \{f \in \mathcal{O}(G) : \forall \sigma \in \Sigma : \partial^\sigma f \in \mathcal{H}^\infty(G)\}, \quad \Sigma \subset (\mathbb{Z}_+)^n,$$

$$\Sigma_k := \{\sigma \in (\mathbb{Z}_+)^n : |\sigma| = k\}, \quad k \in \mathbb{Z}_+.$$

Let e_1, \dots, e_n denote the canonical basis of \mathbb{R}^n .

PROPOSITION 12. *Let $G \subset \mathbb{C}^n$ be an n -circled domain. Then the following conditions are equivalent:*

- (i) *G is an $\mathcal{H}^{\infty,\Sigma_1}(G)$ -domain of holomorphy;*
 - (ii) *there exist $A \subset \mathbb{Z}^n$ and functions $b_1, \dots, b_n : A \rightarrow \mathbb{R}_{>0}$ such that*
- (1)
$$G = \text{int} \bigcap_{\nu \in A} \{z \in \Omega(\nu) : \forall j \in \{1, \dots, n\} : \nu_j \neq 0 \Rightarrow |z^{\nu - e_j}| < b_j(\nu)\}.$$

EXAMPLE 13. Let $G \subset \mathbb{C}^2$ be a 2-circled $\mathcal{H}^{\infty,\Sigma_1}(G)$ -domain of holomorphy. Assume that $\mathcal{E}(\log G) \neq \{0\}$ and that G is not a Cartesian product of two plane domains. Then, by Proposition 12,

$$G := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2|\}$$

up to a permutation and rescaling of coordinates.

Note that G is not an $\mathcal{H}^{\infty,1}(G)$ -domain of holomorphy (Proposition 11).

This example shows that there are domains G and Fréchet spaces $\mathcal{F}_1(G)$ and $\mathcal{F}_2(G)$ of holomorphic functions on G such that G is an $\mathcal{F}_j(G)$ -domain of holomorphy, $j = 1, 2$, but not an $\mathcal{F}_1(G) \cap \mathcal{F}_2(G)$ -domain of holomorphy.

REMARK 14. Let $\mathcal{F}(G)$ be one of the spaces

$$\mathcal{A}^k(G), \quad \mathcal{H}^{\infty,\Sigma}(G), \quad L_h^{p,k}(G), \quad L_h^{\diamond,k}(G).$$

Then $\mathcal{F}(G)$ has a natural structure of a Fréchet space. Consequently, G is an $\mathcal{F}(G)$ -domain of holomorphy iff there exists a function $f \in \mathcal{F}(G)$ such that G is the domain of existence of f .

In [Sic 1,2] J. Siciak characterized those balanced domains of holomorphy $G \subset \mathbb{C}^n$ which are $\mathcal{H}^\infty(G)$ (resp. $\mathcal{H}^\infty(G) \cap \mathcal{A}^\infty(G)$)-domains of holomorphy. Moreover, it is known that any bounded balanced domain of holomorphy $G \subset \mathbb{C}^n$ is an $L_h^2(G)$ -domain of holomorphy (cf. [Jar-Pfl 2]). A general discussion for balanced domains of holomorphy (like the above for n -circled domains) is still lacking.

Proof of Proposition 4

LEMMA 15. Let $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{R} \setminus \{0\})^n$,

$$(2) \quad D_\varepsilon := \{(z_1, \dots, z_n) \in \Omega(\alpha) : |z_1|^{\alpha_1} \cdot \dots \cdot |z_n|^{\alpha_n} < 1 + \varepsilon\}, \quad \varepsilon \geq 0.$$

Then for any $\varepsilon > 0$ there exists a neighborhood U of the set $(\partial D_0) \setminus D_\varepsilon$ such that

$$(3) \quad d_{D_\varepsilon}(z) \geq |z^{\mathbf{2}}|, \quad z \in U \cap D_0,$$

where d_D denotes the distance to ∂D with respect to the maximum norm, i.e. $d_D(z) = \sup\{r > 0 : P(z, r) \subset D\}$, $z \in D$ (where $P(z, r)$ is the polydisc with center at z and radius r), $\mathbf{2} := (2, \dots, 2) \in \mathbb{N}^n$.

Proof. We may assume that $\alpha_1, \dots, \alpha_s > 0$, $\alpha_{s+1}, \dots, \alpha_n < 0$ for some $0 < s < n$. Fix $\varepsilon > 0$ and $a = (a_1, \dots, a_n) \in (\partial D_0) \setminus D_\varepsilon$. Note that $a_1 \cdot \dots \cdot a_s = a_{s+1} \cdot \dots \cdot a_n = 0$.

We have to prove that there exists a neighborhood U of a such that $P(z, |z^{\mathbf{2}}|) \subset D_\varepsilon$ for any $z \in U \cap D_0 \setminus V_0$.

Let U be a neighborhood of a such that $|z^{2-e_j}| < 1$, $j = 1, \dots, n$, and

$$\prod_{j=1}^s (1 + |z^{2-e_j}|)^{\alpha_j} \prod_{j=s+1}^n (1 - |z^{2-e_j}|)^{\alpha_j} < 1 + \varepsilon, \quad z \in U.$$

Then

$$\prod_{j=1}^s (|z_j| + |z^{\mathbf{2}}|)^{\alpha_j} \prod_{j=s+1}^n (|z_j| - |z^{\mathbf{2}}|)^{\alpha_j} < 1 + \varepsilon, \quad z \in U \cap D_0,$$

and therefore $P(z, |z^{\mathbf{2}}|) \subset D_\varepsilon$, $z \in U \cap D_0 \setminus V_0$. ■

Remark 16. The proof shows that, under the assumptions of the lemma, the following slightly stronger assertion holds:

For any $\varepsilon > 0$, $\eta > 1$, there exists a neighborhood U of the set $(\partial D_0) \setminus D_\varepsilon$ such that

$$d_{D_\varepsilon}(z) \geq |z_1 \cdot \dots \cdot z_n|^\eta, \quad z = (z_1, \dots, z_n) \in U \cap D_0.$$

We pass to the proof of Proposition 4. Fix a $k \in \mathbb{Z}_+$. Since G is a fat n -circled domain of holomorphy, there exist a family $A \subset \mathbb{R}^n$ and a function $c : A \rightarrow \mathbb{R}_{>0}$ such that

$$(4) \quad G = \text{int} \bigcap_{\alpha \in A} \{z \in \Omega(\alpha) : |z_1|^{\alpha_1} \cdot \dots \cdot |z_n|^{\alpha_n} < c(\alpha)\}.$$

Consequently, it suffices to consider the case

$$G = \{z \in \Omega(\alpha) : |z_1|^{\alpha_1} \cdot \dots \cdot |z_n|^{\alpha_n} < c\}$$

for some $\alpha \in \mathbb{R}^n$ and $c > 0$. Furthermore, we may also assume that $\alpha \in (\mathbb{R} \setminus \{0\})^n$ (otherwise we can pass to \mathbb{C}^{n-1}) and that $c = 1$. Thus we may assume that $G = D_0$, where D_0 is as in (2).

Suppose that G is not an $\mathcal{A}^k(G)$ -domain of holomorphy and let G_0, \tilde{G} be domains such that $\emptyset \neq G_0 \subset \tilde{G} \cap G, \tilde{G} \not\subset G$, and for each $f \in \mathcal{A}^k(G)$ there exists $\tilde{f} \in \mathcal{O}(\tilde{G})$ with $\tilde{f} = f$ on G_0 . Since G is fat, we may assume that $\tilde{G} \cap V_0 = \emptyset$ and that $\tilde{G} \not\subset \bar{G}$. Let $\varepsilon > 0$ be such that $\tilde{G} \not\subset D_\varepsilon$ (D_ε is given by (2)) and let U be as in Lemma 15.

It is known (cf. [Pfl]) that there exist $N > 0$ and a function $g \in \mathcal{O}(D_\varepsilon)$ such that D_ε is the domain of existence of g and $\delta_{D_\varepsilon}^N |g| \leq 1$, where

$$\delta_{D_\varepsilon}(z) := \min \left\{ \text{dist}_{D_\varepsilon}(z), \frac{1}{\sqrt{1 + \|z\|^2}} \right\},$$

$\text{dist}_{D_\varepsilon}$ denoting the distance to ∂D_ε with respect to the Euclidean norm. (In fact, we know (cf. [Jar-Pfl 1]) that such a function exists for arbitrary $N > 0$.) Let $\mu \in \mathbb{N}$ be such that $\mu \geq 2N + 3k + 1$. We will show that $f := z^{\mu \mathbf{1}} g|_G \in \mathcal{A}^k(G)$ ($\mathbf{1} := (1, \dots, 1) \in \mathbb{N}^n$). Then the function $z^{-\mu \mathbf{1}} \tilde{f} \in \mathcal{O}(\tilde{G})$ extends g and this will be a contradiction.

It suffices to prove that

$$\lim_{G \ni z \rightarrow a} z^{\mu \mathbf{1} - \sigma} \partial^\tau g(z) = 0, \quad a \in (\partial G) \setminus D_\varepsilon, \quad \sigma, \tau \in (\mathbb{Z}_+)^n, \quad |\sigma| + |\tau| \leq k.$$

Fix an $a \in (\partial G) \setminus D_\varepsilon$. It may be easily proved (cf. [Fer]) that

$$\delta_{D_\varepsilon}^{N+k} |\partial^\tau g| \leq c_0, \quad |\tau| \leq k,$$

where c_0 depends only on n, N , and k . Then, by virtue of (3), for $z \in G \cap U$, z near a , we get

$$\begin{aligned} |z^{\mu \mathbf{1} - \sigma} \partial^\tau g(z)| &\leq c_0 |z^{\mu \mathbf{1} - \sigma}| \delta_{D_\varepsilon}^{-(N+k)}(z) \\ &\leq c_1 |z^{\mu \mathbf{1} - \sigma}| d_{D_\varepsilon}^{-(N+k)}(z) \leq c_1 |z^{\mu \mathbf{1} - \sigma - 2(N+k)\mathbf{1}}| \leq c_2 |z^{\mathbf{1}}|, \end{aligned}$$

where c_1, c_2 are independent of z . The proof of Proposition 4 is complete. ■

Proof of Proposition 6

LEMMA 17. *Let $D \subsetneq \mathbb{C}^n$ be n -circled and $\Sigma \subset (\mathbb{Z}_+)^n$ be such that there exists $k_0 \in \mathbb{Z}_+$ with $\Sigma_{k_0} \subset \Sigma$. Assume that D is an $\mathcal{H}^{\infty, \Sigma}(D)$ -domain of holomorphy. Then there exist $A \subset \mathbb{Z}^n$ and functions $a : A \rightarrow \mathbb{R}_{>0}, b : \Sigma \rightarrow \mathbb{R}_+$ such that*

$$(5) \quad D = \text{int} \bigcap_{(\nu, \sigma) \in A \times \Sigma : \binom{\nu}{\sigma} \neq 0} \left\{ z \in \Omega(\nu) : \left| \sigma! \binom{\nu}{\sigma} z^{\nu - \sigma} \right| < a(\nu) b(\sigma) \right\}.$$

Moreover, if $\Sigma = (\mathbb{Z}_+)^n$, then D satisfies (*).

PROOF. Let $f \in \mathcal{H}^{\infty, \Sigma}(D)$ be such that D is the domain of existence of f (cf. Remark 14). Write

$$f(z) = \sum_{\nu \in A} a_\nu z^\nu, \quad z \in D,$$

where $A \subset \mathbb{Z}^n$ is such that $a_\nu \neq 0$ for $\nu \in A$. Note that

$$(6) \quad \partial^\sigma f(z) = \sum_{\nu \in A} \sigma! \binom{\nu}{\sigma} a_\nu z^{\nu-\sigma}, \quad z \in D, \sigma \in \Sigma.$$

Put $a(\nu) := 1/|a_\nu|$, $\nu \in A$, $b(\sigma) := \|\partial^\sigma f\|_{\mathcal{H}^\infty(D)}$, $\sigma \in \Sigma$. By the Cauchy inequalities, we get

$$\left| \sigma! \binom{\nu}{\sigma} z^{\nu-\sigma} \right| < a(\nu)b(\sigma), \quad z \in D, (\nu, \sigma) \in A \times \Sigma, \binom{\nu}{\sigma} \neq 0.$$

Thus $D \subset \tilde{D}$, where \tilde{D} is the domain defined by the right side of (5).

It is clear that for each $\sigma \in \Sigma$ the series (6) is convergent in \tilde{D} . Suppose that $D \subsetneq \tilde{D}$. Since \tilde{D} is connected, there exist $a \in D$, $r > 0$ such that $P(a, r) \subset \tilde{D}$ but $P(a, r) \not\subset D$.

Observe that if $g \in \mathcal{O}(D)$ is such that each derivative $\partial g / \partial z_j$ extends to a function $g_j \in \mathcal{O}(P(a, r))$, $j = 1, \dots, n$, then the function g itself extends to $P(a, r)$. Indeed, the extension may be given by the formula

$$\tilde{g}(z) = g(a) + \sum_{j=1}^n (z_j - a_j) \int_0^1 g_j(a + t(z - a)) dt, \quad z \in P(a, r).$$

The above property and the fact that $\Sigma_{k_0} \subset \Sigma$ easily imply that the function f extends to $P(a, r)$; a contradiction.

Now, suppose that $\Sigma = (\mathbb{Z}_+)^n$ and that $\partial D \cap V_{j_0} \neq \emptyset$ for some $j_0 \in \{1, \dots, n\}$. By virtue of (5), to prove that $\tilde{D}^{(j_0)} = D$ it suffices to show that $\nu_{j_0} \geq 0$ for any $\nu \in A$. Fix a $\nu \in A$ and let $\sigma = (\sigma_1, \dots, \sigma_n)$, $\sigma_j := \max\{0, \nu_j\}$, $j = 1, \dots, n$. Observe that $\binom{\nu}{\sigma} \neq 0$ and therefore

$$z^{\nu-\sigma} = \prod_{j: \nu_j < 0} z_j^{\nu_j}$$

is bounded on \bar{D} . In particular, $\nu_{j_0} \geq 0$. ■

The implications (v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) in Proposition 6 are evident.

(ii) \Rightarrow (i). It is clear that G is fat (cf. Remark 5). Suppose that $\partial G \cap V_{j_0} \neq \emptyset$ for some $j_0 \in \{1, \dots, n\}$. Then for any $r > 0$ the domain $D_r := G \cap P(0, r)$ is an $\mathcal{H}^{\infty, (\mathbb{Z}_+)^n}(D_r)$ -domain of holomorphy. Hence, by Lemma 17, if $\bar{D}_r \cap V_{j_0} \neq \emptyset$, then $\tilde{D}_r^{(j_0)} = D_r$. Consequently, $\tilde{G}^{(j_0)} = G$.

(i) \Rightarrow (iv) (resp. (i) \Rightarrow (v)) provided that G is an $\mathcal{H}^\infty(G)$ -domain of holomorphy). Suppose that G is not an $\mathcal{O}(\bar{G})$ (resp. $\mathcal{H}^\infty(G) \cap \mathcal{O}(\bar{G})$)-domain of

holomorphy and let G_0, \tilde{G} be domains such that $\emptyset \neq G_0 \subset \tilde{G} \cap G, \tilde{G} \not\subset G$, and for each $f \in \mathcal{O}(\tilde{G})$ (resp. $f \in \mathcal{H}^\infty(G) \cap \mathcal{O}(\tilde{G})$) there exists $\tilde{f} \in \mathcal{O}(\tilde{G})$ with $\tilde{f} = f$ on G_0 . We know that G may be represented in the form (4) with $A \subset \mathbb{R}^n$ (resp. $A \subset \mathbb{Z}^n$). Let $\alpha \in A, c > 0, \varepsilon > 0$ be such that

$$G \subset \{z \in \Omega(\alpha) : |z_1|^{\alpha_1} \cdot \dots \cdot |z_n|^{\alpha_n} < c\}$$

$$\subset D_\varepsilon := \{z \in \Omega(\alpha) : |z_1|^{\alpha_1} \cdot \dots \cdot |z_n|^{\alpha_n} < (1 + \varepsilon)c\}, \quad \tilde{G} \not\subset D_\varepsilon.$$

Observe that D_ε is a domain of holomorphy (resp. D_ε is an $\mathcal{H}^\infty(D_\varepsilon)$ -domain of holomorphy). If we prove that $\tilde{G} \subset D_\varepsilon$, then we get a contradiction.

Obviously, $\tilde{G} \setminus V_0 \subset D_\varepsilon$. Suppose that $(\partial G) \cap V_{j_0} \neq \emptyset$ for some $j_0 \in \{1, \dots, n\}$. Since G satisfies (*), we get

$$|\lambda|^{\alpha_{j_0}} |z_1|^{\alpha_1} \cdot \dots \cdot |z_n|^{\alpha_n} < c, \quad z \in G, \lambda \in \bar{E}.$$

Consequently, $\alpha_{j_0} \geq 0$. Thus $(\partial G) \cap V_0 \subset D_\varepsilon$.

The proof of Proposition 6 is complete. ■

Proof of Proposition 9

LEMMA 18. *Let $D \subset \mathbb{C}^n$ be n -circled. Then $\mathcal{H}^{\infty, \Sigma_1}(D) \subset \mathcal{A}^0(D)$. In particular, $\mathcal{H}^{\infty, k}(D) \subset \mathcal{A}^{k-1}(D)$, $k \in \mathbb{N}$.*

PROOF. Note that D has univalent $\mathcal{H}^\infty(D)$ -envelope of holomorphy. Therefore, we may assume that D is a domain of holomorphy. Fix $f \in \mathcal{H}^{\infty, \Sigma_1}(D)$. Let ϱ_D denote the arc-length distance on D . Obviously,

$$|f(z') - f(z'')| \leq \sup_{z \in D} \{\|f'(z)\|\} \cdot \varrho_D(z', z''), \quad z', z'' \in D.$$

For $J = (j_1, \dots, j_s), 1 \leq j_1 < \dots < j_s \leq n$ with $0 \leq s \leq n$, let $p_J : \mathbb{C}^n \rightarrow \mathbb{C}^s$ denote the natural projection $(z_1, \dots, z_n) \rightarrow (z_{j_1}, \dots, z_{j_s})$, where $p_\emptyset := 0$.

To show that f extends continuously to \bar{D} it suffices to prove that for any point $a = (a_1, \dots, a_n) \in \partial D$ there exist a constant $c > 0$ and a neighborhood U of a such that

$$\varrho_D(z', z'') \leq c(\|z' - z''\| + \|p_J(z')\| + \|p_J(z'')\|), \quad z', z'' \in U \cap D,$$

where J is such that $a_j = 0$ iff $j \in J$. Fix an a . We may assume that $J = (1, \dots, s)$. Let $w' := (|z'_1|, \dots, |z'_s|, z'_{s+1}, \dots, z'_n)$. Since D is n -circled, $w' \in D$ and

$$\varrho_D(z', w') \leq 2\pi(|z'_1| + \dots + |z'_s|).$$

Let w'' be defined in the same way for z'' . Thus it remains to prove that there exist a constant $c' > 0$ and a neighborhood U of a such that

$$\varrho_D(w', w'') \leq c'\|z' - z''\|, \quad z', z'' \in U \cap D.$$

By continuity, it suffices to consider only the case where $0 \neq |z'_j| \neq |z''_j| \neq 0$, $j = 1, \dots, n$. Let $L_1 = \dots = L_s := \text{Log} =$ the principal branch of the logarithm. Furthermore, for $j \geq s + 1$, let L_j be a branch of the logarithm. Put $\gamma = (\gamma_1, \dots, \gamma_n) : [0, 1] \rightarrow \mathbb{C}^n$, $\gamma_j(t) := \exp((1 - t)L_j(w'_j) + tL_j(w''_j))$, $j = 1, \dots, n$. Since D is logarithmically convex, $\gamma([0, 1]) \subset D$. We only need to show that for each j there exists $c'_j > 0$ such that the length l_j of γ_j is $\leq c'_j |z'_j - z''_j|$ provided that z'_j, z''_j are near a_j .

If $j \leq s$, then $l_j \leq ||z'_j| - |z''_j|| \leq |z'_j - z''_j|$.

If $j \geq s + 1$, then let U_j be a neighborhood of a_j such that $|z_j - a_j| < |a_j|/2$, $z \in U_j$. Consequently, for $z'_j, z''_j \in U_j$ we get

$$l_j = \int_0^1 |\gamma'_j(t)| dt = \int_0^1 |z'_j|^{1-t} |z''_j|^t |L_j(z'_j) - L_j(z''_j)| dt$$

$$\leq 2|a_j|(2/|a_j|)|z'_j - z''_j| = 4|z'_j - z''_j|. \blacksquare$$

(iii) \Rightarrow (ii) in Proposition 9 follows from Remark 5.

(ii) \Rightarrow (i). Let $f = \sum_{\nu \in \mathbb{Z}^n} a_\nu z^\nu \in L_h^p(G)$, $f \neq 0$. Then

$$\int_G |a_\nu z^\nu|^p d\Lambda_{2n}(z) = (2\pi)^n \int_{|G|} \left| \frac{1}{(2\pi i)^n} \int_{\substack{|\zeta_j|=r_j \\ j=1, \dots, n}} \frac{f(\zeta)}{\zeta^{\nu+1}} d\zeta \right|^p r^{p\nu+1} d\Lambda_n(r)$$

$$\leq (2\pi)^{n(1-p)} \int_{|G|} \left(\int_{[0, 2\pi]^n} |f(re^{i\theta})| d\Lambda_n(\theta) \right)^p r^1 d\Lambda_n(r)$$

$$\leq \int_{|G|} \int_{[0, 2\pi]^n} |f(re^{i\theta})|^p d\Lambda_n(\theta) r^1 d\Lambda_n(r)$$

$$= \int_G |f|^p d\Lambda_{2n},$$

where $|G| := \{(|z_1|, \dots, |z_n|) : (z_1, \dots, z_n) \in G\}$ and Λ_n denotes Lebesgue measure in \mathbb{R}^n . Consequently, there exists $\nu_0 \in \mathbb{Z}^n$ such that $z^{\nu_0} \in L_h^p(G)$.

Suppose that $F := \mathcal{E}(\log G) \neq \{0\}$. Let $m := \dim F$ and let $Y \subset F^\perp$ be a convex domain such that $\log G = Y + F$. We have

$$\int_G |z^{\nu_0}|^p d\Lambda_{2n}(z) = (2\pi)^n \int_{\log G} e^{\langle x, p\nu_0 + \mathbf{2} \rangle} d\Lambda_n(x)$$

$$= \int_Y e^{\langle x', p\nu_0 + \mathbf{2} \rangle} d\Lambda_{n-m}(x') \int_F e^{\langle x'', p\nu_0 + \mathbf{2} \rangle} d\Lambda_m(x'') = \infty,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in \mathbb{R}^n . We have got a contradiction.

(i) \Rightarrow (iii). Fix $k \in \mathbb{Z}_+$. Suppose that there exist domains $G_0, \tilde{G} \subset \mathbb{C}^n$ such that $\emptyset \neq G_0 \subset 1G \cap \tilde{G}$, $\tilde{G} \not\subset G$, and for each $f \in L_h^{\diamond, k}(G)$ there exists $\tilde{f} \in \mathcal{O}(\tilde{G})$

with $\tilde{f} = f$ on G_0 . We may assume that $\tilde{G} \not\subset \bar{G}$ and that $\tilde{G} \cap V_0 = \emptyset$.

Since $\mathcal{E}(\log G) = \{0\}$ and G is fat, there exist \mathbb{R} -linearly independent vectors $\alpha_j = (\alpha_{j,1}, \dots, \alpha_{j,n}) \in \mathbb{Z}^n$, $j = 1, \dots, n$, and $c > 0$ such that

$$G \subset D_0 := \{z \in \Omega : |z^{\alpha_j}| < c, j = 1, \dots, n\}$$

$$\subset D_\varepsilon := \{z \in \Omega : |z^{\alpha_j}| < (1 + \varepsilon)c, j = 1, \dots, n\}, \quad \tilde{G} \not\subset D_\varepsilon,$$

where $\Omega := \Omega(\alpha_1) \cap \dots \cap \Omega(\alpha_n)$ (cf. [Jar–Pfl 1]). We may assume that $c = 1$. Fix an $a \in \tilde{G} \setminus D_\varepsilon$ and let $j_0 \in \{1, \dots, n\}$ be such that $|a^{\alpha_{j_0}}| \geq 1 + \varepsilon$.

Put $\alpha := \alpha_1 + \dots + \alpha_n$. For $N \in \mathbb{N}$ define

$$f_N(z) := \frac{z^{N\alpha}}{z^{\alpha_{j_0}} - a^{\alpha_{j_0}}}, \quad z \in D_\varepsilon.$$

Obviously, $f_N \in \mathcal{O}(D_\varepsilon)$. We will show that there exists $N \in \mathbb{N}$ such that $f_N \in L_h^{\diamond, k}(D_0)$. Then $\tilde{f}_N(z)(z^{\alpha_{j_0}} - a^{\alpha_{j_0}}) = z^{N\alpha}$, $z \in \tilde{G}$, which will give a contradiction.

Observe that any derivative $\partial^\sigma f_N$, $\sigma \in (\mathbb{Z}_+)^n$, $|\sigma| \leq k$, is a finite sum of terms of the form

$$d \frac{z^{N\alpha + l\alpha_{j_0} - \sigma}}{(z^{\alpha_{j_0}} - a^{\alpha_{j_0}})^{l+1}},$$

where $d \in \mathbb{Z}$, $l \in \{0, \dots, k\}$. Thus it suffices to find N such that $\|z^{N\alpha - \sigma}\|_{L^p(D_0)} \leq 1$, $|\sigma| \leq k$, $1 \leq p \leq \infty$.

Let

$$A := [\alpha_{j,l}]_{j,l=1,\dots,n}, \quad B := A^{-1},$$

$$T_j(x) := (xB)_j = \sum_{l=1}^n B_{l,j}x_l, \quad j = 1, \dots, n, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$

For $p \in [1, \infty)$ and $\nu \in \mathbb{Z}^n$ we have

$$\begin{aligned} \int_{D_0} |z^\nu|^p d\Lambda_{2n}(z) &= (2\pi)^n \int_{\log D_0} e^{\langle x, p\nu + \mathbf{2} \rangle} d\Lambda_n(x) \\ &= (2\pi)^n \int_{\{\xi < 0\}} e^{\langle B(\xi), p\nu + \mathbf{2} \rangle} |\det B| d\Lambda_n(\xi) \\ &= \frac{(2\pi)^n}{|\det A| T_1(p\nu + \mathbf{2}) \cdots T_n(p\nu + \mathbf{2})} \end{aligned}$$

provided that $T_j(p\nu + \mathbf{2}) > 0$, $j = 1, \dots, n$. In particular, if

$$T_j(\nu) \geq \frac{1}{p} \left(\frac{2\pi}{|\det A|^{1/n}} - T_j(\mathbf{2}) \right), \quad j = 1, \dots, n,$$

then $\|z^\nu\|_{L^p(D_0)} \leq 1$. Hence, if $\nu = N\alpha - \sigma$ and if

$$N \geq N_0 := \sup \left\{ T_j(\sigma) + \frac{1}{p} \left(\frac{2\pi}{|\det A|^{1/n}} - T_j(\mathbf{2}) \right) : \right. \\ \left. j = 1, \dots, n, \sigma \in (\mathbb{Z}_+)^n, |\sigma| \leq k, p \in [1, \infty) \right\},$$

then $\|z^{N\alpha - \sigma}\|_{L^p(D_0)} \leq 1$ for arbitrary $p \in [1, \infty)$ and $|\sigma| \leq k$.

Moreover, $N_0 \geq T_j(\sigma)$, $j = 1, \dots, n$, and therefore $N\alpha - \sigma \in \mathbb{R}_+\alpha_1 + \dots + \mathbb{R}_+\alpha_n$, which shows that $\|z^{N\alpha - \sigma}\|_{\mathcal{H}^\infty(D_0)} \leq 1$ for arbitrary $|\sigma| \leq k$. ■

Proof of Proposition 11. (i) \Rightarrow (iii) follows from Proposition 9. (iii) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (i). Let $F := \mathcal{E}(\log G)$, $m := \text{codim } F$. The cases $m = 0$ and $m = n$ are trivial. Assume $1 \leq m \leq n - 1$. By Lemma 17 (with $\Sigma := \{0\} \cup \Sigma_1$) we know that there exist $A \subset \mathbb{Z}^n$ and functions $b_0, \dots, b_n : A \rightarrow \mathbb{R}_{>0}$ such that

$$G = \text{int} \bigcap_{\nu \in A} \{z \in \Omega(\nu) : |z^\nu| < b_0(\nu), \forall_{j \in \{1, \dots, n\}} : \nu_j \neq 0 \Rightarrow |z^{\nu - e_j}| < b_j(\nu)\}.$$

Hence if $\nu \in A$ and $\nu_j \neq 0$, then $\nu, \nu - e_j \in F^\perp$, and, consequently, $e_j \in F^\perp$. Since $\dim F^\perp = m$, we may assume that $e_{s+1}, \dots, e_n \notin F^\perp$ for some $0 \leq s \leq m$. Hence $G = D \times \mathbb{C}^{n-s}$. Clearly, $F = \mathcal{E}(\log D) \times \mathbb{R}^{n-s}$. Hence $s = m$ and therefore $\mathcal{E}(\log D) = \{0\}$. ■

Proof of Proposition 12. The implication (i) \Rightarrow (ii) follows from Lemma 17. To prove that any domain G of the form (1) is an $\mathcal{H}^{\infty, \Sigma_1}(G)$ -domain of holomorphy it suffices to consider only the case where

$$G = \{z \in \Omega(\nu) : \forall_{j \in \{1, \dots, n\}} : \nu_j \neq 0 \Rightarrow |z^{\nu - e_j}| < b_j\}$$

for some $\nu \in \mathbb{Z}^n$ and $b_1, \dots, b_n > 0$. We may assume that $\nu_j \neq 0$, $j = 1, \dots, n$ (otherwise we can pass to \mathbb{C}^{n-1}). It is enough to prove that for any point $a \notin \bar{G} \cup V_0$ there exists a function $f \in \mathcal{H}^{\infty, \Sigma_1}(G)$ such that f cannot be continued across a . Fix such an a and let $j_0 \in \{1, \dots, n\}$ be such that $|a^{\nu - e_{j_0}}| > b_{j_0}$. Then the function

$$f(z) := \frac{z^\nu}{z^{\nu - e_{j_0}} - a^{\nu - e_{j_0}}}, \quad z \in G,$$

belongs to $\mathcal{H}^{\infty, \Sigma_1}(G)$ (cf. the proof of Proposition 9) and evidently cannot be continued across a . ■

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