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On continuous solutions of a functional equation

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Abstract. This paper discusses continuous solutions of the functional equation $\varphi[f(x)] = g(x, \varphi(x))$ in topological spaces.

Let us consider the equation

$$\varphi[f(x)] = g(x, \varphi(x)) \tag{1}$$

with $\varphi : X \to Y$ as unknown function.

In order to obtain a solution of equation (1), it is enough to extend a function defined on a set which for every x contains exactly one element of the form $f^k(x)$, where $k = 0, \pm 1, \pm 2, \ldots$ and $f^k(x)$ denotes the kth iterate of the function f (cf. [3] and [4]). In the case when X is an open interval and Y is a Banach space, it is well known under what conditions these extensions are continuous (cf. [5]). Paper [6] by M. Sablik brings theorems which answer the above question for X and Y contained in some Banach spaces ([6, Th. 2.1, Th. 2.2]). In the case when X and Y are locally convex vector spaces the continuity of similar extensions was examined by W. Smajdor in [7] but for the Schröder equation (i.e. $\varphi[f(x)] = s\varphi(x)$, $0 < |s| < 1$). We are going to adopt the method given in that paper to the more general situation.

We shall employ Baron's Extension Theorem proved in [1] (cf. also [2]). This theorem concerns extending solutions of functional equations from a neighbourhood of a distinguished point (Lemma 7).

We shall deal with the following hypotheses:

(i) X is a Hausdorff topological space; ξ is a given (and fixed) point of X; Y is a topological space.

(ii) The function f maps X into X in such a manner that

(2) f is homeomorphism of X onto $f(X)$;

(3) $\xi \in \operatorname{int} f(X)$;

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(4) $\lim_{n\to\infty} f^n(x) = \xi$ for every $x \in X$;

(5) each neighbourhood U of the point ξ contains a neighbourhood W of ξ such that $\operatorname{cl} f(W) \subset W \subset U$.

(iii) The function $g : X \times Y \to Y$ is continuous; for every $x \in X \setminus \{\xi\}$ the function $g(x,\cdot)$ is a bijection and the function $h : (X \setminus \{\xi\}) \times Y \to Y$ defined by

$$h(x,y) = g(x,\cdot)^{-1}(y)$$

is continuous.

Evidently

$$f(\xi) = \xi. \tag{6}$$

According to (3) and (5) we can find a neighbourhood W of ξ such that $W \subset \operatorname{int} f(X)$ and $\operatorname{cl} f(W) \subset W$. Obviously $f^2(W) \subset f(W)$, thus $\operatorname{cl} f^2(W) \subset \operatorname{cl} f(W) \subset W \subset f(X)$. By (2) we have

$$\operatorname{cl} f^2(W) = \operatorname{cl} f^2(W) \cap f(X) = f(\operatorname{cl} f(W)) \subset f(W).$$

Putting $V_0 := f(W)$ we obtain an open set with the following properties:

$$\xi \in V_0, \quad \operatorname{cl} V_0 \subset \operatorname{int} f(X), \tag{7}$$

$$\operatorname{cl} f(V_0) \subset V_0. \tag{8}$$

Moreover, by induction we have

$$f^k(V_0) \text{ is open}, \quad k = 0,1,2,\ldots, \tag{9}$$

$$\operatorname{cl} f^{k+1}(V_0) \subset f^k(V_0), \quad k = 0,1,2,\ldots \tag{10}$$

Fix an open set V_0 satisfying (7) and (8) and put

$$A_0 := \operatorname{cl} V_0 \setminus \operatorname{cl} f(V_0), \tag{11}$$

$$C_0 := \operatorname{cl} V_0 \setminus V_0. \tag{12}$$

We have the following

LEMMA 1.

$$A_0 = C_0 \cup \operatorname{int} A_0, \tag{13}$$

$$\operatorname{cl} A_0 \subset A_0 \cup f(C_0). \tag{14}$$

Proof of (13). Recalling (11) and (12) we have $A_0 \subset C_0 \cup (A_0 \setminus C_0) \subset C_0 \cup (V_0 \setminus \operatorname{cl} f(V_0)) \subset C_0 \cup \operatorname{int} A_0$. The converse inclusion follows immediately from (11), (12) and (8).

Proof of (14). Let $x \in \operatorname{cl} A_0 \setminus A_0$. Then from the definition of A_0 we infer that $x \in \operatorname{cl} f(V_0)$. Since, by (9) and (11), $f(V_0)$ is an open set disjoint from A_0, it follows that $x \notin f(V_0)$. Applying (8), (7) and (2) we get $x \in \operatorname{cl} f(V_0) \setminus f(V_0) = \operatorname{cl} f(V_0) \cap f(X) \setminus f(V_0) = f(\operatorname{cl} V_0) \setminus f(V_0) = f(\operatorname{cl} V_0 \setminus V_0) = f(C_0)$, which was to be proved.

Put

$$A_k = f^k(A_0), \quad k = 0, 1, 2, \ldots, \tag{15}$$

$$C_k = f^k(C_0), \quad k = 0, 1, 2, \ldots \tag{16}$$

By continuity of f^k, $k = 0, 1, 2, \ldots$, from (15), (11), (10) and (7) we have

$$\begin{aligned} \operatorname{cl} A_k &\subset \operatorname{cl} f^k(A_0) \subset \operatorname{cl} f^k(\operatorname{cl} V_0) \subset \operatorname{cl}\operatorname{cl} f^k(V_0) \subset \operatorname{cl} f^k(V_0) \\ &\subset \operatorname{cl} V_0 \subset \operatorname{int} f(X) \subset f(X). \end{aligned} \tag{17}$$

Using the above inclusions and induction we can derive from Lemma 1 the next one:

LEMMA 2.

$$A_k = C_k \cup \operatorname{int} A_k, \quad k = 0, 1, 2, \ldots, \tag{18}$$

$$\operatorname{cl} A_k \subset A_k \cup C_{k+1}, \quad k = 0, 1, 2, \ldots \tag{19}$$

We have

LEMMA 3.

$$A_k \cap A_l = \emptyset \quad \textit{for } k \neq l,\ k, l = 0, 1, 2, \ldots \tag{20}$$

Proof. Fix $l, k \in \{0, 1, 2, \ldots\}$, $l \neq k$. Let $l \geq k+1$. Then, by (2) and (10) we get $A_l \subset f^l(\operatorname{cl} V_0) = \operatorname{cl} f^l(V_0) \subset \operatorname{cl} f^{k+1}(V_0) = f^k(\operatorname{cl} f(V_0))$. Now, (20) follows from the fact that $A_k \cap f^k(\operatorname{cl} f(V_0)) = \emptyset$.

Put

$$P := \bigcap_{k=0}^{\infty} f^k(V_0). \tag{21}$$

LEMMA 4.

$$P \textit{ is closed}; \tag{22}$$

$$\xi \in P; \tag{23}$$

$$f(P) = P; \tag{24}$$

$$f(V_0 \setminus P) \subset V_0 \setminus P; \tag{25}$$

$$P \neq X \quad \textit{implies} \quad \xi \notin \operatorname{int} P; \tag{26}$$

$$X \setminus P = \bigcup_{k=0}^{\infty} [f^{-k}(V_0) \setminus P]. \tag{27}$$

Proof. It follows from (10) that $\bigcap_{n=0}^{\infty} f^n(V_0) = \bigcap_{n=0}^{\infty} \operatorname{cl} f^n(V_0)$ thus (22) is true. (23) follows from (6) and (7), and (24) results from (10). Since $f(V_0 \setminus P) = f(V_0) \setminus f(P)$, (25) follows from (8) and (24).

To prove (26) let $x \in X \setminus P$. Then, by (24), $f^k(x) \in X \setminus P$, $k = 0, 1, 2, \ldots$ and $\xi = \lim_{k\to\infty} f^k(x) \in X \setminus \operatorname{int} P$.

Finally, (27) follows from (4) and (7).

LEMMA 5.

$$\operatorname{cl} V_0 \setminus P = \bigcup_{k=0}^{\infty} A_k.$$

Proof. Fix $k \in \{0,1,2,\ldots\}$ and $x \in A_k$. Then $x \in \operatorname{cl} V_0$ by (17). Using the definition of A_k we infer that $x \notin f^k[\operatorname{cl} f(V_0)]$. This implies that $x \notin f^{k+1}(V_0)$ and, consequently, $x \notin P$. Now, fix $x \in \operatorname{cl} V_0 \setminus P$. Take the smallest non-negative k such that $x \notin f^k(V_0)$. If $k = 0$, then $x \in \operatorname{cl} V_0 \setminus V_0 \subset A_0$. If $k > 0$, then either $x \in \operatorname{cl} f^k(V_0)$ or not. In the first case, recalling (15), we have $x \in \operatorname{cl} f^k(V_0) \setminus f^k(V_0) \subset A_k$. In the other case we have $x \in \operatorname{cl} f^{k-1}(V_0) \setminus \operatorname{cl} f^k(V_0) = A_{k-1}$. This implies that $x \in \bigcup_{k=0}^{\infty} A_k$.

LEMMA 6. *For every $x \in X \setminus P$ the set A_0 contains exactly one element of the orbit $C(x) := \{f^k(x) : k = 0, \pm 1, \pm 2, \ldots$ and $f^k(x)$ is defined$\}$.*

Proof. First we prove the uniqueness. Suppose that for some $x \in X \setminus P$, x_0 and y_0 are two different elements of $A_0 \cap C(x)$. Then there exists $k > 0$ such that $y_0 = f^k(x_0)$ (otherwise we interchange x_0 and y_0). Since $x_0 \in \operatorname{cl} V_0$ we infer that $y_0 \in f^k(\operatorname{cl} V_0) = \operatorname{cl} f^k(V_0) \subset \operatorname{cl} f(V_0)$, which is impossible.

To prove the existence suppose that $A_0 \cap C(x) = \emptyset$ for some $x \in X \setminus P$. In view of (4) there exists an integer $n \geq 0$ such that $f^n(x) \in V_0$. Defining $r := f^n(x)$ we have $r \in V_0 \cap C(x)$. Since $A_0 \cap C(x) = \emptyset$ we obtain $r \in \operatorname{cl} f(V_0)$, i.e. $r \in f(X)$ in view of (8) and (7). This implies that $f^{-1}(r)$ is defined. We have

$$f^{-1}(r) \in f^{-1}(\operatorname{cl} f(V_0)) \subset f^{-1}(\operatorname{cl} f(V_0) \cap f(X)) = f^{-1}(f(\operatorname{cl} V_0)) = \operatorname{cl} V_0.$$

Hence $f^{-1}(r) \in \operatorname{cl} V_0 \cap C(x)$, which again implies that $f^{-1}(r) \in \operatorname{cl} f(V_0) \subset V_0 \subset f(X)$. By induction we can prove that $f^{-i}(r)$ is defined for every integer $i \geq 0$ and $f^{-i}(r) \in V_0$. This together with the equation $r = f^i[f^{-i}(r)]$, $i = 0,1,2,\ldots$, implies that $r \in P$. This yields $x \in P$, which is impossible. Thus $A_0 \cap C(x) = \emptyset$.

LEMMA 7 (K. Baron). *Let X and Y be topological spaces, $U \subset X$ an open set, $h : X \times Y \to Y$ and $f : X \to X$ continuous functions. If $f(U) \subset U$ and for every $x \in X$ there exists a positive integer k such that $f^k(x) \in U$, then for every solution $\varphi_0 : U \to Y$ of the functional equation*

$$\varphi(x) = h(x, \varphi[f(x)])$$

there exists exactly one solution $\varphi : X \to Y$ of this equation such that $\varphi(x) = \varphi_0(x)$, $x \in U$. If φ_0 is continuous then so is φ.

THEOREM. *Let hypotheses* (i)–(iii) *be satisfied. Let V_0 be an open set satisfying* (7) *and* (8) *and let the sets P, A_0, C_1 be defined by* (21), (11) *and* (16). *Then for every continuous function $\psi : A_0 \cup C_1 \to Y$ such that*

$$\psi(x) = g(f^{-1}(x), \psi[f^{-1}(x)]) \quad \text{for } x \in C_1 \tag{28}$$

there exists exactly one solution $\varphi : X \setminus P \to Y$ *of equation* (1) *such that*

$$\varphi|_{A_0 \cup C_1} = \psi. \tag{29}$$

Proof. In view of Lemma 6 the Theorem from [3] (cf. also [4, Theorem 1.1]) may be applied. It follows from that theorem and Lemma 5 that the function $\Phi : \operatorname{cl} V_0 \setminus P \to Y$ defined by

$$\Phi(x) = \psi_n(x), \quad x \in A_n,\ n \geq 0, \tag{30}$$

where the functions $\psi_n : A_n \to Y$ are given by

$$\psi_0 = \psi|_{A_0}, \quad \psi_{n+1}(x) = g(f^{-1}(x), \psi_n[f^{-1}(x)]), \tag{31}$$

is a unique solution of equation (1) on $\operatorname{cl} V_0 \setminus P$ such that

$$\Phi|_{A_0} = \psi_0. \tag{32}$$

We are going to prove that Φ is continuous on $\operatorname{cl} V_0 \setminus P$. By definition of Φ and Lemma 3 it follows that Φ is continuous on $\bigcup_{k=0}^{\infty} \operatorname{int} A_k$. We shall show that it is also continuous on C_1. First observe that

$$\Phi(x) = \psi(x) \quad \text{for } x \in A_0 \cup C_1. \tag{33}$$

Indeed, if $x \in C_1$ then $f^{-1}(x) \in C_0 \subset A_0$ and by (30), (31) and (28) we have

$$\Phi(x) = \psi_1(x) = g(f^{-1}(x), \psi_0[f^{-1}(x)]) = g(f^{-1}(x), \psi[f^{-1}(x)]) = \psi(x).$$

Next, fix an $x_0 \in C_1$ and a neighbourhood U of $\Phi(x_0)$. From the continuity of ψ on $A_0 \cup C_1$ and (33) there exists a neighbourhood $V_{x_0}^1$ of x_0 such that

$$\Phi(V_{x_0}^1 \cap (A_0 \cup C_1)) \subset U. \tag{34}$$

By the continuity of $g(\cdot, \psi(\cdot))$ on $A_0 \cup C_1$ and since $f^{-1}(x_0) \in A_0$ and $g(f^{-1}(x_0), \psi[f^{-1}(x_0)]) = \Phi(x_0)$ we can find a neighbourhood W of $f^{-1}(x_0)$ such that

$$g(\cdot, \psi(\cdot))[W \cap (A_0 \cup C_1)] \subset U. \tag{35}$$

Putting $V_{x_0}^2 := f(W) \cap V_0$ we obtain a neighbourhood of x_0 such that

$$\Phi(V_{x_0}^2 \cap (A_1 \cup C_2)) \subset U. \tag{36}$$

Indeed, for $x \in V_{x_0}^2 \cap (A_1 \cup C_2)$ we have $f^{-1}(x) \in W \cap (A_0 \cup C_1)$ and by (33) and (35), $\Phi(x) = g(f^{-1}(x), \Phi[f^{-1}(x)]) = g(f^{-1}(x), \psi[f^{-1}(x)]) \in U$. Now $x_0 \in C_1$ implies that $x_0 \notin f(V_0)$ and by (10), $x_0 \notin \operatorname{cl} f^2(V_0)$. Hence $V_{x_0} := V_0 \cap V_{x_0}^1 \cap V_{x_0}^2 \setminus \operatorname{cl} f^2(V_0)$ is an open neighbourhood of x_0. Moreover, since $V_{x_0} \subset \operatorname{cl} V_0 \setminus f^2(V_0) \subset A_0 \cup C_1$ by (34) and (36) we get $\Phi(V_{x_0}) \subset U$. This proves the continuity of Φ at points of C_1. Hence the continuity of Φ on C_k, $k = 0, 1, 2, \ldots$ may be obtained by induction. From (18) and Lemma 5 we see that Φ is continuous on $\operatorname{cl} V_0 \setminus P$.

Hypothesis (iii) implies that $\Phi|_{V_0\setminus P}$ is a solution of the equation

$$\Phi(x) = h(x, \Phi[f(x)]) \tag{37}$$

on $V_0 \setminus P$. Observe that by (22) the set $V_0 \setminus P$ is open in $X \setminus P$ and that for every $x \in X \setminus P$ there exists $k \in \{0, 1, 2, \ldots\}$ such that $f^k(x) \in V_0 \setminus P$ (by (4) and (7)). Thus from Lemma 7 it follows that there exists exactly one solution $\varphi : X \setminus P \to Y$ of (37). It is easy to verify that the function φ satisfies equation (1) and condition (29).

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