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On the increasing solutions of the translation equation

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Abstract. Let M be a non-empty set endowed with a dense linear order without the smallest and greatest elements. Let $(G,+)$ be a group which has a non-trivial uniquely divisible subgroup. There are given conditions under which every solution $F : M\times G \to M$ of the translation equation is of the form $F(a,x) = f^{-1}(f(a)+c(x))$ for $a \in M$, $x \in G$ with some non-trivial additive function $c : G \to \mathbb{R}$ and a strictly increasing function $f : M \to \mathbb{R}$ such that $f(M) + c(G) \subset f(M)$. In particular, a problem of J. Tabor is solved.

This paper is motivated by the following problem raised by J. Tabor during his talk at the Mathematics Department of the Pedagogical University in Rzeszów:

Let M be a non-empty set endowed with a dense linear order $\leq$ without the smallest and greatest elements and let $\mathbb{R}$ stand for the set of all reals. Find conditions such that a function $F : M \times \mathbb{R} \to M$ satisfies them and the translation equation

$$F(F(a,x),y) = F(a,x+y) \tag{1}$$

if and only if there exist an additive function $c : \mathbb{R} \to \mathbb{R}$, a non-empty set $T \subset \mathbb{R}$, and a strictly increasing function f mapping M onto the set $T+c(\mathbb{R})$ such that $c(\mathbb{R})$ is dense in $\mathbb{R}$ and

$$F(a,x) = f^{-1}(f(a)+c(x)) \tag{2}$$

for $x \in \mathbb{R}$, $a \in M$.

The problem is connected with some iteration groups. For the details we refer to [4].

We present a solution of the problem. We also give some results concerning monotonic solutions of equation (1). Such solutions of the translation equation have already been studied e.g. in [1]–[3]. However, the literature devoted to them is not very vast.

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Throughout this paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ stand for the sets of all positive integers, all integers, and rationals, respectively. $(G,+)$ denotes a group (not necessarily commutative) and H is a non-trivial uniquely divisible subgroup of G, unless explicitly stated otherwise.

Let us start with the following

LEMMA 1. *Let $x_0 \in H \setminus \{0\}$ and $b \in M$. Suppose that a function $F : M \times G \to M$ satisfies the following two conditions*:

(3) *the set* $\{F(b,px_0) : p \in \mathbb{Q}\}$ *is dense in* M,

(4) $F(b,px_0) < F(b,qx_0)$ *for every* $p,q \in \mathbb{Q},\ p<q$.

Then, for each $t \in \mathbb{R}$, $t>0$ ($t<0$ respectively), the function $f_t : M \to \mathbb{R}$ given by

$$f_t(a) = t \lim_{n\to\infty} q_n^a \quad \text{for } a \in M, \tag{5}$$

where $\{q_n^a : n \in \mathbb{N}\} \subset \mathbb{Q}$ is any strictly increasing sequence such that $a = \lim_{n\to\infty} F(b,q_n^a x_0)$ (with respect to the order in M), is strictly increasing (decreasing resp.).

Proof. Fix $t \in \mathbb{R}$, $t>0$ (in the case $t<0$ the proof is analogous). First, we prove that f_t is well defined.

Fix $a \in M$. Since M is without a greatest element, by (3) there is $p \in \mathbb{Q}$ with $a < F(b,px_0)$. Thus, according to (4), there is a real number $s := \sup\{q \in \mathbb{Q} : F(b,qx_0) < a\} < p$. Let $\{q_n^a : n \in \mathbb{N}\} \subset \mathbb{Q}$ be a strictly increasing sequence with $s = \lim_{n\to\infty} q_n^a$. Then $\{F(b,q_n^a x_0) : n \in \mathbb{N}\}$ is a strictly increasing sequence in M. Take $d \in M$ with $d<a$. There is $r \in \mathbb{Q}$ with $d < F(b,rx_0) < a$. Further, there exists $m \in \mathbb{N}$ with $r < q_m^a$. Consequently, $d < F(b,rx_0) < F(b,q_n^a x_0) < a$ for every $n \in \mathbb{N}$ with $n>m$. This means that $a = \lim_{n\to\infty} F(b,q_n^a x_0)$.

Next, suppose that $\{p_n^a : n \in \mathbb{N}\} \subset \mathbb{Q}$ is also a strictly increasing sequence such that $a = \lim_{n\to\infty} F(b,p_n^a x_0)$. Then, for every $n \in \mathbb{N}$, there are $k,m \in \mathbb{N}$ with $F(b,q_n^a x_0) < F(b,p_k^a x_0)$ and $F(b,p_n^a x_0) < F(b,q_m^a x_0)$. Hence (4) implies $\lim_{n\to\infty} q_n^a = \lim_{n\to\infty} p_n^a$. In this way we have proved that the definition of f_t is correct.

To complete the proof, fix $a,d \in M$, $a<d$. Let $\{q_n^a : n \in \mathbb{N}\}$, $\{q_n^d : n \in \mathbb{N}\} \subset \mathbb{Q}$ be strictly increasing sequences with $a = \lim_{n\to\infty} F(b,q_n^a x_0)$ and $d = \lim_{n\to\infty} F(b,q_n^d x_0)$. Since $a<d$, there is $n_0 \in \mathbb{N}$ such that

$$F(b,q_k^a x_0) < F(b,q_n^d x_0) \quad \text{for every } k,n \in \mathbb{N},\ n>n_0.$$

Thus, by (4), $q_k^a < q_n^d$ for every $k,n \in \mathbb{N}$, $n>n_0$, which means that $\lim_{n\to\infty} q_n^a < \lim_{n\to\infty} q_n^d$. Hence $f_t(a) < f_t(d)$. This ends the proof.

LEMMA 2. *Let $a,b \in M$ and $x_0 \in H\setminus\{0\}$. Suppose that $F : M\times G \to M$ is a function and $\{q_n^a : n \in \mathbb{N}\} \subset \mathbb{Q}$ is a strictly increasing sequence such* *that* $a = \lim_{n\to\infty} F(b, q_n^a x_0)$, (3) *and* (4) *hold*,

(6) *the set* $\{F(d,x) : d \in M\}$ *is dense in* M *for every* $x \in G$,

and

(7) $F(d,x) < F(c,x)$ *for every* $d, c \in M$, $d < c$, $x \in G$.

Then $F(a,x) = \lim_{n\to\infty} F(F(b, q_n^a x_0), x)$ *for every* $x \in G$.

P r o o f. Fix $x \in G$. Since, according to (3), the set $\{F(b, px_0) : p \in \mathbb{Q}\}$ is dense in M, so is the set $\{F(F(b, px_0), x) : p \in \mathbb{Q}\}$ by (6) and (7). Moreover, by (7), for every $p \in \mathbb{Q}$ with $F(F(b, px_0), x) < F(a,x)$, there exists $m \in \mathbb{N}$ such that

$$F(F(b, px_0), x) < F(F(b, q_n^a x_0), x) < F(a,x) \quad \text{for } n \in \mathbb{N},\ n > m,$$

because $a = \lim_{n\to\infty} F(b, q_n^a x_0)$ and, by (4) and (7), the sequence $\{F(F(b, q_n^a x_0), x) : n \in \mathbb{N}\}$ is strictly increasing. This yields the assertion.

LEMMA 3. *Let* $b, d \in M$, $\{a_k : k \in \mathbb{N}\} \subset M$, $x_0 \in H \setminus \{0\}$, $F : M \times G \to M$ *and let* $\{p_n : n \in \mathbb{N}\}$, $\{q_n : n \in \mathbb{N}\} \subset \mathbb{Q}$ *be bounded strictly increasing sequences such that* $a_k = \lim_{n\to\infty} F(b, (q_k + p_n)x_0)$ *for* $k \in \mathbb{N}$ *and* $d = \lim_{k\to\infty} a_k$. *Suppose that condition* (4) *holds. Then* $d = \lim_{n\to\infty} F(b, (q_n + p_n)x_0)$.

P r o o f. Fix $k \in \mathbb{N}$. Then there exists $n_0 \in \mathbb{N}$, $n_0 > k$, with

$$\begin{aligned} q_k + p_j &< q_{k+1} + p_j < q_{k+1} + \lim_{n\to\infty} p_n < q_m + p_m \\ &< q_m + p_{m+j} \quad \text{for } j, m \in \mathbb{N},\ m > n_0, \end{aligned}$$

because the sequences $\{p_n : n \in \mathbb{N}\}$ and $\{q_n : n \in \mathbb{N}\}$ are strictly increasing and bounded. Thus, according to (4), $a_k \le a_{k+1}$ and

$$F(b, (q_k + p_n)x_0) < F(b, (q_m + p_m)x_0) < a_m \quad \text{for } n, m \in \mathbb{N},\ m > n_0.$$

In this way we have proved that the sequence $\{a_k : k \in \mathbb{N}\}$ is increasing and, for each $k \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with

$$a_k \le F(b, (q_m + p_m)x_0) \le a_m,$$

which yields the assertion.

LEMMA 4. *Let* $x_0 \in H \setminus \{0\}$, $z \in G$, $b \in M$ *and let* $F : M \times G \to M$ *be a function satisfying*

$$(1') \qquad F(F(b,x), y) = F(b, x+y) \quad \text{for } x, y \in G,$$

$$(8) \qquad F(b, x+y) = F(b, y+x) \quad \text{for } x, y \in G$$

and conditions (3), (4), (6), (7). *Suppose that* $\{q_n^z : n \in \mathbb{N}\} \subset \mathbb{Q}$ *is a strictly increasing sequence such that* $F(b,z) = \lim_{n\to\infty} F(b, q_n^z x_0)$. *Then*

$$F(a,z) = \lim_{n\to\infty} F(a, q_n^z x_0) \quad \text{for every } a \in M.$$

P r o o f. Fix $a \in M$. Then, by (3) and (4), there is a strictly increasing bounded sequence $\{q_n^a : n \in \mathbb{N}\} \subset \mathbb{Q}$ such that $a = \lim_{n\to\infty} F(b, q_n^a x_0)$. According to Lemma 2 and conditions (1′) and (8),

$$\begin{aligned} F(a,z) &= \lim_{n\to\infty} F(F(b,q_n^a x_0),z) = \lim_{n\to\infty} F(F(b,z),q_n^z x_0) \\ &= \lim_{n\to\infty} (\lim_{k\to\infty} F(F(b,q_k^z x_0),q_n^a x_0)) \\ &= \lim_{n\to\infty} (\lim_{k\to\infty} F(b,(q_n^a + q_k^z)x_0)) =: I_1 \end{aligned}$$

and

$$\begin{aligned} \lim_{k\to\infty} F(a,q_k^z x_0) &= \lim_{k\to\infty} (\lim_{n\to\infty} F(F(b,q_n^a x_0),q_k^z x_0)) \\ &= \lim_{k\to\infty} (\lim_{n\to\infty} F(b,(q_n^a + q_k^z)x_0)) =: I_2. \end{aligned}$$

Since, by Lemma 3, $I_1 = I_2$, we obtain the statement.

LEMMA 5. *Let $b \in M$, $x_0 \in H\setminus\{0\}$ and let $F : M\times G \to M$ be a function satisfying conditions* (1′), (3), (4), *and* (6)–(8). *Then, for each $t \in \mathbb{R}$, $t > 0$ ($t < 0$ resp.), the function $c_t : G \to \mathbb{R}$, $c_t(x) = f_t(F(b,x))$ for $x \in G$, is additive, $c_t(x_0) > 0$ ($c_t(x_0) < 0$ resp.), and*

$$F(a,x) = f_t^{-1}(f_t(a) + c_t(x)) \quad \text{for } a \in M,\ x \in G, \tag{9}$$

where the function $f_t : M \to \mathbb{R}$ is given by (5).

P r o o f. Fix $a \in M$, $x, y \in G$, $t \in \mathbb{R}$, $t \neq 0$, and strictly increasing bounded sequences $\{q_n^x : n \in \mathbb{N}\}, \{q_n^y : n \in \mathbb{N}\}, \{q_n^a : n \in \mathbb{N}\} \subset \mathbb{Q}$ with $a = \lim_{n\to\infty} F(b, q_n^a x_0)$, $F(b,x) = \lim_{n\to\infty} F(b, q_n^x x_0)$, and $F(b,y) = \lim_{n\to\infty} F(b, q_n^y x_0)$ (we construct these sequences e.g. as in the proof of Lemma 1). According to Lemmas 2–4 and (1′),

$$\begin{aligned} F(b,x+y) = F(F(b,x),y) &= \lim_{n\to\infty} F(F(b,x),q_n^y x_0) \\ &= \lim_{n\to\infty} (\lim_{k\to\infty} F(F(b,q_k^x x_0),q_n^y x_0) \\ &= \lim_{n\to\infty} (\lim_{k\to\infty} F(b,(q_k^x + q_n^y)x_0)) = \lim_{n\to\infty} F(b,(q_n^x + q_n^y)x_0). \end{aligned}$$

Thus

$$\begin{aligned} c_t(x+y) = f_t(F(b,x+y)) &= t \lim_{n\to\infty} (q_n^x + q_n^y) \\ &= f_t(F(b,x)) + f_t(F(b,y)) = c_t(x) + c_t(y). \end{aligned}$$

Consequently, c_t is additive. Further, by Lemmas 2–4,

$$\begin{aligned} F(a,x) &= \lim_{k\to\infty} F(a,q_k^x x_0) = \lim_{k\to\infty} (\lim_{n\to\infty} F(F(b,q_n^a x_0),q_k^x x_0)) \\ &= \lim_{n\to\infty} F(b,(q_n^a + q_n^x)x_0). \end{aligned}$$

Hence

$$f_t(F(a,x)) = t \lim_{n\to\infty} (q_n^a + q_n^x) = f_t(a) + f_t(F(b,x)) = f_t(a) + c_t(x).$$

Since, in view of Lemma 1, f_t is one-to-one, this implies (9).

To complete the proof, note that, by (4), $F(b,0) < F(b,x_0)$. Thus, on account of Lemma 1, for $t > 0$,

$$0 = c_t(0) = f_t(F(b,0)) < f_t(F(b,x_0)) = c_t(x_0),$$

and for $t < 0$,

$$0 = c_t(0) = f_t(F(b,0)) > f_t(F(b,x_0)) = c_t(x_0).$$

So, we have proved Lemma 5.

Now, we are in a position to prove the following

THEOREM 1. *Assume that* $(G,+)$ *is a group* (*not necessarily commutative*) *and has a uniquely divisible subgroup* H. *Let* $b \in M$ *and* $x_0 \in H \setminus \{0\}$. *Then a function* $F : M \times G \to M$ *satisfies conditions* $(1')$, (3), (4), *and* (6)–(8) *if and only if there exist a non-empty set* $T \subset \mathbb{R}$, *an additive function* $c : G \to \mathbb{R}$, $c(x_0) > 0$ ($c(x_0) < 0$ *resp.*), *and an increasing* (*decreasing resp.*) *bijection* $f : M \to K$, *where* $K = T + c(G)$, *such that* (2) *holds for every* $x \in G$ *and* $a \in M$. *Furthermore, every function* $F : M \times G \to M$ *of the form* (2) *is a solution of equation* (1).

P r o o f. Assume that $F : M \times G \to M$ satisfies conditions $(1')$, (3), (4), and (6)–(8). Fix $t \in \mathbb{R}$, $t > 0$ ($t < 0$ resp.). Then, according to Lemmas 1 and 5, the function $f_t : M \to \mathbb{R}$ given by (5) is strictly increasing (decreasing resp.), the function $c_t : G \to \mathbb{R}$, $c_t(z) = f_t(F(b,z))$ for $z \in G$, is additive, $c_t(x_0) > 0$ ($c_t(x_0) < 0$ resp.), and (9) holds. Thus it suffices to put $T = f_t(M)$.

Now, assume that $F : M \times G \to M$ is of the form (2). It is easy to check that F is a solution of equation (1) and satisfies (8). Conditions (4) and (7) result from the fact that f is increasing (decreasing resp.) and $c(x_0) > 0$ ($c(x_0) < 0$ resp.). Further, since c is additive and H is uniquely divisible, $c(qx_0) = qc(x_0)$ for $q \in \mathbb{Q}$. Thus the sets $c(G)$ and K are dense in $\mathbb{R}$, which means that $s + c(G)$ and $K + r$ are dense in K for every $s \in K$ and $r \in c(G)$. Consequently, by (2), conditions (3) and (6) hold, because f is a monotonic bijection. This completes the proof.

From Theorem 1 we obtain the following corollary, which gives a solution of the problem of J. Tabor.

COROLLARY 1. *Suppose that* G *is a uniquely divisible group. Then a function* $F : M \times G \to M$ *satisfies equation* (1) *and there exist* $b \in M$, $x_0 \in G \setminus \{0\}$ *such that conditions* (3), (4), *and* (6)–(8) *are valid if and only if there exist a non-empty set* $T \subset \mathbb{R}$, *an additive function* $c : G \to \mathbb{R}$, $c(G) \neq \{0\}$, *and an increasing bijection* $f : M \to K$, *where* $K = T + c(G)$, *such that* (2) *holds for* $x \in G$ *and* $a \in M$.

Proof. It suffices to note that $c(G) \neq \{0\}$ iff there is $x_0 \in G$ with $c(x_0) > 0$.

Remark 1. The assumption of Theorem 1 that G is a group can be weakened. Namely, it suffices to suppose that G is a groupoid (i.e. a non-empty set endowed with a binary operation) and has a subgroupoid H which is a uniquely divisible group.

The representation (2) of a solution $F : M \times G \to M$ of equation (1) is not unique:

Proposition 1. *Suppose that G is as in Theorem 1, $c_1, c_2 : G \to \mathbb{R}$ are additive functions such that $c_i(H) \neq \{0\}$ for $i = 1, 2$, $T_1, T_2 \subset \mathbb{R}$ are non-empty sets, $f_i : M \to K_i$, where $K_i = T_i + c_i(G)$, for $i = 1, 2$ are monotonic bijections, and*

$$F_i(a, x) = f_i^{-1}(f_i(a) + c_i(x)) \quad \text{for } x \in G,\ a \in M,\ i = 1, 2. \tag{10}$$

Then $F_1 = F_2$ iff there exist $u, v \in \mathbb{R}$, $u \neq 0$, such that $f_1(a) = uf_2(a) + v$ for $a \in M$ and $c_1(x) = uc_2(x)$ for $x \in G$.

Proof. First suppose that there are $u, v \in \mathbb{R}$, $u \neq 0$, with $f_1 = uf_2 + v$ and $c_1 = uc_2$. Then, for every $a \in M$ and $x \in G$,

$$\begin{aligned} F_1(a, x) &= f_1^{-1}(f_1(a) + c_1(x)) = f_1^{-1}(u(f_2(a) + c_2(x)) + v) \\ &= f_1^{-1}(uf_2(F_2(a, x)) + v) = F_2(a, x). \end{aligned}$$

Now, assume that $F_1 = F_2$. Fix $t \in \mathbb{R}$, $t > 0$, $b \in M$, and $x_0 \in H \setminus \{0\}$ with $c_1(x_0) > 0$. Then, by (10), we also have $c_2(x_0) \neq 0$. Put $F = F_1$. According to Theorem 1, (3) and (4) hold. Thus, by Lemma 1, the function $f_t : M \to \mathbb{R}$ given by (5) is well defined.

Fix $i \in \{1, 2\}$ and $a \in M$. Let $\{q_n^a : n \in \mathbb{N}\} \subset \mathbb{Q}$ be a strictly increasing sequence with $a = \lim_{n\to\infty} F(b, q_n^a x_0)$. Note that $f_i(F(b, x)) = f_i(b) + c_i(x)$ for $x \in G$. Thus the set $\{f_i(F(b, px_0)) : p \in \mathbb{Q}\}$ is dense in $\mathbb{R}$ and the sequence $\{f_i(F(b, q_n^a x_0)) : n \in \mathbb{N}\}$ is strictly monotonic. Moreover, by (4), for each $p \in \mathbb{Q}$ with $F(b, px_0) < a$, there is $n_0 \in \mathbb{N}$ such that $F(b, px_0) < F(b, q_n^a x_0) < a$ for $n \in \mathbb{N}$, $n > n_0$. Consequently, since f_i is monotonic, $f_i(a) = \lim_{n\to\infty} f_i(F(b, q_n^a x_0))$. Hence, for each $a \in M$,

$$\begin{aligned} f_i(a) &= \lim_{n\to\infty} (f_i(b) + c_i(q_n^a x_0)) = f_i(b) + c_i(x_0) \lim_{n\to\infty} q_n^a \\ &= f_i(b) + t^{-1} c_i(x_0) f_t(a) \end{aligned}$$

and, for each $z \in G$,

$$f_i(b) + c_i(z) = f_i(F(b, z)) = f_i(b) + t^{-1} c_i(x_0) f_t(F(b, z)).$$

So, we have proved that there are $u_1, u_2, v_1, v_2 \in \mathbb{R}$, $u_1 u_2 \neq 0$, with $f_t(a) =$

$u_i f_i(a) + v_i$ and $f_t(F(b, z)) = u_i c_i(z)$ for $a \in M$, $z \in G$, $i = 1, 2$. Hence it suffices to put $u = u_2 u_1^{-1}$ and $v = u_1^{-1}(v_2 - v_1)$. This ends the proof.

Remark 2. Let $x_0 \in H \setminus \{0\}$ and $b \in M$. A function $F : M \times G \to M$ satisfies equation (1) and condition (4) iff the function $F_0 : M \times G \to M$, $F_0(a, x) = F(a, -x)$ for $a \in M$, $x \in G$, satisfies (1) and

$$(4') \qquad F_0(b, px_0) > F_0(b, qx_0) \quad \text{for } p, q \in \mathbb{Q},\ p < q.$$

Thus, from Theorem 1 we can also get a description of solutions of (1) satisfying conditions (3), (6), (7), and (4′).

Remark 3. Suppose that $x_0 \in H \setminus \{0\}$, $b \in M$, and a function $F : M \times G \to M$ satisfies (4), equation (1), and the condition

$$(7') \qquad F(a, x) < F(c, x) \quad \text{for } a, c \in M,\ a > c,\ x \in G.$$

Fix $p, q \in \mathbb{Q}$ with $p < q$. Then, by (4), $F(b, px_0) < F(b, qx_0)$. Thus, on account of (7′),

$$F(b, (p + 1)x_0) = F(F(b, px_0), x_0) > F(F(b, qx_0), x_0) = F(b, (q + 1)x_0),$$

which, in view of (4), means that $p + 1 > q + 1$. This gives a contradiction. Consequently, there are no solutions of (1) satisfying conditions (4) and (7′), and similarly for (4′) and (7′) according to Remark 2.

Remark 4. Suppose that H is endowed with a linear order such that, for every $x, y, z \in H$,

$$x < y \quad \text{iff} \quad z + x < z + y \text{ and } x + z < y + z.$$

Let $x_0 \in H$, $x_0 > 0$, and $b \in M$. Then every function $F : M \times G \to M$ such that $F(b, x) < F(b, y)$ for every $x, y \in H$, $x < y$, also satisfies (4).

In fact, let $p, q \in \mathbb{Q}$, $p < q$. Then there are $j, k, m, n \in \mathbb{Z}$, $k > 0$, $n > 0$, with $p = jk^{-1}$ and $q = mn^{-1}$. Note that $jn < km$. Thus $knpx_0 = njx_0 < kmx_0 = knqx_0$, whence $px_0 < qx_0$. Hence $F(b, px_0) < F(b, qx_0)$.

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