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Evolution equations with parameter in the hyperbolic case

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Abstract. The purpose of this paper is to give theorems on continuity and differentiability with respect to (h,t) of the solution of the initial value problem $du/dt = A(h,t)u + f(h,t), u(0) = u_0(h)$ with parameter $h \in \Omega \subset \mathbb{R}^m$ in the "hyperbolic" case.

1. Introduction. We consider the initial value problem

$$\begin{cases} \dfrac{du}{dt} = A(h,t)u + f(h,t), & t \in [0,T],\ h \in \Omega, \\ u(0) = u_0(h). \end{cases} \tag{1}$$

It is known that under some assumptions on the family of the operators $\{A(h,t)\}$ and on the function f, the problem (1) has the unique solution given by

$$u(h,t) = U(h,t,0)u_0(h) + \int_0^t U(h,t,s)f(h,s)\,ds, \tag{2}$$

where, for each $h \in \Omega$, U is the fundamental solution (or evolution system) for problem (1) (cf. [3, Ch. 5]).

Analogously to the papers [5] and [6], where the "parabolic" case of problem (1) was studied, we investigate the continuity and differentiability of the mapping

$$\Omega \times [0,T] \ni (h,t) \to u(h,t) \in X, \tag{3}$$

where the mapping u is given by (2).

2. Stable approximations of the family of operators. This section is based on Krein's monograph [2, Ch. II] and it has the auxiliary character. To simplify notations we assume that the family $\{A(h,t)\}$ considered in the introduction is independent of the parameter h.

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Assuming that X is a Banach space we let $B(X)$ be the Banach space of all linear bounded operators and $\mathcal{C}(X)$ be the vector space of all linear closed operators from X into itself. If $A : X \to X$ is a linear operator then $D(A)$, $N(A)$, $R(A)$, $\overline{A}$, $P(A)$ denote the domain, kernel, range, closure and resolvent set of A, respectively.

In this section we consider a family of operators $\{A(t)\}$, $t \in [0,T]$, where $A(t) \in \mathcal{C}(X)$, $D(A(t)) = D$, $\overline{D} = X$ and $0 \in P(A(t))$ for every $t \in [0,T]$.

We investigate the Cauchy problem

$$\frac{du}{dt} = A(t)u, \quad u(s) = x, \quad 0 \le s \le t \le T, \tag{4}$$

where $x \in D$.

Definition 1 ([2, p. 193]). The Cauchy problem (4) is said to be *uniformly correct* if:

(i) for each $s \in [0,T]$ and any $x \in D$ there exists a unique solution $u = u(t,s)$ of (4) on the interval $[s,T]$,

(ii) the function $u = u(t,s)$ and its derivative u'_t are continuous in the triangle $\Delta_T := \{(t,s) : 0 \le s \le t \le T\}$,

(iii) the solution depends continuously on the initial data.

If the Cauchy problem is uniformly correct, then it is possible to introduce a linear operator $U(t,s)$ for $(t,s) \in \Delta_T$ by the formula

$$U(t,s)x := u(t,s), \qquad (t,s) \in \Delta_T,\ x \in D, \tag{5}$$

where $u(s,s) = x$. The formula (5) defines the operator $U(t,s)$ on the set D dense in X. Since for fixed $(t,s) \in \Delta_T$ it is a bounded operator, it admits a continuous extension to the entire space X.

It is known (cf. [2, pp. 193–194]) that if for each $x \in D$ the mapping $[0,T] \ni t \to A(t)x$ is continuous (i.e. the mapping $t \to A(t)$ is strongly continuous on D) and the Cauchy problem (4) is uniformly correct, then the fundamental solution U has the following properties:

(a) the mapping $\Delta_T \ni (t,s) \to U(t,s) \in B(X)$ is strongly continuous and $\|U(t,s)\| \le M$ for $(t,s) \in \Delta_T$,

(b) $U(t,t) = I$ and $U(t,s) = U(t,r)U(r,s)$ for $0 \le s \le r \le t \le T$,

(c) $\frac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x$, $\frac{\partial}{\partial s}U(t,s)x = -U(t,s)A(s)x$ for $(t,s) \in \Delta_T$, $x \in D$,

(d) the mappings $\Delta_T \ni (t,s) \to \frac{\partial}{\partial t}U(t,s)$ and $\Delta_T \ni (t,s) \to \frac{\partial}{\partial s}U(t,s)$ are strongly continuous on D.

Definition 2 ([4, p. 89]). An operator-valued function $U : \Delta_T \ni (t,s) \to U(t,s) \in B(X)$ satisfying the above conditions (a)–(d) is called the *fundamental solution* of problem (4).

It is known (see [2, Ch. II, §2]) that if the operator $A(t)$ is bounded for each $t \in [0,T]$ and the mapping $[0,T] \ni t \to A(t)$ is strongly continuous, then problem (4) is uniformly correct and so the fundamental solution U for this problem exists.

DEFINITION 3 ([2, p. 199]). If there exists a sequence of bounded and strongly continuous operators $A_n(t), t \in [0,T]$, for which

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|[A(t) - A_n(t)]A(t)^{-1}x\| = 0, \qquad x \in X, \tag{6}$$

and the fundamental solutions of the problems

$$\frac{du}{dt} = A_n(t)u, \qquad u(s) = x,$$

are uniformly bounded, i.e.,

$$\|U_n(t,s)\| \le M, \tag{7}$$

where M does not depend on $n \in \mathbb{N}$ and $(t,s) \in \Delta_T$, then we say that the family $\{A(t)\}$, $t \in [0,T]$, is *stably approximated* by the sequence $\{A_n(t)\}$.

In [2, Ch. II] the following sufficient conditions are given for the family $\{A(t)\}$, $t \in [0,T]$, to be stably approximated:

$$\text{the mapping } [0,T] \ni t \to A(t) \text{ is strongly continuous in } D, \tag{8}$$

$$\|R(\lambda; A(t))\| := \|(A(t) - \lambda I)^{-1}\| \le \frac{1}{\lambda+1} \qquad \text{for } \lambda \ge 0. \tag{9}$$

The sequence $\{A_n(t)\}$ approximating the family $\{A(t)\}, t \in [0,T]$, has the form

$$A_n(t) := -nA(t)R(n; A(t)) \tag{10}$$

(cf. [2, p. 204]).

Our nearest purpose is to give other sufficient conditions for the family $\{A(t)\}$, $t \in [0,T]$, to be stably approximated (see Theorems 1 and 2).

DEFINITION 4 ([3, p. 130]). A family $\{A(t)\}, t \in [0,T]$, is called *stable* if there are constants $M \ge 1$ and ω (called the *stability constants*) such that

$$(\omega, \infty) \subset P(A(t)) \qquad \text{for } t \in [0,T] \tag{11}$$

and

$$\Big\| \prod_{j=1}^{k} R(\lambda; A(t_j)) \Big\| \le M(\lambda - \omega)^{-k} \qquad \text{for } \lambda > \omega \tag{12}$$

and for every finite sequence $0 \le t_1 \le \ldots \le t_k \le T$, $k \in \mathbb{N}$.

LEMMA 1. *Let $\{A(t)\}$, $t \in [0,T]$, be a stable family in the sense of Definition 4. Then the sequence $\{A_n(t)\}$, where $A_n(t)$ is defined by* (10), *is* *uniformly stable*, *i.e.*, *the stability constants for the operators* $A_n(t)$ *do not depend on* $n \in \mathbb{N}$.

Proof. From the identity

$$R(\lambda; A_n(t)) = \frac{n^2}{(n+\lambda)^2} R\left(\frac{n\lambda}{n+\lambda}; A(t)\right) - \frac{1}{n+\lambda} I$$

we have

$$\begin{aligned}
&\Big\| \prod_{j=1}^{k} R(\lambda; A_n(t_j)) \Big\| \\
&\le \Big\| \prod_{j=1}^{k} \left[\frac{n^2}{(n+\lambda)^2} R\left(\frac{n\lambda}{n+\lambda}; A(t_j)\right) - \frac{1}{n+\lambda} I \right] \Big\| \\
&\le \bigg| \left[\frac{n^2}{(n+\lambda)^2}\right]^k M \left(\frac{\lambda n}{n+\lambda} - \omega\right)^{-k} \\
&\quad + \binom{k}{1} \left(\frac{n^2}{(n+\lambda)^2}\right)^{k-1} \frac{1}{n+\lambda} M \left(\frac{\lambda n}{n+\lambda} - \omega\right)^{-k+1} \\
&\quad + \binom{k}{2} \left(\frac{n^2}{(n+\lambda)^2}\right)^{k-2} \frac{1}{(n+\lambda)^2} M \left(\frac{\lambda n}{n+\lambda} - \omega\right)^{-k+2} + \ldots + \frac{1}{(n+\lambda)^k} \bigg| \\
&\le M \left(\frac{n}{n+\lambda}\right)^k \left(\lambda - \frac{n+\lambda}{n}\omega\right)^{-k} \\
&\quad \times \left[1 + \left(\frac{n^2}{(n+\lambda)^2}\right)^{-1} \frac{1}{n+\lambda} \left(\frac{\lambda n}{n+\lambda} - \omega\right)\right]^k \\
&= M \left(\lambda - \frac{n+\lambda}{n}\omega\right)^{-k} \left(1 - \frac{\omega}{n}\right)^k = M \left(\lambda - \frac{n}{n-\omega}\omega\right)^{-k}.
\end{aligned}$$

It follows that for $n \ge 2\omega$, the family $\{A_n(t)\}$, $t \in [0,T]$, is stable with stability constants M and 2ω ($n \ge 2\omega$ is fixed).

Lemma 2. *Let* $\{A(t)\}$, $t \in [0,T]$, *be a stable family with stability constants* M *and* ω. *If the mapping* $[0,T] \ni t \to A(t) \in B(X)$ *is strongly continuous*, *then the fundamental solution* U *corresponding to* $A(t)$ *is strongly continuous in the triangle* Δ_T *and*

$$\|U(t,s)\| \le M e^{\omega T} \quad \text{for } (t,s) \in \Delta_T, \tag{13}$$

where M *and* ω *are the stability constants.*

Proof. Existence and strong continuity of U follow from boundedness and strong continuity of the mapping $[0,T] \ni t \to A(t)$.

In order to prove (13), we start by approximating the family $\{A(t)\}$, $t \in [0,T]$, by piecewise constant families $\{A_\nu(t)\}$, $t \in [0,T]$, defined as follows. Let $t_k^\nu := (k/\nu)T$, $k = 0,1,\ldots,\nu$, $\nu \in \mathbb{N}$, and let (cf. [3, p. 135])

$$A_\nu(t) := \begin{cases} A(t_k^\nu) & \text{for } t_k^\nu \le t < t_{k+1}^\nu,\ k = 0,1,\ldots,\nu-1, \\ A(T) & \text{for } t = T. \end{cases} \tag{14}$$

From the strong continuity of $t \to A(t)$ it follows that

$$\|[A(t) - A_\nu(t)]x\| \to 0 \quad \text{as } \nu \to \infty \tag{15}$$

uniformly with respect to $t \in [0,T]$ for each $x \in X$.

Denote by $S_t(s)$, $s \ge 0$, the C_0-semigroup generated by $A(t)$ for $t \in [0,T]$ and let

$$U_\nu(t,s) := \begin{cases} S_{t_j^\nu}(t-s) \quad \text{for } t_j^\nu \le s \le t \le t_{j+1}^\nu, \\ S_{t_k^\nu}(t-t_k^\nu)[\prod_{j=l+1}^{k-1} S_{t_j^\nu}(T/\nu)]S_{t_l^\nu}(t_{l+1}^\nu - s) \\ \qquad\qquad \text{for } k > l,\ t_k^\nu \le t \le t_{k+1}^\nu,\ t_l^\nu \le s \le t_{l+1}^\nu. \end{cases} \tag{16}$$

From (16) and Theorem 3.1 of [3, p. 135] it follows that $U_\nu(t,s)$ is the fundamental solution corresponding to $A_\nu(t)$, the mapping

$$\varDelta_T \ni (t,s) \to U_\nu(t,s) \tag{17}$$

is strongly continuous and

$$\|U_\nu(t,s)\| \le Me^{\omega(t-s)} \quad \text{for } (t,s) \in \varDelta_T, \tag{18}$$

where M and ω are the constants from (12).

From the equality

$$\frac{\partial}{\partial t} U(t,s)x = A(t)U(t,s)x, \quad x \in X,$$

we obtain

$$\frac{\partial}{\partial t} U(t,s)x = A_\nu(t)U(t,s)x + [A(t) - A_\nu(t)]U(t,s)x.$$

Hence

$$U(t,s)x = U_\nu(t,s)x + \int_s^t U_\nu(t,\tau)[A(\tau) - A_\nu(\tau)]U(\tau,s)x\,d\tau \tag{19}$$

(cf. [2, p. 195, Th. 3.1]) and so we have

$$\|[U(t,s) - U_\nu(t,s)]x\| \le Me^{\omega T} \int_0^T \|[A(\tau) - A_\nu(\tau)]U(\tau,s)x\|\,d\tau.$$

From (15) and from Lemma 3.7 of [1, p. 151] it follows that $\|[U(t,s) - U_\nu(t,s)]x\| \to 0$ as $\nu \to \infty$ uniformly in $(t,s) \in \varDelta_T$. By (18), this implies (13), i.e. the conclusion of Lemma 2.

THEOREM 1. *Suppose that*

(i) $\{A(t)\}$, $t \in [0,T]$, *is a stable family in the sense of Definition* 4,
(ii) $D(A(t)) = D$ *does not depend on* $t \in [0,T]$,
(iii) *the mapping* $[0,T] \ni t \to A(t)$ *is strongly continuous*,
(iv) $0 \in P(A(t))$ *for* $t \in [0,T]$.

Then the family $\{A(t)\}$, $t \in [0,T]$, *is stably approximated* (cf. Def. 3).

Proof. Define $A_n(t)$ by (10) for $n \in \mathbb{N}$. For each fixed $n \in \mathbb{N}$ and $t \in [0,T]$ the operator $A_n(t)$ commutes with $A(t)$ on D and $A_n(t)$ is a bounded operator on X.

Let $x \in D$ be fixed. We have

$$\begin{aligned}
&\|[A(t) - A_n(t)]A(t)^{-1}x\| \\
&\quad = \|[A(t) + nA(t)(A(t) - n)^{-1}]A(t)^{-1}x\| \\
&\quad = \|x + n(A(t) - n)^{-1}x\| = \|(A(t) - n)^{-1}A(t)x\| \\
&\quad \le \|(A(t) - n)^{-1}\| \cdot \|A(t)x\| \\
&\quad \le \frac{M}{n-\omega}\|A(t)x\| \le \frac{M}{n-\omega}K, \qquad \text{where } K = \sup\{\|A(t)x\| : t \in [0,T]\}.
\end{aligned}$$

This shows that

$$\|[A(t) - A_n(t)]A(t)^{-1}x\| \le M_1,$$

where M_1 does not depend on $n > \omega$ or $t \in [0,T]$. From this estimate we get

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|[A(t) - A_n(t)]A(t)^{-1}x\| = 0 \tag{20}$$

for each $x \in D$, where $\overline{D} = X$. By (19) and (20) in view of the Banach–Steinhaus theorem (cf. [2, p. 9]), the condition (6) of Definition 3 is satisfied.

From Lemma 2 it follows that the sequence $\{A_n(t)\}$ is uniformly stable with stability constants M and 2ω for $n \ge 2\omega$. Using Lemma 2 for each fixed $n \ge 2\omega$, we obtain

$$\|U_n(t,s)\| \le Me^{2\omega(t-s)} \le Me^{2\omega T}. \tag{21}$$

Theorem 1 is proved.

LEMMA 3. *Suppose that*

(i) *the mapping* $[0,T] \ni t \to A(t)x \in X$ *is of class* C^1 *for* $x \in D$,
(ii) $A(t)^{-1} \in B(X)$ *exists for* $t \in [0,T]$,
(iii) *the family* $\{A(t)\}$, $t \in [0,T]$, *is stably approximated by the sequence* $\{A_n(t)\}$, *where* $A_n(t)$ *is defined by* (10).

Then there exists a constant K independent of $n \in \mathbb{N}$ and $(t,s) \in \Delta_T$ such that

$$\|A(t)U_n(t,s)A(s)^{-1}\| \le K, \tag{22}$$

where $U_n(t,s)$ is the fundamental solution corresponding to $A_n(t)$.

Proof. According to Definition 3,

$$\|U_n(t,s)\| \le M, \tag{23}$$

where M does not depend on $n \in \mathbb{N}$ and $(t,s) \in \Delta_T$.

Consider the equation (cf. [2, p. 200])

$$\frac{dy}{dt} = A_n(t)y + A'(t)A(t)^{-1}y. \tag{24}$$

By (i) and (ii), the mapping $[0,T] \ni t \to A'(t)A(t)^{-1} \in B(X)$ is strongly continuous. In view of the Banach–Steinhaus theorem we get

$$\|A'(t)A(t)^{-1}\| \le C, \tag{25}$$

where C does not depend on $t \in [0,T]$.

Let $V_n(t,s)$ be the fundamental solution of (24). We have

$$V_n(t,s) = A(t)U_n(t,s)A(s)^{-1}, \qquad (t,s) \in \Delta_T \tag{26}$$

(cf. [2, p. 201]). From (23), (25) and (26) it follows that

$$\|V_n(t,s)\| = \|A(t)U_n(t,s)A(s)^{-1}\| \le Me^{CMT} = K \tag{27}$$

(see [2, p. 191]).

THEOREM 2. *Suppose that*

(i) *$\{A(t)\}$, $t \in [0,T]$, is a stable family in the sense of Definition* 4,
(ii) *$D(A(t)) = D$ does not depend on $t \in [0,T]$,*
(iii) *the mapping $[0,T] \ni t \to A(t)x \in X$ is of class C^1 for $x \in D$,*
(iv) *$A(t)^{-1} \in B(X)$ exists for $t \in [0,T]$.*

Then the family $\{A(t)\}$, $t \in [0,T]$, is stably approximated by the sequence $\{A_n(t)\}$ defined by (10), *and the sequence $\{U_n(t,s)\}$ of the fundamental solutions corresponding to $\{A_n(t)\}$ is strongly and uniformly convergent to $U(t,s)$ in Δ_T.*

Proof. Upon using Theorem 1 and Lemmas 2–4, the proof is analogous to the proof of Theorem 3.11 of [2, p. 208]. We omit the details and refer the reader to [2, Ch. II].

From Theorem 2 and [2, Th. 3.6, p. 200] it follows that if the family $\{A(t)\}$, $t \in [0,T]$, satisfies the assumptions of Theorem 2, then the Cauchy problem

$$\frac{du}{dt} = A(t)u, \quad u(s) = x, \quad x \in D,\ 0 \le s \le t \le T, \tag{28}$$

has the unique solution given by

$$u(t) = U(t,s)x, \tag{29}$$

where $U(t,s)$ is the fundamental solution for (28) defined in Theorem 2.

Remark 1. The set of assumptions (i)–(iii) of Theorem 2 is usually referred to as the "hyperbolic" case in contrast to the "parabolic" case where each $A(t)$, $t \geq 0$, is assumed to be the infinitesimal generator of an analytic semigroup. This terminology is justified by applications of the abstract results to partial differential equations (cf. [3, p. 134]).

3. Dependence of the fundamental solution on parameters. Let Ω be a compact subset of $\mathbb{R}^m$. We shall consider the following initial value problem with a parameter $h \in \Omega$:

$$\begin{cases} \dfrac{du}{dt} = A(h,t)u, \quad t \in [0,T], \ h \in \Omega, \\ u(s) = x, \quad 0 \leq s \leq t \leq T, \end{cases} \tag{30}$$

where $A : \Omega \times [0,T] \ni (h,t) \to A(h,t) \in \mathcal{C}(X)$, $D(A(h,t)) = D$, $\overline{D} = X$, $0 \in P(A(h,t))$ for $(h,t) \in \Omega \times [0,T]$ and $x \in D$.

THEOREM 3. *If, for any $(h,t) \in \Omega \times [0,T]$, $A(h,t)$ is bounded and, for each $x \in X$, the mapping*

$$\Omega \times [0,T] \ni (h,t) \to A(h,t)x \in X \text{ is continuous}, \tag{31}$$

then the mapping

$$\Omega \times \Delta_T \ni (h,t,s) \to U(h,t,s)x \in X \text{ is continuous.} \tag{32}$$

Proof. It follows from [2, p. 189] that the mapping $\Delta_T \ni (t,s) \to U(h,t,s)x \in X$ is continuous for any fixed $h \in \Omega$ and $x \in X$. Hence, by the Banach–Steinhaus theorem there exists $M_1 = M_1(h) \geq 0$ such that

$$\|U(h,t,s)\| \leq M_1 \quad \text{for } (t,s) \in \Delta_T.$$

To prove the theorem it is enough to show that

$$U(h,t,s)x \to U(h_0,t,s)x \quad \text{as } h \to h_0,$$

uniformly in $(t,s) \in \Delta_T$, for any $x \in X$. Since

$$\frac{\partial}{\partial t}U(h,t,s)x = A(h,t)U(h,t,s)x \quad \text{for } h \in \Omega, \ (t,s) \in \Delta_T, \ x \in X,$$

and $U(h,t,t)x = x$ for $h \in \Omega$, $t \in [0,T]$, $x \in X$, we have

$$\begin{aligned}\|[U(h,t,s) - U(h_0,t,s)]x\| &\\ &\le \int_s^t \|[A(h,\tau)U(h,\tau,s) - A(h_0,\tau)U(h_0,\tau,s)]x\|\,d\tau\\ &\le \int_s^t \|A(h,\tau)\| \cdot \|[U(h,\tau,s) - U(h_0,\tau,s)]x\|\,d\tau\\ &\quad + \int_s^t \|[A(h,\tau) - A(h_0,\tau)]U(h_0,\tau,s)x\|\,d\tau.\end{aligned}$$

By (31) and the Banach–Steinhaus theorem there exists $M > 0$ such that $\|A(h,t)\| \le M$. Thus,

$$\begin{aligned}\|[U(h,t,s) - U(h_0,t,s)]x\| \le M \int_0^T \|[U(h,\tau,s) - U(h_0,\tau,s)]x\|\,d\tau&\\ + \int_0^T \|[A(h,\tau) - A(h_0,\tau)]U(h_0,\tau,s)x\|\,d\tau.&\end{aligned}$$

By Gronwall's inequality

$$\|[U(h,t,s) - U(h_0,t,s)]x\| \le e^{TM} \int_0^T \|[A(h,\tau) - A(h_0,\tau)]U(h_0,\tau,s)x\|\,d\tau.$$

By (31) the operators $A(h,\tau) - A(h_0,\tau)$ converge strongly and uniformly in $\tau \in [0,T]$ to zero as $h \to h_0$, on the entire space X. This means that they converge to zero on the compact set of values of the continuous functions $U(h_0,\tau,s)x$. It follows that the functions

$$[A(h,\tau) - A(h_0,\tau)]U(h_0,\tau,s)x$$

converge to zero uniformly in $(\tau,s) \in \Delta_T$ (cf. [1, p. 151]). Hence $\lim_{h\to h_0} U(h,t,s)x = U(h_0,t,s)x$ uniformly in $(t,s) \in \Delta_T$.

DEFINITION 5. A family $\{A(h,t)\}, (h,t) \in \Omega \times [0,T]$, is said to be *uniformly stably approximated* with respect to $h \in \Omega$ if there exists a sequence $\{A_n(h,t)\}$ of bounded linear operators $A_n(h,t) : X \to X$, $n = 1,2,\ldots$, such that

(i) the mapping $\Omega \times [0,T] \ni (h,t) \to A_n(h,t)x \in X$ is continuous for $x \in X$, $n = 1,2,\ldots$,

(ii) $\lim_{n\to\infty}\{\sup \|[A_n(h,t) - A(h,t)]A(h,t)^{-1}x\| : (h,t) \in \Omega \times [0,T]\} = 0$ for $x \in X$ and the sequence $\{U_n(h,t,s)\}$ of fundamental solutions of (30) with $A(h,t) = A_n(h,t)$, $n = 1,2,\ldots$, is uniformly bounded, i.e. there exists

$K > 0$ such that

$$\|U_n(h,t,s)\| \le K \quad \text{for } h \in \Omega,\ (t,s) \in \Delta_T,\ n = 1,2,\ldots$$

DEFINITION 6. We say that a family $\{A(h,t)\}$, $(h,t) \in \Omega \times [0,T]$, is *uniformly stable* in Ω if

(i) $\{A(h,t)\}$ is stable (in the sense of Def. 4) for any $h \in \Omega$,
(ii) the stability constants M, ω are independent of h.

THEOREM 4. *Suppose that*

(i) *the family* $\{A(h,t)\}$, $(h,t) \in \Omega \times [0,T]$ *is uniformly stably approximated by* $\{A_n(h,t)\}$, $(h,t) \in \Omega \times [0,T]$,
(ii) *the mapping* $\Omega \times [0,T] \ni (h,t) \to A(h,t)x \in X$ *is continuous for* $x \in D$,
(iii) *the mapping* $[0,T] \ni t \to A(h,t)x \in X$ *is of class* C^1 *for* $h \in \Omega$, $x \in D$,
(iv) $A_n(h,t)$ *commutes with* $A(h,t)$ *for* $n \in \mathbb{N}$, $(h,t) \in \Omega \times [0,T]$,
(v) $\{U_n(h,t,s)\}$ *strongly and uniformly converges to* $U(h,t,s)$ *in* $\Omega \times \Delta_T$.

Then $U(h,t,s)$ *is the fundamental solution of the problem* (30) *and the mapping* $(h,t,s) \to U(h,t,s)x$ *is continuous.*

Proof. It follows from Theorem 3.6 of [2, p. 200] that the problem (30) is uniformly correct and, for $h \in \Omega$, $U(h,t,s)$ is its fundamental solution. By (i), the assumptions of Theorem 3 are satisfied. Thus, for $n \in \mathbb{N}$, the mapping $(h,t,s) \to U_n(h,t,s)x$ is continuous and the assumption (v) ends the proof.

THEOREM 5. *Suppose that*

(i) $\{A(h,t)\}, (h,t) \in \Omega \times [0,T]$, *is stable uniformly in* $h \in \Omega$,
(ii) *the mapping* $\Omega \times [0,T] \ni (h,t) \to A(h,t)x \in X$ *is continuous for* $x \in D$,
(iii) *the mapping* $[0,T] \ni t \to A(h,t)x \in X$ *is of class* C^1 *for* $h \in \Omega$, $x \in D$.

Then the problem (30) *has, for any* $h \in \Omega$, *exactly one solution* $u(h,\cdot)$ *which is given by* $u(h,t) = U(h,t,s)x$, *where* $U(h,t,s)$ *is the fundamental solution of* (30). *Moreover, the mapping* $\Omega \times \Delta_T \ni (h,t,s) \to U(h,t,s)x \in X$ *for* $x \in X$ *is continuous.*

Proof. Since for any $h \in \Omega$, the family $\{A(h,t)\}$ satisfies the assumptions of Theorem 2, it is stably approximated and the approximating sequence is given by

$$A_n(h,t) = -nA(h,t)R(n;A(h,t)) = -nI - n^2R(n;A(h,t)). \tag{33}$$

By (i),

$$\|R(n;A(h,t))\| \le \frac{M}{n-\omega}$$

and so $R(n;A(h,t))$ is bounded uniformly in $(h,t)\in \Omega\times[0,T]$, for any fixed $n\in\mathbb{N}$. Hence the mapping $(h,t)\to A_n(h,t)x$ for $x\in X$ is continuous (see [2, p. 176]), where $A_n(h,t)$ is given by (33). By Theorem 3 the mapping

$$(h,t,s)\to U_n(h,t,s)x \quad \text{for } x\in X,\ n=1,2,\ldots,$$

is continuous, where $U_n(h,t,s)$ is the fundamental solution of (30) with $A(h,t)=A_n(h,t)$ given by (33). By Theorem 2 the sequence $\{U_n(h,t,s)\}$ is strongly and uniformly convergent to $U(h,t,s)$ in Δ_T, for $h\in\Omega$. Since the family $\{A(h,t)\}$ is uniformly stably approximated with respect to $h\in\Omega$, similarly to the proof of Theorem 3.11 in [2] we conclude that $U_n(h,t,s)x\to U(h,t,s)x$ uniformly in $(h,t,s)\in\Omega\times\Delta_T$.

4. Dependence on parameter of solutions to problem (1). It is well known that under suitable assumptions the solution of problem (1) is given by

$$u(h,t)=U(h,t,0)u_0(h)+\int_0^t U(h,t,s)f(h,s)\,ds. \tag{34}$$

THEOREM 6. *Suppose that*

(i) *the family* $\{A(h,t)\}$ *satisfies the assumptions of Theorem* 4,
(ii) *the mapping* $\Omega\ni h\to u_0(h)\in X$ *is continuous*,
(iii) *the mapping* $\Omega\times[0,T]\ni(h,t)\to f(h,t)\in X$ *is continuous.*

Then the function u given by (34) *is continuous in* $\Omega\times[0,T]$.

Proof. By Theorem 4 the mapping $\Omega\times\Delta_T\ni(h,t,s)\to U(h,t,s)x\in X$ for $x\in X$ is continuous and so Theorem 6 is now a simple consequence of Theorem 1 of [5].

COROLLARY. *If the family* $\{A(h,t):(h,t)\in\Omega\times[0,T]\}$ *satisfies the assumptions of Theorem* 5 *and the mappings* $\Omega\ni h\to u_0(h)\in X$ *and* $\Omega\times[0,T]\ni(h,t)\to f(h,t)\in X$ *are continuous then the function given by* (34) *is continuous in* $\Omega\times[0,T]$.

Indeed, it is a simple consequence of Theorems 5 and 6.

THEOREM 7. *Let the assumptions of Theorem* 4 *be satisfied. Suppose that* $\Omega\subset\mathbb{R}$, h_0 *is an interior point of* Ω *and*

(i) $u(h,\cdot)\in C([0,T];X)$ *is a solution of the problem* (1),
(ii) *the mappings* $\Omega\ni h\to A(h,\cdot)x\in C([0,T];X)$, $\Omega\ni h\to f(h,\cdot)\in C([0,T];X)$ *and* $\Omega\ni h\to u_0(h)\in X$ *are differentiable at* h_0.

Then the mapping $\Omega \ni h \to u(h,\cdot) \in C([0,T];X)$ is differentiable at h_0 and

$$u'(h_0,t) = U(h_0,t,0)u_0'(h_0) + \int_0^t U(h_0,t,s)[f'(h_0,s) - A'(h_0,s)u(h_0,s)]\,ds, \tag{35}$$

where "$'$" denotes differentiation with respect to h.

Proof. Since $u(h,\cdot)$ is a solution of the problem (1), the function

$$\omega(h,t) = \frac{u(h,t)-u(h_0,t)}{h-h_0} \quad \text{for } h \neq h_0 \tag{36}$$

is for $h \neq h_0$ a solution of the problem

$$\begin{cases} \dfrac{dv}{dt} = A(h,t)v + F(h,t), \\ v(0) = \omega_0(h), \end{cases} \tag{37}$$

where

$$F(h,t) = \begin{cases} \dfrac{f(h,t)-f(h_0,t)}{h-h_0} - \dfrac{A(h,t)-A(h_0,t)}{h-h_0}u(h_0,t) & \text{for } h \neq h_0, \\ f'(h_0,t) - A'(h_0,t)u(h_0,t) & \text{for } h = h_0, \end{cases}$$

$$\omega_0(h) = \begin{cases} \dfrac{u_0(h)-u_0(h_0)}{h-h_0} & \text{for } h \neq h_0, \\ u_0'(h_0) & \text{for } h = h_0. \end{cases}$$

By (ii) the mapping

$$(h,t) \to \begin{cases} \dfrac{f(h,t)-f(h_0,t)}{h-h_0} & \text{for } h \neq h_0, \\ f'(h_0,t) & \text{for } h = h_0, \end{cases}$$

is continuous. We have

$$\frac{A(h,t)-A(h_0,t)}{h-h_0}u(h_0,t) = \frac{A(h,t)-A(h_0,t)}{h-h_0}A(h_0,0)^{-1}A(h_0,0)A(h_0,t)^{-1}A(h_0,t)u(h_0,t).$$

Since

$$A(h_0,t)u(h_0,t) = \frac{du(h_0,t)}{dt} - f(h_0,t)$$

and by Definition 1, the mapping

$$[0,T] \ni t \to A(h_0,t)u(h_0,t)u$$

is continuous. Also, the mapping

$$[0,T] \ni t \to A(h_0,t)A(h_0,t)^{-1}u$$

is continuous (cf. [2, Lemma 1.5]). Therefore

$$(h,t) \to \begin{cases} \dfrac{A(h,t)-A(h_0,t)}{h-h_0} u(h_0,t) & \text{for } h \neq h_0, \\ A'(h_0,t)u(h_0,t) & \text{for } h = h_0, \end{cases}$$

is continuous. By Theorem 6 the mapping

$$\widetilde{\omega}(h,t) := U(h,t,0)\omega_0(h) + \int_0^t U(h,t,s)F(h,s)\,ds$$

is continuous and

$$\widetilde{\omega}(h,t) = \begin{cases} \omega(h,t) & \text{for } h \neq h_0, \\ u'(h_0,t) & \text{for } h = h_0. \end{cases}$$

Therefore

$$\begin{aligned} u'(h_0,t) = {} & U(h_0,t,0)u_0'(h_0) \\ & + \int_0^t U(h_0,t,s)[f'(h_0,s) - A(h_0,s)u(h_0,s)]\,ds. \end{aligned}$$

COROLLARY 2. *If for any $h \in \Omega$ the assumptions of Theorem 7 are satisfied, then the mapping*

$$\Omega \ni h \to u(h,\cdot) \in C([0,T];X)$$

is differentiable and

$$u'(h,t) = U(h,t,0)u_0'(h) + \int_0^t U(h,t,s)F_1(h,s)\,ds,$$

where $F_1(h,s) = f'(h,s) - A'(h,s)u(h,s)$.

Remark 1. Let the assumptions of Theorem 4 be satisfied. If for any $h \in \Omega$ the mapping $[0,T] \ni t \to f(h,t) \in X$ is of class C^1, then the function u given by (34) is the unique solution of the problem (1) (see [4, Th. 4.52]).

Remark 2. Similarly to [6] one can prove theorems on higher regularity of the solution of problem (1).

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